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$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n, \quad n \geq 1 \quad (1)$$

$$(F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}, \quad n \geq 1).$$

I.

$$\sin r + \sin s = 2 \sin \frac{r+s}{2} \cos \frac{r-s}{2},$$

$$r = s, \quad r = \frac{(n+1)x}{2}, s = \frac{(n-1)x}{2}, n \in \mathbb{N},$$

$$\sin \frac{(n+1)x}{2} + \sin \frac{(n-1)x}{2} = 2 \sin \frac{nx}{2} \cos \frac{x}{2},$$

$$2 \sin \frac{(n+1)x}{2} = 4 \sin \frac{nx}{2} \cos \frac{x}{2} - 2 \sin \frac{(n-1)x}{2}. \quad (2)$$

$$p_n = 2 \sin \frac{nx}{2}, n = 0, 1, 2, 3, \dots \quad q = 2 \cos \frac{x}{2}. \quad (2)$$

$$p_{n+1} = p_n q - p_{n-1}, \quad n = 1, 2, 3, \dots \quad (3)$$

$$(3) \quad (p_n)$$

$$p_1 = q.$$

$$p_1 = p,$$

$$p_2 = p_1 q - p_0 = p_1 q,$$

$$p_3 = p_2 q - p_1 = (p_1 q) q - p_1 = p_1 (q^2 - 1),$$

$$p_4 = p_3 q - p_2 = p_1 (q^2 - 1) q - p_1 q = p_1 (q^3 - 2q),$$

$$p_5 = p_4 q - p_3 = p_1 (q^3 - 2q) q - p_1 (q^2 - 1) = p_1 (q^4 - 3q^2 + 1),$$

$$p_6 = p_5 q - p_4 = p_1 (q^4 - 3q^2 + 1) q - p_1 (q^3 - 2q) = p_1 (q^5 - 4q^3 + 3q),$$

$$p_7 = p_6 q - p_5 = p_1 (q^5 - 4q^3 + 3q) q - p_1 (q^4 - 3q^2 + 1) = p_1 (q^6 - 5q^4 + 6q^2 - 1),$$

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$$\frac{p_n}{p_1} \quad q,$$

$n \setminus j$	0	1	2	3	4	Σ
1	1					1
2	1					1
3	1	1				2
4	1	2				3
5	1	3	1			5
6	1	4	3			8
7	1	5	6	1		13
8	1	6	10	4		21

$$B(n, j)$$

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} B(n, j). \tag{4}$$

$$F_7 = B(7,0) + B(7,1) + B(7,3) = 1 + 5 + 6 + 1 = 13.$$

(, , [2]).

$$u_n = 2 \cos \frac{nx}{2} \quad u_{n-1}, u_{n-2} \quad q = 2 \cos \frac{x}{2}.$$

$$L_n,$$

$$L_1 = 1, L_2 = 3, L_{n+2} = L_{n+1} + L_n, \quad n \geq 1$$

([2]).

$$L_n^2 - 5F_n^2 = 4 \cdot (-1)^n.$$

II.

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$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \quad n \geq 1.$$

$$r = \frac{1+\sqrt{5}}{2}, s = \frac{1-\sqrt{5}}{2},$$

$$F_n = \frac{1}{\sqrt{5}}(r^n - s^n), n \geq 1,$$

$$rs = -1, \quad - \quad s = -r^{-1},$$

$$F_n = \frac{1}{\sqrt{5}}(r^n - (-1)^n r^{-n}), n \geq 1. \tag{5}$$

f

$[-a, a]$.

$$f_N(x) = \frac{1}{2}[f(x) - f(-x)] \quad f_P(x) = \frac{1}{2}[f(x) + f(-x)].$$

$$x \in [-a, a] \quad -x \in [-a, a]$$

$$f_N(-x) = \frac{1}{2}[f(-x) - f(-(-x))] = \frac{1}{2}[f(-x) - f(x)]$$

$$= -\frac{1}{2}[f(x) - f(-x)] = -f_N(x),$$

$$f_P(-x) = \frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(-x) + f(x)] = f_P(x),$$

$$f_N \quad , \quad f_P \quad . \quad -$$

$$x \in [-a, a]$$

$$f_N(x) + f_P(x) = \frac{1}{2}[f(x) - f(-x)] + \frac{1}{2}[f(x) + f(-x)] = f(x),$$

$[-a, a]$

$[-a, a]$.

$$\text{sh}(x) = \frac{e^x - e^{-x}}{2}, \quad -$$

$$\text{ch}(x) = \frac{e^x + e^{-x}}{2} \quad .$$

$$\text{sh}(x) \quad , \quad \text{ch}(x) \quad . \quad ,$$

$$t = \ln r, \dots e^t = r, \tag{5} \quad k$$

$$F_{2k} = \frac{1}{\sqrt{5}}(r^{2k} - (-1)^{2k} r^{-2k}) = \frac{1}{\sqrt{5}}(r^{2k} - r^{-2k})$$

$$= \frac{1}{\sqrt{5}}((e^t)^{2k} - (e^t)^{-2k}) = \frac{1}{\sqrt{5}}(e^{2kt} - e^{-2kt}) \tag{6}$$

$$= \frac{2}{\sqrt{5}} \text{sh}(2kt),$$

$$\begin{aligned}
F_{2k+1} &= \frac{1}{\sqrt{5}} (r^{2k+1} - (-1)^{2k+1} r^{-(2k+1)}) = \frac{1}{\sqrt{5}} (r^{2k+1} + r^{-(2k+1)}) \\
&= \frac{1}{\sqrt{5}} ((e^t)^{2k+1} + (e^t)^{-(2k+1)}) = \frac{1}{\sqrt{5}} (e^{(2k+1)t} + e^{-(2k+1)t}) \quad (7) \\
&= \frac{2}{\sqrt{5}} \operatorname{ch}((2k+1)t).
\end{aligned}$$

(6) (7)

$$\begin{aligned}
& A_n \operatorname{sh}(nt) + B_n \operatorname{ch}(nt), \quad A_n \quad B_n \\
n \quad A_n = \frac{2}{\sqrt{5}} \quad n \quad A_n = 0 \quad n \quad , \quad B_n = \frac{2}{\sqrt{5}} \\
n \quad B_n = 0 \quad n \quad .
\end{aligned}$$

$$e^{ix} = \cos x + i \sin x,$$

$$e^{-ix} = \cos x - i \sin x. \quad :$$

$$\operatorname{ch}(ix) + \operatorname{sh}(ix) = \frac{e^{ix} + e^{-ix}}{2} + \frac{e^{ix} - e^{-ix}}{2} = e^{ix} = \cos x + i \sin x, \quad (8)$$

$$\operatorname{ch}(ix) - \operatorname{sh}(ix) = \frac{e^{ix} + e^{-ix}}{2} - \frac{e^{ix} - e^{-ix}}{2} = e^{i(-x)} = \cos x - i \sin x. \quad (9)$$

(8) (9),

$$\operatorname{ch}(ix) = \cos x, \quad (10)$$

(8) (9),

$$\operatorname{sh}(ix) = i \sin x. \quad (11)$$

$$\begin{aligned}
\operatorname{ch}(x)\operatorname{ch}(y) + \operatorname{sh}(x)\operatorname{sh}(y) &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} \\
&= \frac{e^{x+y} + e^{-(x+y)} + e^{-x+y} + e^{x-y}}{4} + \frac{e^{x+y} + e^{-(x+y)} - e^{-x+y} - e^{x-y}}{4} \\
&= \frac{e^{x+y} + e^{-(x+y)}}{2} = \operatorname{ch}(x+y),
\end{aligned}$$

$$\operatorname{ch}(x+y) = \operatorname{ch}(x)\operatorname{ch}(y) + \operatorname{sh}(x)\operatorname{sh}(y), \quad (12)$$

$$\begin{aligned}
\operatorname{ch}(x)\operatorname{sh}(y) + \operatorname{sh}(x)\operatorname{ch}(y) &= \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} \\
&= \frac{e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}}{4} + \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)}}{4} \\
&= \frac{e^{x+y} - e^{-(x+y)}}{2} = \operatorname{sh}(x+y).
\end{aligned}$$

$$\operatorname{sh}(x+y) = \operatorname{ch}(x)\operatorname{sh}(y) + \operatorname{sh}(x)\operatorname{ch}(y). \quad (13)$$

$$\cos \frac{nf}{2} = \begin{cases} 0, & n = 2k+1, \\ (-1)^k, & n = 2k, \end{cases} \quad (14)$$

$$\sin \frac{nf}{2} = \begin{cases} 0, & n = 2k, \\ (-1)^k, & n = 2k+1. \end{cases} \quad (15)$$

$$(15), \quad (10) \quad (11), \quad (14)$$

$$t = \ln r = \ln \frac{1+\sqrt{5}}{2},$$

$$\begin{aligned} \frac{2}{\sqrt{5}} i^n \operatorname{sh} n(t - i \frac{f}{2}) &= \frac{2}{\sqrt{5}} i^n (\operatorname{sh} nt \cdot \operatorname{ch}(-i \frac{nf}{2}) + \operatorname{sh}(-i \frac{nf}{2}) \cdot \operatorname{ch}(nt)) \\ &= \frac{2}{\sqrt{5}} i^n (\operatorname{sh} nt \cdot \operatorname{ch}(i \frac{nf}{2}) - \operatorname{ch}(nt) \cdot \operatorname{sh}(i \frac{nf}{2})) \\ &= \frac{2}{\sqrt{5}} i^n (\operatorname{sh} nt \cdot \cos \frac{nf}{2} - i \operatorname{ch}(nt) \cdot \sin \frac{nf}{2}) \\ &= \begin{cases} \frac{2}{\sqrt{5}} i^{2k} (\operatorname{sh}(2kt) \cdot \cos \frac{2kf}{2} - i \operatorname{ch}(2kt) \cdot \sin \frac{2kf}{2}), & n = 2k \\ \frac{2}{\sqrt{5}} i^{2k+1} (\operatorname{sh}((2k+1)t) \cdot \cos \frac{(2k+1)f}{2} - i \operatorname{ch}((2k+1)t) \cdot \sin \frac{(2k+1)f}{2}), & n = 2k+1 \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{5}} (-1)^k \operatorname{sh}(2kt) \cdot \cos kf, & n = 2k \\ \frac{2}{\sqrt{5}} (-1)^k \operatorname{ch}((2k+1)t) \sin \frac{(2k+1)f}{2}, & n = 2k+1 \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{5}} \operatorname{sh}(2kt), & n = 2k \\ \frac{2}{\sqrt{5}} \operatorname{ch}((2k+1)t), & n = 2k+1 \end{cases} \\ &= F_n. \end{aligned}$$

$$F_n = \frac{2}{\sqrt{5}} i^n \operatorname{sh} n(t - i \frac{f}{2}), \quad t = \ln r = \ln \frac{1+\sqrt{5}}{2}.$$

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