

The 16th Romanian Master of Mathematics Competition

Day 1: 12 February, 2025, Bucharest

Language: English

Problem 1. Let $n > 10$ be an integer, and let A_1, A_2, \dots, A_n be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j, k)$ to be the 10th smallest of the distances from A_j to A_1, A_2, \dots, A_k , excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j, n) \geq \frac{1}{2}f_{10}(n, n)$ for all j in the range $1 \leq j \leq n-1$.

Problem 2. Consider an infinite sequence of positive integers a_1, a_2, a_3, \dots such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n . Is it possible for two terms of such a sequence to be equal?

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

For any tree T with vertices v_1, v_2, \dots, v_n and any pairwise distinct complex numbers z_1, z_2, \dots, z_n , there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j .

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 because $7 = 3 + 4$.

Each problem is worth 7 marks.

Time allowed: $4\frac{1}{2}$ hours.

The 16th Romanian Master of Mathematics Competition

Day 2: 13 February, 2025, Bucharest

Language: English

Problem 4. Let \mathbb{Z} denote the set of integers, and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100} . Fix a positive integer c . Determine all functions $f: S \rightarrow \mathbb{Z}$ satisfying $f(xy + c) = f(x) + f(y)$ for all $x, y \in S$.

Problem 5. Let ABC be an acute triangle with $AB < AC$, and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circumcircle of triangle BOC . Circle Γ intersects line AO at points O and A' , and Γ intersects the circle of radius AO with centre A at points O and F . Prove that the circle which has diameter AA' , the circumcircle of triangle AFH , and Γ pass through a common point.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets S_1, S_2, \dots, S_k , each of which has exactly $m + 1$ elements: one red and m blue. Let \mathcal{F} be the family of all subsets T of $S_1 \cup S_2 \cup \dots \cup S_k$ such that, for every i , the intersection $T \cap S_i$ is monochromatic. Determine the largest possible number of sets in a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that no two sets in \mathcal{G} are disjoint.

A set is monochromatic if all of its elements have the same colour. In particular, the empty set is monochromatic.

Each problem is worth 7 marks.

Time allowed: $4\frac{1}{2}$ hours.

The 16th Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Let $n > 10$ be an integer, and let A_1, A_2, \dots, A_n be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j, k)$ to be the 10th smallest of the distances from A_j to A_1, A_2, \dots, A_k , excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j, n) \geq \frac{1}{2}f_{10}(n, n)$ for all j in the range $1 \leq j \leq n-1$.

IRAN, MORTEZA SAGHAFIAN

Solution 1. For every i , denote $a_i = f_{10}(i, i-1)$ and $b_i = f_{10}(i, n)$. So, we need to show that $b_n \leq 2b_i$ for all i . Notice that $a_i \geq b_i$ for all i .

To prove this, choose an arbitrary $i < n$, and let $A_i A_{j_1}, A_i A_{j_2}, \dots, A_i A_{j_{10}}$ be the ten smallest numbers among the $A_i A_j$ with $j \neq i$, ordered so that $j_1 < j_2 < \dots < j_{10}$.

If $j_{10} < i$, then $i > 10$, and the problem condition yields

$$b_i = \max_{1 \leq k \leq 10} A_i A_{j_k} = a_i \geq a_n \geq b_n,$$

which is even stronger than we need.

Otherwise, set $j = j_{10} > i$ (in this case we also have $j_{10} > 10$), and denote $m = b_i = \max_{1 \leq k \leq 10} A_i A_{j_k}$. By the problem condition, we have $a_j \geq a_n = b_n$. On the other hand, we have

$$a_j \leq \max \left(A_j A_i, \max_{1 \leq k \leq 9} A_j A_{j_k} \right) \leq \max \left(A_j A_i, \max_{1 \leq k \leq 9} (A_j A_i + A_i A_{j_k}) \right) \leq 2m,$$

as $A_j A_i, A_i A_{j_k} \leq m$. So $b_n \leq a_j \leq 2m = 2b_i$, as desired.

Solution 2. Let $d_j = f_{10}(j, n)$, $j = 1, \dots, n$. To prove that $2d_j \geq d_n$ for every $j = 1, \dots, n-1$, induct on n .

Consider the base case, $n = 11$. Note that each $d_j = \max_{i \neq j} A_i A_j$, as f_{10} is 10-variate. Let $d_{11} = A_{11} A_k$ for some index $k \leq 10$. Clearly, $2d_k \geq 2A_k A_{11} = 2d_{11} \geq d_{11}$ and if $j \neq k$ then $2d_j \geq A_j A_k + A_j A_{11} \geq A_k A_{11} = d_{11}$, by the triangle inequality.

For the induction step, let $n > 11$ and note that

$$\max_{n \geq \ell \geq k} f_{10}(\ell, k-1) = \max_{n-1 \geq \ell \geq k} f_{10}(\ell, k-1),$$

as both maxima are achieved at $\ell = k$, by the condition in the statement. Hence A_1, A_2, \dots, A_{n-1} also satisfy this condition.

Let $d'_j = f_{10}(j, n-1)$, $j = 1, \dots, \leq n-1$. By the induction hypothesis, $2d'_j \geq d'_{n-1}$ for all $j \leq n-2$. Note that $d'_{n-1} = f_{10}(n-1, n-2) \geq f_{10}(n, n-2) \geq f_{10}(n, n-1) = d_n$; the first inequality holds by the condition in the statement for $k = n-1$ and the second because adding more variables to f_{10} does not increase its value.

Let now Δ_i be the closed disc (interior and boundary) of radius d_i , centred at A_i . By the definition of d_i , each Δ_i contains at least 11 points, of which at most 10 (A_i , inclusive) lie strictly inside.

Finally, suppose, if possible, that $2d_j < d_n$ for some index $j < n$. If $A_j A_n$ is not among the first 10 distances from A_j to the other points, then $d_j = d'_j$ and this leads to a contradiction with the induction hypothesis. So $A_j A_n$ has to be among the first 10 distances from A_j to the other points. This means that $d_j \geq A_j A_n$, so $d_n > 2d_j \geq 2A_j A_n$. Hence Δ_j lies strictly inside Δ_n . This is a contradiction, as Δ_j contains at least 11 points, whereas Δ_n contains at most 10 strictly inside. The conclusion follows.

Problem 2. Consider a sequence of integers a_1, a_2, a_3, \dots such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n . Is it possible that two terms of such a sequence be equal?

RUSSIA, PAVEL KOZLOV

Solution. The answer is in the negative. Notice first that, if $a_n > 1$, then $2^{a_n} - 1 \equiv 3 \pmod{4}$; since $(2^{a_n} - 1)a_{n+1}$ is a perfect square, we should have $a_{n+1} \equiv 0 \pmod{4}$ or $a_{n+1} \equiv 3 \pmod{4}$, so in particular $a_{n+1} > 1$. As $a_1 > 1$, we conclude that all terms of the sequence are greater than 1.

Denote the largest prime divisor of an integer $k > 1$ by $g(k)$. We will show that $g(a_{n+1}) > g(a_n)$ for all n , which yields the desired result. To this end, usage is made of the lemma below.

Lemma: For any prime p , each prime divisor of $2^p - 1$ is greater than p .

Proof. Let q be a prime factor of $2^p - 1$; then q is odd. The multiplicative order d of 2 modulo q divides p and is larger than 1, so $d = p$. On the other hand, by Fermat's little theorem, $2^{q-1} \equiv 1 \pmod{q}$, so $p = d \mid q - 1$ and the lemma follow.

Choose now any positive integer n , and denote, for convenience, $k = a_m$ and $\ell = a_{n+1}$. Let $p = g(k)$; then $2^p - 1 \mid 2^k - 1$. Since $2^p - 1 \equiv 3 \pmod{4}$, this number is not a square, so there exists a prime q such that $v_q(2^p - 1)$ is odd. By the Lemma, $q > p$, so in particular $q \nmid k$. Therefore, by the Lifting Exponent Lemma,

$$v_q(2^k - 1) = v_q(2^p - 1) + v_q(k/p) = v_q(2^p - 1) + 0,$$

so $v_q(2^k - 1)$ is odd as well. Since $(2^k - 1)\ell$ is a perfect square, we should then have $q \mid \ell$, so $g(\ell) \geq q > p = g(k)$, as desired.

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

For any tree T with vertices v_1, v_2, \dots, v_n and any pairwise distinct complex numbers z_1, z_2, \dots, z_n , there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j .

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 because $7 = 3 + 4$.

ROMANIA, ANDREI CHIRIȚĂ

Solution 1. First we provide a proof that $k \geq n - 1$. Let T be the path where v_i and v_{i+1} are adjacent for all $1 \leq i \leq n - 1$. Let ω be a primitive root of unity of order n and let $a_i = \omega^i$ for all $1 \leq i \leq n$.

If $f(X) = P(X, \omega X)$, then for all $1 \leq i \leq n - 1$ we have $f(\omega^i) = P(a_i, a_{i+1}) = 0$. Since $f(1) = P(a_n, a_1) \neq 0$, f is non-zero and has at least $n - 1$ roots. This means that $\deg P \geq \deg f \geq n - 1$, proving $k \geq n - 1$.

It remains to prove that $k = n - 1$ is sufficient i.e. for any tree T and any a_1, a_2, \dots, a_n we can find a polynomial P of degree at most $n - 1$. For brevity, we call a two-variable polynomial $A(X, Y)$ *symmetric* if $A(X, Y) = A(Y, X)$.

We begin with the following observation. Suppose that A and B are two variable polynomials of degree at most d . Then we can find $\alpha \in \mathbb{C}$ such that for any $1 \leq i, j \leq n$, $A(a_i, a_j) + \alpha B(a_i, a_j) = 0$ if and only if $A(a_i, a_j) = B(a_i, a_j) = 0$. This means that we can "merge" two conditions of degree at most d into a condition of degree at most d (note that this produces a symmetric polynomial if the initial polynomials are symmetric).

For any integer $t \geq 2$, let a *star* of size t be a collection of t edges for which there is a vertex which belongs to all edges. We will prove the following claims.

Claim 1. Let G be a graph with vertices v_1, v_2, \dots, v_n and E edges. Suppose that we can partition the edges of G into a number of stars. Then for any distinct complex numbers a_1, a_2, \dots, a_n we can find a symmetric polynomial P of degree at most E such that for all $1 \leq i, j \leq n, i \neq j$, $P(a_i, a_j) = 0$ if and only if there is an edge between v_i and v_j in G .

Proof. We will first prove the claim when G consists of a star of size $E \leq n - 1$ and some isolated vertices. Without loss of generality, let $v_1v_2, v_1v_3, \dots, v_1v_{E+1}$ be the edges of G . Also let $s_1 = a_1 + a_2, s_2 = a_1 + a_3, \dots, s_E = a_1 + a_{E+1}$.

Consider merging the polynomials $(X - a_1)(Y - a_1)$ and $(X + Y - s_1)(X + Y - s_2) \dots (X + Y - s_E)$ into a polynomial of degree at most E (which is of course symmetric). They both vanish at a pair a_i, a_j if and only if $1 \in \{i, j\}$ and $a_i + a_j \in \{s_1, s_2, \dots, s_E\}$. These two happen if and only if v_i and v_j are adjacent, so this produces a valid polynomial.

For the general case, let $S_1 \cup S_2 \cup \dots \cup S_k$ be the partition of the edges of G . For each $1 \leq i \leq k$, we can find a two variable polynomial P_i of degree at most $|S_i|$ which vanishes only at the edges of S_i . Then we can let $P = P_1P_2 \dots P_k$, which satisfies the claim as $\deg P \leq |S_1| + |S_2| + \dots + |S_k| = E$, as desired.

Claim 2. Any tree Γ with odd number of vertices can be partitioned into stars of size 2.

Proof. We prove this by induction on the number of the vertices of Γ . The base case is clear, since Γ is a star of size 2 when Γ has three vertices.

For the inductive step, let Γ be a tree with $2m + 1$ vertices, where $m \geq 2$. Let $u_1u_2 \dots u_t$ be a path of maximal length in Γ (of course, $t \geq 3$). Then any neighbour of u_2 except for

maybe u_3 must have degree 1, otherwise we can delete u_1 and insert two edges, contradicting the maximality of t . If $\deg u_2 = 2$, we can form the star u_1u_2, u_2u_3 and apply the inductive hypothesis on $\Gamma \setminus \{u_1, u_2\}$. If $\deg u_2 \geq 3$, let $u \neq u_1, u_3$ be a neighbour of u_2 . Then create the star u_1u_2, uu_2 and apply the inductive hypothesis on $\Gamma \setminus \{u, u_1\}$. This proves the claim.

The case where n is odd becomes trivial, since T has $n - 1$ edges. Assume that n is even. Without loss of generality, let $\deg v_n = 1$ and let v_n be adjacent to v_{n-1} .

Consider the graph T' formed by deleting the edge $v_{n-1}v_n$ from T (but not perturbing the n vertices). Clearly, T' consists of v_n and a tree on v_1, v_2, \dots, v_{n-1} . As $n - 1$ is odd, the edges of T' can be partitioned into stars of size 2 from Claim 2. From Claim 1 it follows that there is a symmetric polynomial $Q(X, Y)$ of degree at most $n - 2$ such that $Q(a_i, a_j) = 0$ if and only if $i, j \leq n - 1$ and v_i, v_j are adjacent in T .

Let $g(X)$ be the polynomial of degree $n - 1$ such that $g(a_1) = g(a_2) = \dots = g(a_{n-1}) = 0$ and $g(a_n) = -Q(a_{n-1}, a_n) = -Q(a_n, a_{n-1})$. Let P be the polynomial of degree at most $n - 1$ obtained by merging $F_1(X, Y) = Q(X, Y) + g(X) + g(Y)$ and $F_2(X, Y) = Q(X, Y)(X + Y - s)$, where $s = a_n + a_{n-1}$. It is easy to see that P vanishes at each (a_i, a_j) for which v_i, v_j are adjacent in T .

Suppose that v_i and v_j are not adjacent in T . If $i, j \leq n - 1$, then $F_1(a_i, a_j) = Q(a_i, a_j) \neq 0$. If $n \in \{i, j\}$, then $a_i + a_j \neq s$ and $Q(a_i, a_j) \neq 0$, so $F_2(a_i, a_j) \neq 0$. Hence $P(a_i, a_j) \neq 0$. This proves that P satisfies the required conditions, as desired.

Solution 2. Establish the lower bound as in Solution 1. We now address the upper bound differently. Let G be a graph on vertices v_1, \dots, v_n . Say that a polynomial $P(X, Y)$ is G -good if it satisfies the conditions in the statement of the problem. We prove the more general fact below:

Claim. Let d_i be the degree of v_i . Assume that these degrees satisfy $d_i \leq n - i$ for all $i \leq n - 1$, and $d_n = 1$. Then there is a G -admissible polynomial of degree at most $n - 1$.

Notice here that, if G is a tree with vertices ordered so that $d_1 \geq d_2 \geq \dots \geq d_n$, then it satisfies the conditions in the Claim. Indeed, we have $d_n = 1$, and if $d_i > n - i$ for some $i \leq n - 1$, then we have

$$2n - 2 \sum_{j=1}^n d_j \geq \sum_{j=1}^i (n - i + 1) + \sum_{j=i+1}^n 1 = i(n - i + 1) + (n - i) = (i + 1)(n - i + 1) - 1 \geq 2n - 1,$$

which is a contradiction. So, it suffices to prove the Claim.

Proof of the Claim. Let

$$P(X, Y) = \sum_{j=0}^{n-1} R_j(Y)X^j$$

be the sought polynomial; set $R_{n-1}(X) = 1$. Denote

$$Q_i(X) = P(X, a_i) = X^{n-1} + \sum_{j=0}^{n-2} q_{ij}X^j, \quad \text{where } q_{ij} = R_j(a_i).$$

So, we will seek for the sequences $C_j = (q_{1j}, q_{2j}, \dots, q_{nj})$ such that there exists a polynomial R_j with $\deg R_j \leq n - j - 1$ such that $q_{ij} = R_j(a_i)$. Notice that the first $n - j$ terms of such a sequence determine it uniquely; in particular, there are no restrictions on the sequence C_0 .

We know that the polynomial Q_i has d_i prescribed roots. For every $i \leq n - 1$, augment this list by some numbers not from the set $A = \{a_1, a_2, \dots, a_n\}$ to the list $b_{i1}, b_{i2}, \dots, b_{i, n-i}$. Also, denote by b_{n1} the unique prescribed root of Q_n . Thus, we should have

$$Q_i(X) = S_i(X) \prod_{j=1}^{\max(n-i, 1)} (X - b_{ij}), \quad (1)$$

where S_i is a monic polynomial with $\deg S_i = i - 1$ for $i \leq n - 1$ and $\deg S_n = n - 2$. The only extra conditions we have are that each S_i should achieve non-zero values at the prescribed finite subset A_i of A (containing no prescribed roots of Q_i).

The polynomial Q_1 is uniquely determined by (1).

Assume that the polynomials Q_1, \dots, Q_{i-1} have already been determined, for some $i \leq n - 2$. This means that the sequences S_1, \dots, S_{i-1} are also determined. This determines the polynomial S_i up to the constant term. So we may choose this constant term so that S_i has no prohibited roots, thus defining Q_i .

It remains to deal with the indices $i = n - 1, n$. At this moment, both polynomials S_{n-1} and S_n are determined up to constant terms. The conditions they must obey are: (i) they should not have roots inside A_{n-1} and A_n , respectively; and (ii) the sequence C_1 should be a sequence of values of a polynomial R_1 with $\deg R_1 \leq n - 2$. Notice that all such polynomials R_1 are determined up to an additive polynomial of the form $\alpha \prod_{j \leq n-2} (X - a_j)$. The coefficients $q_{n-1,1}$ and $q_{n,1}$ depend linearly on α with a non-zero linear term. Hence the constant terms of S_{n-1} and S_n depend linearly on α . Now, again, there are only finitely many restrictions which remove finitely many values of α ; any other value fits the bill.

Solution 3. The answer is $n - 1$. Use the same argument as the official solution for the lower bound.

We can construct such a polynomial via a method that uses no graph theory other than the fact that T has $n - 1$ edges.

Lemma 1. Let k be a positive integer and let $A, B \subseteq \mathbb{C}^2$ be two disjoint finite sets. Suppose $|A| = 2k$ and no line intersects $A \cup B$ in more than $k + 1$ points. Then there exists a polynomial of degree k that vanishes on A and is non-zero on B .

Proof. We first argue that we may reduce to the case when $|B| = 1$. This is since if P and P' are polynomials that are zero on A and nonzero on B and B' , then a generic linear combination of P and P' is nonzero on A and nonzero on $B \cup B'$. Now suppose $B = \{b\}$.

For $a, a' \in A$, write $a \sim a'$ if a, a', b are collinear. This is an equivalence relation, and each equivalence class has at most k elements. Thus we may pair up the elements of A such that no two paired elements are collinear with b . Now let P be the polynomial vanishing on the union of the k lines determined by the pairs, which is nonzero at b by construction.

Lemma 2. Let a_1, a_2, \dots, a_n be distinct complex numbers. Then a line can intersect at most n points of the form (a_i, a_j) .

Proof. If not, then by the pigeonhole principle such a line must contain two points with the same x -coordinate. But then it is vertical and thus can only contain n points.

We are done by applying Lemma 1 with

$$A = \{(a_i, a_j) : v_i v_j \in E(T)\} \quad \text{and} \quad B = \{(a_i, a_j) : i \neq j, v_i v_j \notin E(T)\}.$$

The 16th Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let \mathbb{Z} denote the set of integers and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100} . Fix a positive integer c . Determine all functions $f: S \rightarrow \mathbb{Z}$ satisfying

$$f(xy + c) = f(x) + f(y) \quad \text{for all } x, y \in S.$$

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Solution. Observe that if $x_1, y_1, x_2, y_2 \in S$ with $x_1 y_1 = x_2 y_2$ then

$$f(x_1) + f(y_1) = f(x_2) + f(y_2). \tag{1}$$

This tells us that for $u, v, w \in S$,

$$f(uv) + f(w) = f(u) + f(vw), \quad \text{so} \quad f(uv) - f(u) - f(v) = f(vw) - f(w) - f(v).$$

Notice the RHS is independent of u so the same must be true of the LHS. By replicating the argument with u and v switched, we also see the LHS is independent of v so in fact

$$f(uv) - f(u) - f(v) = k \quad \text{for some constant } k \in \mathbb{Z} \tag{2}$$

Using (1) again we have, for $y, z \in S$

$$\begin{aligned} f(cz) + f(y) &= f(z) + f(cy), \\ \text{so } f(cy) - f(y) &= l \quad \text{for some constant } l \in \mathbb{Z}. \end{aligned} \tag{3}$$

Setting $x = cz$ in the original functional equation for $z \in S$ shows

$$\begin{aligned} f(c(yz + 1)) &\stackrel{(3)}{=} f(yz + 1) + l = f(cz) + f(y) \stackrel{(3)}{=} f(z) + f(y) + l, \\ \text{so } f(yz + 1) &= f(y) + f(z). \end{aligned}$$

Let $x \in S$ and set $y = x, z = x + 2$ in the above to get

$$\begin{aligned} f((x + 1)^2) &\stackrel{(2)}{=} 2f(x + 1) + k = f(x) + f(x + 2) \\ \Rightarrow f(x) + f(x + 2) - 2f(x + 1) &= -k = \text{constant} \end{aligned}$$

which forces f to be a quadratic. By setting $x = y$ in the original functional equation and considering the degree of both sides, we see f must be in fact be constant. The only constant function that satisfies the condition is $f \equiv 0$.

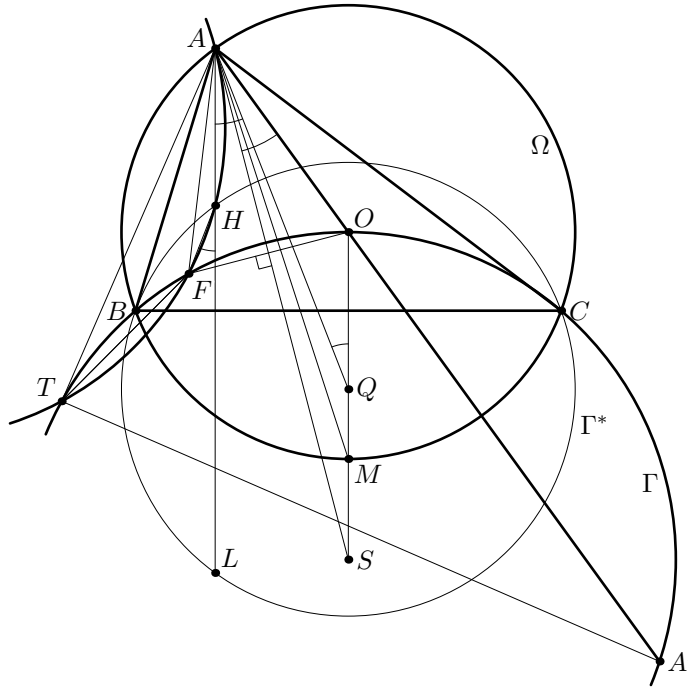
Problem 5. Let ABC be an acute triangle with $AB < AC$ and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circle BOC . The line AO and the circle of radius AO centred at A cross Γ again at A' and F , respectively. Prove that Γ , the circle on diameter AA' and the circle AFH are concurrent.

ROMANIA, RADU-ANDREI LECOIU

Solution 1. Let Ω denote the circle (ABC) (centred at O), and let M be the midpoint of the minor arc BC of that circle.

Consider the composition ι of an inversion centered at A and the reflection in the bisector AM that swaps B and C . Then ι swaps the circle Ω with the line BC , hence it swaps O with the reflection L of A in BC . Hence $\iota(\Gamma)$ is the circle $\Gamma^* = (BCL)$, i.e., the reflection of Ω in BC which passes through H . Let S and Q be the centers of Γ and Γ^* , respectively; then AM is the angle bisector of $\angle QAS$.

Since ι swaps Γ and Γ^* , they are seen from A at the same angle, so there exists a rotational homothety h centred at A mapping Γ to Γ^* ; the angle of h is $\angle SAQ$. Notice that the rays AH and AF are obtained from AO by reflections in AM and AS , respectively, so $\angle HAF = 2\angle MAS = \angle QAS$. This easily yields that $H = h(F)$. Hence the triangles AHF and AQS are similar, and $\angle AHF = \angle AQS$.



Now, let the circle (AHF) meet Γ again at T . Using directed angles, we get $\angle ATF = \angle AHF = \angle AQS$; next, since $FO \perp AS$, we have

$$\angle FTA' = \angle FOA' = \pi/2 - \angle OAS = \pi/2 - \angle QAL = \pi/2 - \angle AQS,$$

(here the equality $\angle OAS = \angle QAL$ holds because these angles are symmetric to each other with respect to AM), so $\angle ATA' = \angle ATF + \angle FTA' = \pi/2$, as desired.

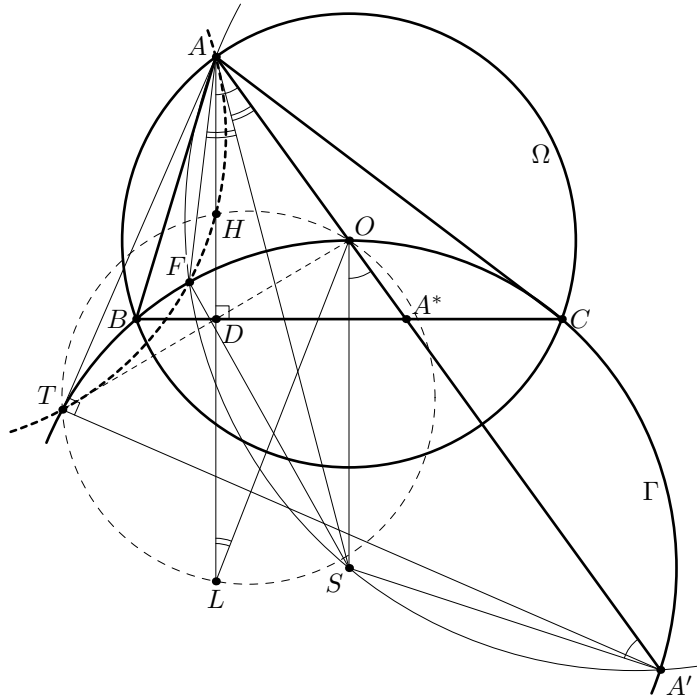
Remark. Existence of the rotational homothety h may be shown in various ways. E.g., one may notice that Ω is an Apollonius circle of the segment QS , so the ratio of the radii of Γ and Γ^* is $BS/BQ = AS/AQ$, which also yields that h exists.

Solution 2. Let S be the centre of Γ and let the circle on diameter AA' cross Γ again at T . It is sufficient to prove that T lies on circle AFH .

The points O and F are reflections of one another in AS ; hence S lies on the internal angle bisectrix of $\angle FAA'$. On the other hand, since $SF = SA'$, it lies on the perpendicular bisectrix of FA' ; so S is the midpoint of the arc $A'F$ on circle $AA'F$ not containing A . In particular, $AFSA'$ is cyclic.

Let D be the orthogonal projection of A on BC . We will prove that O, D, T are collinear. Invert from O with radius OB . This fixes B and C , so Γ maps to line BC . It follows that A maps to $A^* = AA' \cap BC$. Note that A is fixed under this inversion, as $OA = OB$, so the image of the circle on diameter AA' is a circle δ through A and A^* — and, in fact, δ is the circle of diameter AA^* , as AA' passes through O . Hence T maps to one of the points where line BC crosses δ . As $T \neq A'$, its image is D , so O, D, T are indeed collinear.

Letting L be the reflection of A in BC , we now prove that $HTLO$ is cyclic. As circle HBC is the reflection of Γ in BC , the quadrangle $HLBC$ is cyclic, so $HD \cdot DL = BD \cdot DC = OD \cdot DT$, whence $HTLO$ is indeed cyclic.



Next, we show that triangles ALO and $A'SA$ are similar. Let $\angle BAC = \alpha$ and $\angle CBA = \beta$. As $OS \parallel AL$, it follows that $\angle OAL = \angle A'OS = \angle AA'S$, so by the sine law:

$$\frac{AL}{AA'} = 2 \cdot \frac{AD}{AB} \cdot \frac{AB}{AA'} = 2 \cdot \sin \beta \cdot \frac{\cos \alpha}{\sin \beta} = 2 \cos \alpha \quad \text{and} \quad \frac{AO}{A'S} = \frac{BO}{BS} = \frac{\sin 2\alpha}{\sin \alpha} = 2 \cos \alpha.$$

Consequently, $AL/AA' = AO/A'S$, so $AL/AO = AA'/A'S$, implying the desired similarity. In particular, $\angle ALO = \angle A'SA$.

Finally, combine the properties established above to chase angles and write successively

$$\begin{aligned} \angle FTH &= \angle FTO - \angle HTO = \frac{1}{2} \angle FSO - \angle HLO = \angle ASO - \angle A'SA \\ &= \angle ASO + \angle A'SA - 2\angle A'SA = \angle SOA' - \angle FAO = \angle HAO - \angle FAO \\ &= \angle HAF \end{aligned}$$

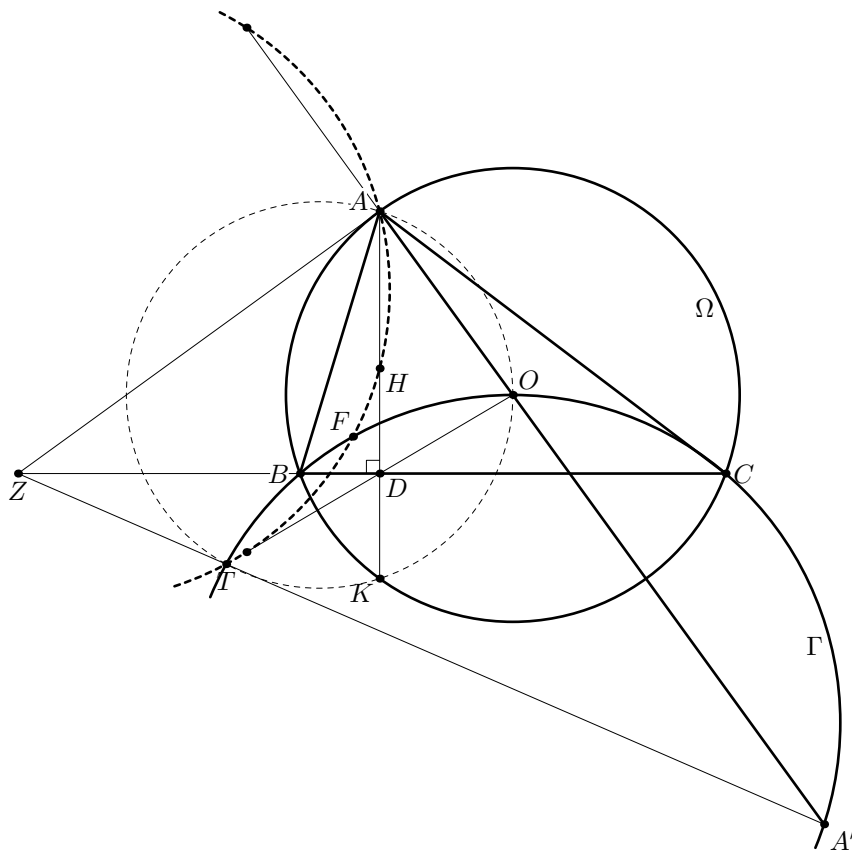
and conclude that T lies on circle AFH , as stated in the first paragraph. This completes the solution.

Remark. Let T be the desired intersection point. The property that the points O, D , and T is also useful in other approaches to the problem; in fact, it may be proved for both definitions of point T (as in Solutions 1 and 2), thus providing a different solution.

Here we list several other properties of the figure which appear to be useful in other approaches to the problem.

Let the tangent to the circle (ABC) at A meet BC at Z . Then the desired common intersection point T lies on ZA' .

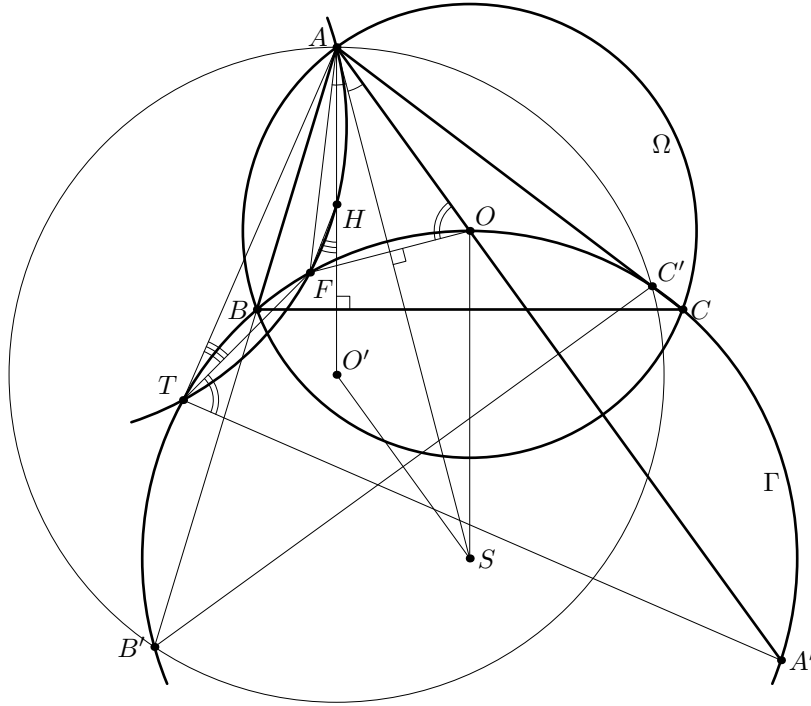
Let AH meet the circle (ABC) again at K . Then the points $A, O, K,$ and T are concyclic. Finally, the circle $(AHFT)$ also passes through K , as well as through the reflection of O in D .



Solution 3. Let S be the centre of Γ and let circle AFH meet Γ again at $T \neq F$. In the sequel, angles are all orientated.

We are to prove that $\angle ATF + \angle FTA' = 90^\circ$. To this end, note the equivalences below:

$$\begin{aligned}
 \angle ATF + \angle FTA' = 90^\circ &\Leftrightarrow \angle AHF + \angle FOA = 90^\circ \Leftrightarrow \angle AHF = \angle OAS \quad (\text{as } AS \perp OF) \\
 &\Leftrightarrow \angle AHF = \angle SAF \Leftrightarrow AS \text{ is tangent to circle } AFH \\
 &\Leftrightarrow \frac{AH}{\sin \angle HAS} = \frac{AF}{\sin \angle FAS} \\
 &\Leftrightarrow \frac{AH}{AO} = \frac{\sin \angle HAS}{\sin \angle SAO}. \tag{*}
 \end{aligned}$$



To prove (*), let AB and AC meet Γ again at B' and C' , respectively. An easy angle chase shows that O is the orthocentre of triangle $AB'C'$.

As triangles ABC and $AC'B'$ are similar, AH passes through the centre O' of circle $AB'C'$; and as circles $AB'C'$ and $BOCC'B'$ are reflections of one another in $B'C'$ and $AO'SO$ is a parallelogram, it follows that

$$\frac{\sin \angle HAS}{\sin \angle SAO} = \frac{\sin \angle O'AS}{\sin \angle ASO'} = \frac{O'S}{AO'} = \frac{AO}{AO'}. \quad (**)$$

Further on, as triangles ABC and $AC'B'$ are similar, (**) implies equal corresponding length ratios, so $AO/AO' = AH/AO$. This establishes (*) and concludes the solution.

Remark. Relation (*) is equivalent to AS being the A -symmedian of triangle AOH . This might very well be known and can actually be proved in several different ways.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets S_1, S_2, \dots, S_k ; each of these sets has exactly $m + 1$ elements, one of which is red and the other m are all blue. Let \mathcal{F} be the family of all subsets F of $S_1 \cup S_2 \cup \dots \cup S_k$ such that, for every i , the intersection $F \cap S_i$ is monochromatic; the empty set is monochromatic. Determine the largest possible cardinality of a subfamily $\mathcal{G} \subseteq \mathcal{F}$, no two sets of which are disjoint.

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Solution. The required maximum is $2^{m-1}(2^m + 1)^{k-1}$ and is achieved if, for instance, \mathcal{G} consists of all sets in \mathcal{F} containing a fixed blue element.

We now prove that $|\mathcal{G}| \leq 2^{m-1}(2^m + 1)^{k-1}$ for any \mathcal{G} satisfying the conditions in the statement. For convenience, write $M = 2^m + 1$. Let r_i denote the red element of S_i , and let B_i be the set of blue elements in S_i .

For every subset $X_i \subset B_i$ and every $j \in \mathbb{Z}_M$, define the sets

$$T_{X_i, j} = \begin{cases} \{r_i\}, & \text{if } j = 0; \\ X_i, & \text{if } j \neq 0 \text{ and } j \text{ is even (considered as a number in } [1, M-1]); \\ B_i \setminus X_i, & \text{if } j \neq 0 \text{ and } j \text{ is odd (considered as a number in } [1, M-1]). \end{cases}$$

Note that, for every i and every j , the sets $T_{X_i, j}$ and $T_{X_i, j+1}$ are disjoint. Now, for every sets $X_i \subset B_i$ and every elements $j_i \in \mathbb{Z}_M$, $i = 1, 2, \dots, k$, denote

$$F(X_1, X_2, \dots, X_k, j_1, j_2, \dots, j_k) = \bigcup_{i=1}^k T_{X_i, j_i}. \quad (*)$$

Claim. Every set $F \in \mathcal{F}$ has exactly 2^{mk} representations of the form $(*)$.

Proof. Set $F_i = F \cap S_i$. If $F_i = \{r_i\}$, then there are 2^m possible choices for X_i , and one should necessarily have $j_i = 0$. Otherwise, there are only two possible choices for X_i , namely $X_i = F_i$ and $X_i = B_i \setminus F_i$, and for each of them there are 2^{m-1} possible choices for j_i . So, whatever F , there are 2^m possible choices for each pair (X_i, j_i) all of which can be made independently, whence a total of 2^{mk} possible tuples $(X_1, X_2, \dots, X_k, j_1, j_2, \dots, j_k)$. This proves the Claim.

The Claim implies that each $F \in \mathcal{F}$ has the same number of representations of the form $(*)$. Thus, it suffices to show that, among all $N = 2^{km}(2^m + 1)^k$ tuples $(X_1, X_2, \dots, X_k, j_1, j_2, \dots, j_k)$, at most $\frac{2^{m-1}}{2^m + 1}N$ satisfy $F(X_1, X_2, \dots, X_k, j_1, j_2, \dots, j_k) \in \mathcal{G}$.

To this end, split all these tuples into length M cycles

$$\left(F(X_1, X_2, \dots, X_k, j_1, j_2, \dots, j_k), F(X_1, X_2, \dots, X_k, j_1 + 1, j_2 + 1, \dots, j_k + 1), \dots, F(X_1, X_2, \dots, X_k, j_1 + M - 1, j_2 + M - 1, \dots, j_k + M - 1) \right),$$

and note that any two adjacent sets of a cycle are disjoint. Hence each cycle contains at most $\lfloor M/2 \rfloor = 2^{m-1}$ sets from \mathcal{G} . This provides the desired upper bound and completes the solution.