29th Junior Balkan Mathematical Olympiad

June 24-29, 2025
Ohrid and Struga, North Macedonia

SHORLISTED PROBLEMS WITH SOLUTIONS

jbmo2025.mk

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until JBMO 2026

Contributing Countries

The Organizing Committee and the Problem Selection Committee of the JBMO 2025 thank the following countries for contributing with problem proposals:

- Albania
- Bulgaria
- Croatia
- Cyprus
- Saudi Arabia
- Serbia
- Turkey
- Turkmenistan
- Ukraine
- Uzbekistan

Problem Selection Committee

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PROBLEMS

Legend	Acceptable	Good	Excellent
Easy			
Medium			
Hard			

Algebra

A1 For all positive real numbers a, b, c with $a+b+c=3$, prove that

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\geq 3.$$

A2 Let a,b,c,d be non-negative integers such that $a>b>c>d$ and $a^2+b^2-c^2-d^2<2025$. Find the maximum value of $a+b+c+d$.

A3 For all positive real numbers a, b, c, prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq \frac{2abc(a+b+c)^2}{ab+bc+ca}.$$

A4 Points $A_1, A_2, \dots, A_n$ are chosen on one of two parallel lines, and points $B_1, B_2, \dots, B_m$ on the other. To each point A_i and B_j we assign a positive real numbers a_i and b_j, respectively.
All segments A_xB_y for every $1\leq x\leq n$, and $1\leq y\leq m$ are drawn. No three segments intersect at one point; each point of intersection of two segments A_iB_j and A_kB_l (if they intersect) is colored and assigned the value

$$\frac{a_i+a_k}{b_j+b_l}.$$

Rabbit Mykola jumps across colored intersection points each time moving along one of the drawn segments. After making 2025 jumps Mykola returns to the starting point. With this the path he took forms 2025-gon (not necessarily simple). After this Mykola realises that each ratio $\frac{a_i}{b_j}$, where $1\leq i\leq n, 1\leq j\leq m$ is either strictly greater or strictly smaller than the value assigned to every vertex of the polygon.
Prove that some three consecutive vertices of the polygon are collinear.

A5 Let $a_1, a_2, a_3, \dots, a_{2n}$ be positive real numbers such that their sum is equal to 2. If

$$\min(a_1;a_2)+\min(a_2;a_3)+\min(a_3;a_4)+\cdots+\min(a_{2n};a_1)\leq 1,$$

then find the minimum possible value of the expression

$$a_1^2+a_2^2+\cdots+a_{2n}^2.$$

A6 Determine the smallest positive integer m for which there are at least 100 positive integers k such that the greatest integer less than or equal to

$$\frac{m(m+1)(2m+1)}{k^2-(4m+1)k+5m^2+3m}$$

equals k.

A7 Find all real numbers k such that there exists some tuple (a,b,c) satisfying

$$a^2+b^2+c^2=a+b+c=k$$

and for each such tuple (a,b,c), values a,b,c are not the side lengths of triangle.

29th JUNIOR BALKAN MATHEMATICAL OLYMPIAD

Combinatorics

C1 Consider 28 positive integers with sum 2027, such that the product of any 7 of them is a perfect square. Prove that one can select some of these numbers, such that the sum of the selected numbers, multiplied by the product of the remaining numbers is divisible by 8100.

C2 The numbers $2, 4, 8, \ldots, 2^{2024}$ are written on the board. At every step we can perform one of the following:

(i) We delete an even number from the board, say $2k$, and in its place we write the number k.

(ii) We delete an odd number from the board, say $2k+1$, and in its place we write the numbers k and 1.

(iii) We delete two (possibly equal) numbers from the board, say k and ℓ, and in their place we write the number $k + \ell \cdot 2^k$. (For example, if the board contains the numbers $1, 3, 5$ and we choose $k = 5$ and $\ell = 1$, then after this steps the board will contain the numbers $3, 37$.)

Determine whether the following numbers can be obtained after a series of steps:

(a) $2^{2025} - 1$

(b) $2^{2025} - 3$

Harder Formulation: Which numbers less than 2^{2025} can be the last remaining, after a series of steps?

C3 Consider the set Σ of all quadrilaterals whose interior angles are in the set $\{30°, 60°, 120°, 150°\}$. Find all natural numbers $n \geqslant 3$ such that a regular n–gon can be completely tiled using only quadrilaterals from the set Σ.

C4 Ana and Bob are playing a game. Bob chooses natural numbers n, m with $n \geq 3$. After that, Ana selects natural numbers $a_1, a_2, \ldots, a_n$ as she wishes. Then she forms n pairs (a_i, a_j), such that $i \neq j$ and each number appears in exactly two pairs. Finally, Ana computes:

$$\sum \gcd(a_i^2, a_j^2)$$

and records the result.
If the calculated result is exactly 2025^m, Ana wins; otherwise, Bob wins.
Find the minimum and maximum values of n for which Ana has a winning strategy, regardless of Bob's choice for the number m.

C5 Anna and Bob play the following game: In the first phase of the game Anna chooses a number a from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and then Bob chooses a different number b from the same set. In the second phase of the game, with Anna starting first, they alternately pick any number they wish from the set $\{a, b\}$ and keep the running sum of all choices in that phase. If, after the choice of any one of them, the running sum is a multiple of 1111, then this person wins and the game ends. Determine which one of them, if any, has a winning strategy.

C6 Given a set $\{x_1, x_2, \ldots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$ we say that its *ascending power sum* is

$$x_1 + x_2^2 + x_3^3 + \cdots + x_k^k\,.$$

The ascending power sum of the empty set is considered to be equal to 0.
Determine the number of subsets of $\{1, 2, \ldots, 10\}$ whose ascending power sum is a multiple of 3.

Harder Formulation: One could make the problem harder by asking about the number of subsets of $\{1, 2, \ldots, 2025\}$ with ascending power sum a multiple of 3.

C7 Let n be a positive integer. The integers from 1 to n are written in the cells of an $n \times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by r_x the number of pairs (a, b) of numbers in row x, such that $a > b$, but the column position of a is less than the column position of b. Similarly, denote by c_y the number of pairs (a, b) of numbers in column y, such that $a > b$, but the row position of a is less than the row position of b. Determine the largest possible value of the sum

$$r_1 + r_2 + \cdots + r_n + c_1 + c_2 + \cdots + c_n.$$

Geometry

G1 Let I_A and ω be the excenter opposite to A and circumcircle, respectively of the triangle $\triangle ABC$. Let B_1 be the midpoint of $arc\widehat{AC}$ on circle ω which doesn't contain point B. Let C_1 be the midpoint of $arc\widehat{AB}$ on circle ω which doesn't contain point C. Let B_2 and C_2 be the points of intersection of the line B_1C_1 with the sides AC and AB, respectively. Let line parallel to AB which passes through B_2 intersect the line BC_1 at point C'. Let line parallel to AC which passes through C_2 intersect the line CB_1 at point B'. Prove that the circumcircles of the triangles $\triangle BC_1C_2$, $\triangle CB_1B_2$ and $\triangle B'I_AC'$ share a common point.

G2 Let ABC be a triangle with circumcircle (c) and let D and E be points on the arc AB not containing C and on the arc AC not containing B respectively. Let F and G be the intersections of DE with AB and AC respectively. Let O_1, O_2, O_3 and O_4 be the centers of the circumcircles of triangles DFB, EFB, DGC and EGC respectively, and let ℓ be the perpendicular from A to DE.
Prove that O_1O_3, O_2O_4 and ℓ concur at a point on (c).

G3 Let $(O_1), (O_2)$ be two circles that intersect at A, B with the common tangent line CD (that closer to B than A) with $C \in (O_1), D \in (O_2)$ and $AC > AD$. Construct H so that triangle HCD has B as its orthocenter. Draw diameter CR of (O_1) and diameter DS of (O_2). Lines BR, HC meet at P and lines BS, HD meet at Q. Take I, J such that P, Q are midpoints of segments HI, HJ respectively. Prove that the circumradius of triangles AIC and AJD are equal.

G4 In triangle ABC, where $AB < AC$, the angle bisector of $\angle A$ intersects side BC at point L, and the circumcircle Ω - at point W. On the circumcircle of triangle $\triangle WLC$, a point $P \neq W$ is chosen such that $BP = CP$. The tangent from point A to Ω intersects BC at point T. On line BW, a point Q is chosen such that $TB = TQ$. Prove that points Q, A, P, W lie on the same circle.

G5 Let ABC be a triangle and let D, E be interior points of the sides AB and AC respectively such that B, D, E, C are concyclic. Let F, G be interior points of the side BC such that $\angle BDF = \angle ADE$ and $\angle GEC = \angle AED$. Prove that DF, EG and the perpendicular from A to BC are concurrent.

G6 Let $\triangle ABC$ be right-angled at A and let D be the foot of altitude from A to BC and let E be the midpoint of DC. The circumcircle of $\triangle ABD$ intersects AE again at point F. Let X be the intersection of AB and DF. Prove that $XD = XC$.

G7 Two circles ω_1 and ω_2 with centers O_1 and O_2, respectively, intersect at points A and B. The tangent at point A to the circumcircle of $\triangle AO_1O_2$ intersects ω_1 and ω_2 at points $D \neq A$ and $E \neq A$, respectively. Prove that the center of the circumcircle of $\triangle BO_1O_2$ is equidistant from points D and E.

G8 Let ABC be a triangle with $AC > AB$ have incenter I and A-excenter J. Assume that its incircle touches BC at point K, the A-excircle touches AC at point L, and AI intersects BC at point D. If the intersection of the circumcircles of $\triangle AIC$ and $\triangle KCL$ is P, prove that $\angle APD = 90^\circ$.

G9 Points X and Y are chosen inside quadrilateral $ABCD$ on the internal bisectors of angles A and C respectively such that points A, X, Y, C are lying on one circle and $XY \perp BD$. Prove that either angles of the quadrilateral $ABCD$ can be divided into two groups with equal sums, or sides of the quadrilateral can be divided into two groups with equal products.

Number Theory

N1 Determine all numbers of the form

$$20252025\dots2025$$

(where the block 2025 is repeated one or more times) that are perfect squares of positive integers.

N2 Find all positive integers n, $n \geq 2$, such that there exist infinitely many perfect squares which can be expressed as a sum of n consecutive positive integers.

N3 A set S of positive integers is called *nice* if $\frac{a+b}{a-b}$ is an integer for any different $a, b \in S$. Show that for any positive integer n, there exist infinitely many nice sets with n elements.

N4 Let m be a positive integer. Suppose that there are at least $k \geq 3$ distinct positive integers n with the following property: the sum of n and its largest divisor, different from n, equals m. Prove that $m > 3^{2^{k-2}}$.

N5 Determine all positive integers n such that

$$\frac{1! \cdot 2! \cdot 3! \cdots 2024!}{n!}$$

is a perfect square.

N6 Find all prime numbers $p < 2025$ for which

$$p^2+2p-7=n^3,$$

for some integer n.

SOLUTIONS

Legend	Acceptable	Good	Excellent
Easy			
Medium			
Hard			

Algebra

A1 For all positive real numbers a, b, c with $a+b+c=3$, prove that

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\geq 3.$$

(Turkey)

Solution 1. By Cauchy-Schwarz inequality we have

$$(a+b+c)(\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3})\geq\left(\sqrt{\frac{bc}{a^2}}+\sqrt{\frac{ca}{b^2}}+\sqrt{\frac{ab}{c^2}}\right)^2.$$

By the AM-GM we we have

$$\sqrt{\frac{bc}{a^2}}+\sqrt{\frac{ca}{b^2}}+\sqrt{\frac{ab}{c^2}}\geq 3.$$

Therefore

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\geq\frac{1}{x+y+z}\left(\sqrt{\frac{bc}{a^2}}+\sqrt{\frac{ca}{b^2}}+\sqrt{\frac{ab}{c^2}}\right)^2\geq\frac{9}{a+b+c}=3,$$

as desired. □

Solution 2. By AM-GM we have

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\geq 3\cdot\sqrt[3]{\frac{bc}{a^3}\cdot\frac{ca}{b^3}\cdot\frac{ab}{c^3}}=3\cdot\frac{1}{\sqrt[3]{abc}}\geq 3\cdot\frac{3}{a+b+c}=3,$$

as desired. □

Solution 3. By AM-GM we have

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}=\frac{a+b+c}{3}(\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3})\geq\sqrt[3]{abc}\cdot 3\cdot\sqrt[3]{\frac{bc}{a^3}\cdot\frac{ca}{b^3}\cdot\frac{ab}{c^3}}=3,$$

as desired. □

Solution 4. Using the inequalities between means we get

$$\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}=3abc(\frac{1}{3}(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}))\geq 3abc(\frac{1}{3}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}))^4\geq 3abc(\sqrt[3]{abc})^3\frac{3}{a+b+c}=3,$$

as desired. □

A2 Let a, b, c, d be non-negative integers such that $a > b > c > d$ and $a^2 + b^2 - c^2 - d^2 < 2025$. Find the maximum value of $a + b + c + d$.

(Serbia)

Solution 1. Since the integers a, b, c, d are distinct and strictly decreasing, we must have $a \geqslant b + 1 \geqslant c + 2 \geqslant d + 3$. This gives $a - c \geqslant 2$ i $b - d \geqslant 2$. We can calculate

$$2024 \geqslant (a^2 - c^2) + (b^2 - d^2) = (a - c)(a + c) + (b - d)(b + d) \geqslant 2(a + c) + 2(b + d) = 2(a + b + c + d),$$

and thus $a + b + c + d \leqslant 1012$.
Suppose now that $a + b + c + d \geqslant 1011$ (i.e. it is equal to either 1011 or 1012). First assume $a - c \geqslant 3$. Then from

$$2024 \geqslant (a^2 - c^2) + (b^2 - d^2) \geqslant 3(a + c) + 2(b + d) = a + c + 2(a + b + c + d) \geqslant a + c + 2022,$$

it follows that $a + c \leqslant 2$, which is not possible. A similar contradiction occurs if $b - d \geqslant 3$. Therefore, we must have $a - c = b - d = 2$, which implies the four numbers a, b, c, d are consecutive. However, the sum of four consecutive numbers can not be either 1011 or 2012.
It follows that $a + b + c + d \leqslant 1010$. The numbers $a = 254$, $b = 253$, $c = 252$, $d = 251$ satisfy $a > b > c > d$, $a + b + c + d = 1010$ and $a^2 + b^2 - c^2 - d^2 = 2020 < 2025$. So the maximum value is 1010. ☐

Solution 2. As in the first solution, we prove that $a \geqslant b + 1 \geqslant c + 2 \geqslant d + 3$ and $a + b + c + d \leqslant 1012$. Observe that if quadruple (a, b, c, d) satisfies the given inequality, then also the quadruple $(a, b, b - 1, b - 2)$ satisfies it. In addition to that, this substitution does not decrease the total sum $a + b + c + d$. Therefore, without loss of generality, we can assume that $c = b - 1$ and $d = b - 2$. The given inequality can be simplified as

$$a^2 + b^2 - (b - 1)^2 - (b - 2)^2 = a^2 - b^2 + 6b - 5 = a^2 - (b - 3)^2 + 4 \leqslant 2024.$$

Using the similar argument, we can assume that the numbers a and $b - 3$ are at the smallest possible distance. Otherwise, the sum $a + b + c + d = a + 3b - 3$ can be made larger without contrasting the conditions from the statement. It implies that $a = b + 1$ (i.e. that a, b, c, d are consecutive numbers). Now from

$$2020 \geqslant (b + 1)^2 - (b - 3)^2 = 4(2b - 2) = 8b - 8$$

we conclude that $b \leqslant 253$. The numbers $a = 254$, $b = 253$, $c = 252$, $d = 251$ satisfy $a > b > c > d$, $a + b + c + d = 1010$ and $a^2 + b^2 - c^2 - d^2 = 2020 < 2025$. So the maximum value is 1010. ☐

Solution 3. If (a, b, c, d) satisfies the conditions, then increasing c and d cannot make the second condition fail, but increases the sum. Thus we can WLOG assume $b = c + 1 = d + 2$.
The conditions now become $a > b$ and $a^2 < 2021 + (b - 3)^2$ and we want to maximize $a + 3b - 3$. Decreasing a and increasing b by one preserves the second condition and increases the target sum, so the maximum occurs when $a = b + 1$ or $a = b + 2$.
If $a = b + 1$ we get $8b < 2029 \Longleftrightarrow b \leq 253$ and if $a = b + 2$ then $10b < 2026 \Longleftrightarrow b \leq 202$. Hence the maximum is $253 + 1 + 3 \cdot 253 - 3 = 1010$ and occurs when $a = b + 1 = c + 2 = d + 3 = 254$. ☐

A3 For all positive real numbers a, b, c, prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}.$$

(Turkey)

Solution 1. Apply Cauchy-Schwarz inequality.

$$((b+c)+(c+a)+(a+b))\left(\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\right)\geq(a^2+b^2+c^2+ab+bc+ca)^2.$$

Lemma. $(a^2+b^2+c^2+ab+bc+ca)\geq 2(a+b+c)^2$.
Proof. $3(a^2+b^2+c^2+ab+bc+ca)=2(a^2+b^2+c^2+2ab+2bc+2ca)+(a^2+b^2+c^2-ab-bc-ca)=2(a+b+c)^2+\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2\geq 2(a+b+c)^2$. $\diamond$

Then we have

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{\frac{4}{9}(a+b+c)^4}{2(a+b+c)}=\frac{2}{9}(a+b+c)^3.$$

Again by Cauchy-Schwarz inequality we have

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\geq 9.$$

Finally, using these we get

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2}{9}(a+b+c)^3\geq\frac{2(a+b+c)^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{2abc(a+b+c)^2}{ab+bc+ca},$$

as desired. $\square$

Solution 2. Using Cauchy-Swartz we get

$$\sum_{cyc}\frac{\left(a^2+bc\right)^2}{b+c}=\sum_{cyc}\frac{\left(a^3+abc\right)^2}{a^2(b+c)}\geq\frac{\left(\sum a^3+3abc\right)^2}{\sum a^2(b+c)}=\frac{(a^3+b^3+c^3+3abc)^2}{ab(a+b)+bc(b+c)+ca(c+a)}.$$

Using the Schur's inequality this is greater than

$$ab(a+b)+bc(b+c)+ca(c+a)=(a+b+c)(ab+bc+ca)-3abc\geq -3abc+\frac{(a+b+c)\cdot 3abc(a+b+c)}{ab+bc+ca},$$

where the last inequality is true since $(ab+bc+ca)(\frac{1}{b}+\frac{1}{c}+\frac{1}{a})\geq 3(a+b+c)$.
Finally using

$$\frac{abc(a+b+c)^2}{ab+bc+ca}\geq\frac{3abc(ab+bc+ca)}{ab+bc+ca}=3abc,$$

we get the desired result. $\square$

A4 Points $A_1, A_2, \ldots, A_n$ are chosen on one of two parallel lines, and points $B_1, B_2, \ldots, B_m$ on the other. To each point A_i and B_j we assign a positive real numbers a_i and b_j, respectively.
All segments A_xB_y for every $1 \le x \le n$, and $1 \le y \le m$ are drawn. No three segments intersect at one point; each point of intersection of two segments A_iB_j and A_kB_l (if they intersect) is colored and assigned the value

$$\frac{a_i + a_k}{b_j + b_l}.$$

Rabbit Mykola jumps across colored intersection points each time moving along one of the drawn segments. After making 2025 jumps Mykola returns to the starting point. With this the path he took forms 2025-gon (not necessarily simple). After this Mykola realises that each ratio $\frac{a_i}{b_j}$, where $1 \le i \le n, 1 \le j \le m$ is either strictly greater or strictly smaller than the value assigned to every vertex of the polygon.
Prove that some three consecutive vertices of the polygon are collinear.

(Mykhailo Shtandenko - Ukraine)

Solution. If Mykola used the same segment in any two consecutive jumps, those three vertices are collinear. Suppose that with each successive move, Mykola changed the segment along which he jumped. For convenience, assign to each of the drawn segments, along which Mykola jumped, a number $\frac{a_i}{b_j}$ if it connects point A_i with B_j.
By assumption, each of the numbers assigned to the segments is either greater than, or less than all the numbers assigned to the points Mykola jumped to. The idea of the solution is based on the fact that a number assigned to a certain point lies on the number line strictly between the numbers assigned to the segments whose intersection is this point.
Indeed, if $\frac{a_i}{b_j} \le \frac{a_k}{b_l}$, then

$$\frac{a_i}{b_j} \le \frac{a_i + a_k}{b_j + b_l} \le \frac{a_k}{b_l},$$

which is easy to verify by bringing the fractions to a common denominator. Moreover, if the fractions are not equal, both inequalities will be strict.
However, from the condition it follows that among the points Mykola visited any number assigned to a certain colored point is not equal to the numbers assigned to the segments whose intersection is this point. Therefore, if we proceed step by step along the segments Mykola jumped along, we will get that the numbers assigned to these segments will alternate — one less than, the next is greater than each of the numbers corresponding to the points Mykola jumped to. But the total number of such segments is odd.
This means that he used the same segment on the first and the last jump, implying that those three vertices are collinear. □

A5 Let $a_1, a_2, a_3, \ldots, a_{2n}$ be positive real numbers such that their sum is equal to 2. If

$$\min(a_1; a_2) + \min(a_2; a_3) + \min(a_3; a_4) + \cdots + \min(a_{2n}; a_1) \leq 1,$$

then find the minimum possible value of the expression

$$a_1^2 + a_2^2 + \cdots + a_{2n}^2.$$

(Uzbekistan)

Solution 1.
Lemma. $\sum_{i=1}^{2n} |a_i - a_{i+1}| \geq 2$.
Proof. Since $2\min(a_i; a_{i+1}) = a_i + a_{i+1} - |a_i - a_{i+1}|$, it follows that

$$4 - \sum_{i=1}^{2n} |a_i - a_{i+1}| = 2\sum_{i=1}^{2n} a_i - \sum_{i=1}^{2n} |a_i - a_{i+1}| = \sum_{i=1}^{2n} 2\min(a_i; a_{i+1}) \geq 2.$$

This concludes our lemma. $\diamond$
By Cauchy Bunyakovsky Schwarz inequality, we obtain

$$\sum_{i=1}^{2n} |a_i - a_{i+1}|^2 (1 + 1 + \cdots + 1) \geq \left(\sum_{i=1}^{2n} |a_i - a_{i+1}| \right)^2.$$

The above inequality combined with the given lemma shows that

$$2n \sum_{i=1}^{2n} \geq 4 \Rightarrow \sum_{i=1}^{2n} \geq \frac{2}{n}.$$

It follows that

$$a_1^2 + a_2^2 + \cdots + a_{2n}^2 - a_1a_2 - a_2a_3 - \cdots - a_{2n}a_1 \geq \frac{1}{n}. \tag{1}$$

Again by Cauchy Bunyakovsky Schwarz inequality,

$$\sum_{i=1}^{2n} |a_i - a_{i+1}|^2 (1 + 1 + \cdots + 1) \geq \Big((a_1 + a_2) + (a_2 + a_3) + \cdots + (a_{2n} + a_1) \Big)^2 = 16.$$

From this

$$\sum_{i=1}^{2n} |a_i - a_{i+1}|^2 \geq \frac{8}{n}.$$

We conclude that

$$a_1^2 + a_2^2 + \cdots + a_{2n}^2 + a_1a_2 + a_2a_3 - \cdots + a_{2n}a_1 \geq \frac{4}{n}. \tag{2}$$

Combining (1) and (2), we can find that $a_1^2 + a_2^2 + \cdots + a_{2n}^2 \geq \frac{5}{2n}$.
Finally, choosing $a_{2k} = \frac{3}{2n}$ and $a_{2k-1} = \frac{1}{2n}$ for $k \in \{1, \ldots, n\}$ we get $\frac{9n+n}{4n^2} = \frac{5}{2n}$ achieving equality. $\square$

Solution 2. Let $A = \sum_{i=1}^{2n} \min(a_i; a_{i+1})$, $B = \max(a_i; a_{i+1})$. It is given that $A + B = \sum_{i=1}^{2n} 2a_i = 4$ and $A \leq 1 \Rightarrow B \geq 3$. According to the Cauchy–Bunyakovsky–Schwarz inequality:

$$(\min(a_1^2; a_2^2) + \min(a_2^2; a_3^2) + \cdots + \min(a_{2n}^2; a_1^2))(1 + \cdots + 1) \geq A^2$$

and

$$(\max(a_1^2; a_2^2) + \max(a_2^2; a_3^2) + \cdots + \max(a_{2n}^2; a_1^2))(1 + \cdots + 1) \geq B^2.$$

From these inequalities, we obtain:

$$2(a_1^2 + a_2^2 + \dots + a_{2n}^2) = \min(a_1^2; a_2^2) + \min(a_2^2; a_3^2) + \dots + \min(a_{2n}^2; a_1^2) + \max(a_1^2; a_2^2) + \max(a_2^2; a_3^2) + \dots + \max(a_{2n}^2; a_1^2) \geq \frac{(A^2 + B^2)}{2n}.$$

Therefore, it is sufficient to find the minimum value of the expression:

$$\frac{A^2 + B^2}{4n}.$$

Again, by the Cauchy–Bunyakovsky–Schwarz inequality, we have:

$$(A^2 + B^2/9 + \dots + B^2/9)(1 + 1 + \dots + 1) \geq (A + 3B)^2 = (4 + 2B)^2 \geq 10^2.$$

From this, it follows that $A^2 + B^2 \geq 10$ and hence $\frac{A^2+B^2}{4n} \geq \frac{5}{n}$.
Choosing $a_{2k} = \frac{3}{2n}$ and $a_{2k-1} = \frac{1}{2n}$ for $k \in \{1, \dots, n\}$ we get $\frac{9n+n}{4n^2} = \frac{5}{2n}$ achieving equality. □

Remarks

In Solution 2, the minimum of $A^2 + B^2$ can be proven as follows:

$$A^2 + B^2 = A^2 + (4 - A)^2 = 2A^2 - 8A + 6 + 10 = 2(3 - A)(1 - A) + 10 \geq 10$$

as $A \leq 1$.

A6 Determine the smallest positive integer m for which there are at least 100 positive integers k such that the greatest integer less than or equal to

$$\frac{m(m+1)(2m+1)}{k^2-(4m+1)k+5m^2+3m}$$

equals k.

(Bulgaria)

Answer. 2401

Solution. The denominator equals $\left(k-2m-\frac{1}{2}\right)^2+m(m-1)-\frac{1}{4}$ and so it is positive for $m\geq 2$. We want the main fraction to be greater than or equal to k and less than $k+1$. The former one is equivalent to

$$k^3-(4m+1)k^2+(5m^2+3m)k-m(m+1)(2m+1)\leq 0 \Leftrightarrow (k-2m-1)((k-m)^2+m)\leq 0$$

and so all solutions for it are $k\leq 2m+1$. For the latter one we have

$$k^3-4mk^2+(5m^2-m-1)k-2m^3+2m^2+2m>0 \Leftrightarrow (k-2m)((k-m)^2-m-1)>0.$$

For $k=2m$ the latter is impossible, while for $k=2m+1$ the left-hand side equals m^2+m and so it is positive. For $k\leq 2m-1$ we want $(k-m)^2-m-1<0$, i.e. $k\in(m-\sqrt{m+1},m+\sqrt{m+1})$. For $m=1$ the solutions are $k=1,2$. For $m\geq 2$, since $0<m-\sqrt{m+1}<m+\sqrt{m+1}<2m+1$, all solutions are $2m+1$ and all integers in $(m-\sqrt{m+1},m+\sqrt{m+1})$.
If $m+1$ is not a perfect square, then the latter amount is $2\left\lfloor\sqrt{m+1}\right\rfloor+2$, while if it a square, then the amount is $2\left\lfloor\sqrt{m+1}\right\rfloor$. Note that $49^2=2401$. Hence if $m\leq 2399$, then the amount of solutions does not exceed $2\cdot 48+2=98$. It is also $2\cdot 49=98$ if $m=2400$. However, it equals $2\cdot 49+2=100$ for $m=2401$. □

Remarks

The reason I am asking for a numerical amount rather than an exact description of all solutions is that it might be a bit subjective on what counts as an exact description, e.g. why is "$2m+1$ and all integers in $(m-\sqrt{m+1},m+\sqrt{m+1})$" the only acceptable answer. As always, feel free to vary the formulation a little bit if you figure out a more appropriate one.

A7 Find all real numbers k such that there exists some tuple (a, b, c) satisfying

$$a^2 + b^2 + c^2 = a + b + c = k$$

and for each such tuple (a, b, c), values a, b, c are not the side lengths of triangle.

(Saudi Arabia)

Solution. The answer is all real numbers $k \in [0, 2]$.

Claim 1. No real number $k < 0$ or $k > 2$ satisfy the conditions.
Proof. First, we know that

$$(a + b + c)^2 \le 3(a^2 + b^2 + c^2),$$

so $k^2 \le 3k$, from which to have (a, b, c) satisfying the condition, we need $0 \le k \le 3$.

For $k \in (2, 3)$ we choose $(a, b, c) = \left(t, \frac{k-t}{2}, \frac{k-t}{2}\right)$, with $t > 0$ and $t < 2(\frac{k-t}{2}) \iff k > 2t$, hence side lengths of a triangle. Then $a + b + c = t + 2 \cdot \frac{k-t}{2} = k$. At the same time, we need

$$t^2 + 2\left(\frac{k-t}{2}\right)^2 = k.$$

Expanding it gives $3t^2 - 2kt + k^2 - 2k = 0$. This equation always has two solutions

$$t_1 = \frac{1}{3}\left(k - \sqrt{2k(3-k)}\right), \ t_2 = \frac{1}{3}\left(k + \sqrt{2k(3-k)}\right).$$

If $t_1 > 0 \iff k > \sqrt{2k(3-k)} \iff k > 2$ then we always have $3t_1 < k$ so choosing $a = t_1$ gives the set of three triangle sides. This shows that $k > 2$ are not satisfied. ◇

Claim 2. All real numbers $k \in [0, 2]$ satisfy the conditions.

Proof 1. Suppose that a, b, c are side lengths of some triangle then

$$c(a + b - c) + b(c + a - b) + a(b + c - a) > 0$$

which equivalent to

$$2(ab + bc + ca) > a^2 + b^2 + c^2 \iff (a + b + c)^2 > 2(a^2 + b^2 + c^2).$$

This implies that $k^2 > 2k$, this contradicts to $k \in [0; 2]$. ◇

Proof 2. Consider an arbitrary number k belonging to $[0, 2]$, clearly we only need to care about $k > 0$. Suppose $a \ge b \ge c$ then $(a - b)(a - c) \ge 0$ or $a^2 + bc \ge a(b + c)$. We have

$$2bc = (b + c)^2 - (b^2 + c^2) = (k - a)^2 - (k - a^2) = k^2 - 2ka - k + 2a^2.$$

Substitute to the above inequality, one can get

$$a^2 + \frac{k^2 - k}{2} + a^2 - ka \ge a(k - a) \iff 6a^2 - 4ak + k^2 - k = 0.$$

Solving gives $a = \frac{k}{3} \pm \frac{1}{6}\sqrt{2k(3-k)}$, but since $a \ge \frac{k}{3}$, we have $a = \frac{k}{3} + \frac{1}{6}\sqrt{2k(3-k)}$.

Finally, we see that $a \ge \frac{k}{2}$ since

$$\frac{k}{3} + \frac{1}{6}\sqrt{2k(3-k)} \ge \frac{k}{2} \iff \sqrt{2k(3-k)} \ge k \iff k \le 2.$$

The final inequality is correct so we have $a \ge b + c$, which implies that a, b, c cannot be the lengths of the three sides of the triangle. ◇

Solution 2. First, we know that

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2),$$

so $k^2 \leq 3k$, from which to have (a,b,c) satisfying the condition, we need $0 \leq k \leq 3$.

The triangle inequality is equivalent to $2\max(a,b,c) < a+b+c$. Thus, we want all solutions (a,b,c) to satisfy $\max(a,b,c) \geq \frac{1}{2}(a+b+c) = \frac{k}{2}$, i.e., we want to find $\min(\max\{a,b,c\})$.
WLOG assume that $a = \max\{a,b,c\}$; calculating, we get $b+c = k-a$ and $bc = a^2 - ak + \frac{k(k-1)}{2}$, so b and c are the solutions to $x^2-(k-a)x+a^2-ak+\frac{k(k-1)}{2}$.
These are

$$\frac{k-a \pm \sqrt{2ak+2k-3a^2-k^2}}{2}$$

so

$$a \geq \frac{k-a+\sqrt{2ak+2k-3a^2-k^2}}{2}$$

and $a \geq \frac{a+b+c}{3} = \frac{k}{3}$. Resolving the square root in the second-to-last inequality, we have

$$(3a-k)^2 \geq 2ak+2k-3a^2-k^2 \Leftrightarrow 3a^2-2ak+\frac{k(k-1)}{2} \geq 0.$$

The solutions to

$$3x^2-2xk+\frac{k(k-1)}{2} = 0$$

are

$$\frac{k}{3} \pm \frac{\sqrt{6k-2k^2}}{6}.$$

Since $a \geq \frac{k}{3}$, we have

$$a \geq \frac{k}{3} + \frac{\sqrt{6k-2k^2}}{6}.$$

We may also see that equality is attained when $a = b \geq c$. Thus,

$$\min(\max\{a,b,c\}) \geq \frac{k}{2} \Leftrightarrow \frac{k}{3}+\frac{\sqrt{6k-2k^2}}{6} \geq \frac{k}{2} \Longleftrightarrow 6k \geq 3k^2 \Leftrightarrow \boxed{2 \geq k \geq 0}.$$

We conclude that the conditions are satisfied only when $0 \leq k \leq 2$. $\square$

Combinatorics

C1 Consider 28 positive integers with sum 2027, such that the product of any 7 of them is a perfect square. Prove that one can select some of these numbers, such that the sum of the selected numbers, multiplied by the product of the remaining numbers is divisible by 8100.

(Saudi Arabia)

Solution. Let $a_1, a_2, \ldots, a_{28}$ be the given numbers and

$$a_1 + a_2 + \cdots + a_{28} = 2027.$$

Notice that $8100 = 2^2 \cdot 3^4 \cdot 5^2$. We have $a_1a_2a_3 \ldots a_7$ and $a_2a_3a_4 \ldots a_8$ are both perfect squares. Thus $a_1(a_2a_3 \ldots a_7)^2 a_8$ and a_1a_8 are also perfect squares. Hence, for any two numbers a_i, a_j with $1 \leq i < j \leq 25$, one can choose 6 more arbitrary numbers from the 26 remaining numbers, and combine them with each of a_i, a_j to form perfect squares. This shows that a_ia_j is always a perfect square.

Now suppose that one of the 28 given numbers is not a perfect square. Let's assume that is a_1, which implies that there exists a prime p such that $v_p(a_1)$ is odd. For every $i = 2, 3, \ldots, 28$ we have $v_p(a_1a_i)$ is even so it follows that $v_p(a_i)$ is odd, this shows that $p|a_i$ for every i. So $p|2027$, and 2027 being a prime leads to $p = 2027$, clearly absurd. Therefore all of 28 given numbers are perfect squares.

Thus each of given numbers when divided by 3 will have a remainder 0 or 1. If at most one of them has remainder 0, then the sum of the 28 numbers will have remainder 0 or 1. However $2027 \equiv 2 \pmod 3$ gives a contradiction. Consequently, at least two of the 28 given numbers are divisible by 3. Obviously, each of these numbers is also divisible by 9, so their product is divisible by $81 = 3^4$.

In addition, in these 28 numbers, there must be at least one even number because if all are odd, their sum will be even, which is a contradiction. We remove that even number, combine with the two numbers divisible by 9 above (which can be duplicated). With this we have eliminated no more than 3 numbers, leaving at least 25 numbers, which can be assumed to be $a_1, a_2, \ldots, a_{25}$. We prove that one can select some of this numbers, such that their sum is divisible by 25. Let $s_k = a_1 + a_2 + \cdots + a_k$ with $1 \leq k \leq 25$.

- If there exists k such that $25 \mid s_k$ then the assertion is true.
- Conversely, if all these s_k are not divisible by 25, their remainders will be from 1 to 24. According to the pigeonhole principle, there exists $i < j$ so that $s_i \equiv s_j \pmod{25}$, which shows that $25 | s_j - s_i = a_{i+1} + a_{i+2} + \cdots + a_j$.

After choosing such numbers, having the sum divisible by 25, the remaining will contain at least one number divisible by 4 and at least two numbers divisible by 9, so the product of all of them will be divisible by 8100. $\square$

C2 The numbers $2, 4, 8, \dots, 2^{2024}$ are written on the board. At every step we can perform one of the following:

(i) We delete an even number from the board, say $2k$, and in its place we write the number k.

(ii) We delete an odd number from the board, say $2k+1$, and in its place we write the numbers k and 1.

(iii) We delete two (possibly equal) numbers from the board, say k and ℓ, and in their place we write the number $k + \ell \cdot 2^k$. (For example, if the board contains the numbers $1, 3, 5$ and we choose $k = 5$ and $\ell = 1$, then after this steps the board will contain the numbers $3, 37$.)

Determine whether the following numbers can be obtained after a series of steps:

(a) $2^{2025} - 1$

(b) $2^{2025} - 3$

(Cyprus)

Solution. The sum of the binary digits of all numbers on the board when written in binary remains invariant after performing any step. (For the step (iii) this relies on the fact that when k is written in binary number system, it has at most k digits.)
This shows that (a) is impossible as the original sum of binary digits equals 2024 whereas the sum of binary digits of $2^{2025} - 1$ equals 2025.
We now show how to obtain $2^{2025} - 3$: Performing (i) repeatedly we can arrange that on the board we are left with the numbers $2, 4$ and a total of 2022 copies of the number 1. Note that by (iii) with $k = 1$, we can replace the numbers 1 and ℓ with the number $2\ell + 1$. Applying this repeatedly with each of the 2022 copies of the number 1 we end up with the number $2^{2022} - 1$. (After t applications we are left with $2021 - t$ copies of 1 together with $2^t - 1$. In the next step we apply (iii) with $k = 1$ and $\ell = 2^t - 1$ to reduce the number of copies of 1 by one and replace $2^t - 1$ by $2^{t+1} - 1$.)
So we now have $2, 4, 2^{2022} - 1$ on the board. Applying (iii) with $k = 2, \ell = 2^{2022} - 1$ we end up with 4 and $2^{2024} - 2$. Applying (i) twice we get 1 and $2^{2024} - 2$. Finally, applying (iii) with $k = 1$ and $\ell = 2^{2024} - 2$ we get $2^{2025} - 3$ as required. □

Remarks

Harder Formulation: Which numbers less than 2^{2025} can be the last remaining, after a series of steps?

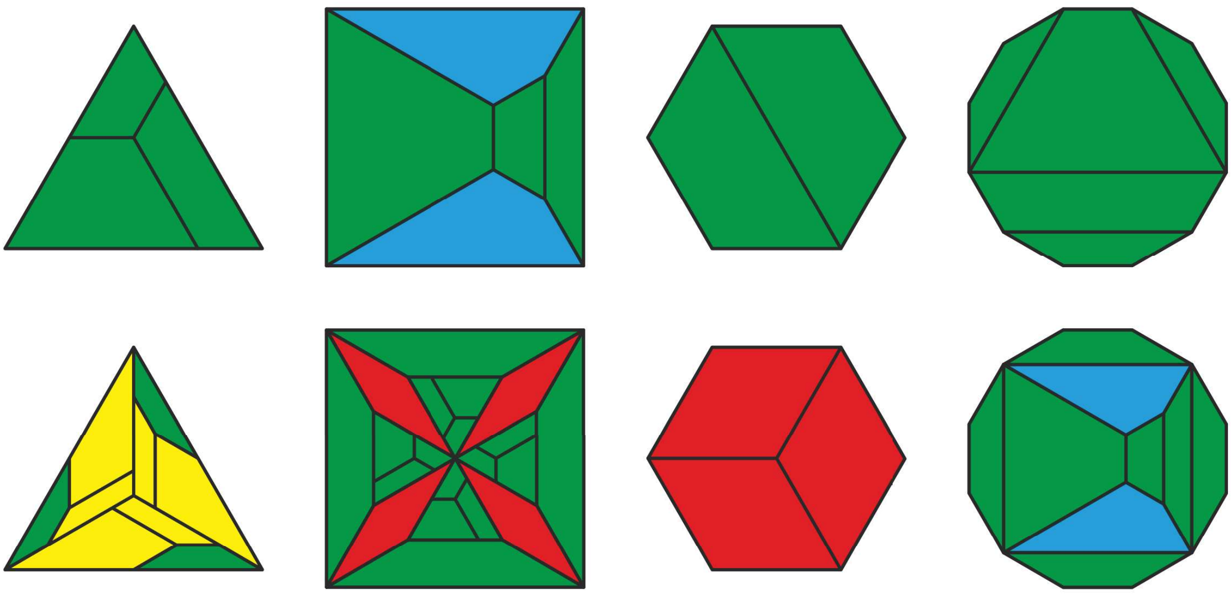
Solution. From the previous solution, the numbers need to have 2024 ones in their binary representation so they need to have exactly one zero in their binary representation (might be the first digit from the left). So they must be of the form $2^{2025} - 1 - 2^m$ for some $m \in \{0, 1, 2, \dots, 2024\}$. All such numbers can be obtained as follows:
As in the previous solution we use (i) repeatedly to create t copies of 1 and then use (iii) repeatedly to create $2^t - 1$. We are left with $2^t - 1$ together with $2024 - t$ positive powers of 2. If $m = 2024$, take $t = 2024$ and we are done. Otherwise, take $t = 2023 - m$ and use (i) repeatedly to change the $2024 - t$ positive powers of 2 into m copies of 1 and one copy of 2. Now apply (iii) with $k = 2$ and $\ell = 2^{2023-m} - 1$ to be left with $2^{2025-m} - 2 = 2^{2025-m} - 2^0 - 1$ and m copies of 1. Now apply again (iii) repeatedly to reduce the copies of 1 by one each time. After t applications we will have $2^{2025-m+t} - 2^t - 1$ together with $m - t$ copies of 1. In the final application we will be left with $2^{2025} - 2^m - 1$ as required. □

C3 Consider the set Σ of all quadrilaterals whose interior angles are in the set $\{30^\circ, 60^\circ, 120^\circ, 150^\circ\}$. Find all natural numbers $n \geqslant 3$ such that a regular n–gon can be completely tiled using only quadrilaterals from the set Σ.

(Serbia)

Solution. In any tiling of the regular n–gon with the elements in Σ, the quadrilaterals must fit around each vertex of the n–gon without gaps or overlaps. This means that the interior angle of the regular n–gon must be exactly the sum of some of the interior angles of the quadrilaterals from Σ. Since all interior angles of the elements from Σ are multiples of 30°, the interior angle of regular n–gon must also be a multiple of 30°. Out of the possible such convex angles (those are 30°, 60°, 90°, 120°, 150°) we recognize the following as the interior angle of the regular n–gons:

- 60°, as the interior angle of the equilateral triangle, $n = 3$;
- 90°, as the interior angle of the square, $n = 4$;
- 120°, as the interior angle of the regular hexagon, $n = 6$;
- 150°, as the interior angle of the regular dodecagon, $n = 12$.



The tilings for all four are indeed possible and can be done in many ways. On the above Figure are shown two ways of tiling for each n–gon. Isosceles trapeziums (with angles 30° and 150°, or 60° and 120°) are depicted in green, trapeziums with angles 30°, 120°, 60°, and 150° (in that order) are depicted in yellow, rhombuses (with angles 30° and 150°, or 60° and 120°) are depicted in red, and quadrilaterals with angles 30°, 60°, 150°, and 120° (in that order) are depicted in blue. $\square$

Remarks

The only nontrivial n–gon to tile is the square.

C4 Ana and Bob are playing a game. Bob chooses natural numbers n, m with $n \geq 3$. After that, Ana selects natural numbers $a_1, a_2, \ldots, a_n$ as she wishes. Then she forms n pairs (a_i, a_j), such that $i \neq j$ and each number appears in exactly two pairs. Finally, Ana computes:

$$\sum \gcd(a_i^2, a_j^2)$$

and records the result.
If the calculated result is exactly 2025^m, Ana wins; otherwise, Bob wins.
Find the minimum and maximum values of n for which Ana has a winning strategy, regardless of Bob's choice for the number m.

(Albania)

Solution. It is well known that $\gcd(x^2, y^2) = (\gcd(x, y))^2$, so by definition, let $\gcd(a_i^2, a_j^2) = x_e^2$. Thus, we can write:

$$\sum \gcd(a_i^2, a_j^2) = \sum_{e=1}^{n} x_e^2.$$

Claim 1. For $n \leq 5$, Bob Wins.
Proof. We must check if the equation:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 2025^m$$

can hold. Taking modulo 8, we analyse possible values:

$$x_i^2 \equiv \{0, 1, 4\} \pmod 8.$$

This sum modulo 8 must be congruent to 1 (as $2025^m \equiv 1 \pmod 8$) , which happens only if four of the terms are congruent to some combinations of $0, 4 \pmod 8$ and one is congruent to $1 \pmod 8$. However, this situation is impossible because we should have all elements x_1, x_2, x_3, x_4 even and x_5 odd, which contradicts the required conditions. ◇

Claim 2. Ana Wins for $n = 6$ and $n = 2025$.
Proof. To show that Ana wins for $n = 6$, set:

$$a_1 = a_2 = 2 \cdot 15 \cdot 45^{m-1},$$

$$a_3 = a_4 = a_5 = a_6 = 15 \cdot 45^{m-1},$$

for any given natural number m.
Similarly, for $n = 2025$, we can take:

$$a_1 = a_2 = \cdots = a_{2025} = 45^{m-1},$$

finishing the proof. ◇
However, Ana could not win for $n > 2025$ because Bob could choose $m = 1$, leading to:

$$\sum \gcd(a_i^2, a_j^2) > 2025.$$

With this we have proven that the minimum value of n for which Ana wins is $n = 6$ and the maximum such value is $n = 2025$. □

Remarks

The fact that 2025 is a perfect square is crucial to solving this problem since it is implicitly useful throughout the entire solution.

C5 Anna and Bob play the following game: In the first phase of the game Anna chooses a number a from the set $\{1,2,3,4,5,6,7,8\}$ and then Bob chooses a different number b from the same set. In the second phase of the game, with Anna starting first, they alternately pick any number they wish from the set $\{a,b\}$ and keep the running sum of all choices in that phase. If, after the choice of any one of them, the running sum is a multiple of 1111, then this person wins and the game ends. Determine which one of them, if any, has a winning strategy. *(Cyprus)*

Solution. Let us denote by A_i, and B_i the running sums up to the i-th chosen number by Anna and Bob respectively. Notice that after choosing $\{a,b\}$, and starting number A_1, Anna can ensure that $A_{i+1} = A_i + (a+b)$, i.e. she gets congruent numbers modulo $a+b$. Similarly Bob can only get the remainders $A_1 + a$ or $A_1 + b$. We give two winning strategies for Anna.

Strategy 1. Anna can win by choosing $a = 1$. She then plays as follows:

Case 1: If $b = 2$, Anna first chooses 1, getting remainder 1 modulo 3, and Bob can only get remainders 2 and 0. Then Anna creates $A_i = 3(i-1)+1$. In particular she wins by creating $1111 \equiv 1 \pmod 3$.

Case 2: If $b = 3$, Anna first chooses 3, with remainder 3 modulo 4, and Bob can only get remainders 0 and 2. Anna creates $A_i = 4(i-1)+3$ and wins by creating $1111 \equiv 3 \pmod 4$ as a running sum.

Case 3: If $b = 4$, Anna first chooses 1, with remainder 1 modulo 5, and Bob can only get remainders 2 and 0. Anna creates the sums $A_i = 5(i-1)+1$ and wins by creating $1111 \equiv 1 \pmod 5$.

Case 4: If $b = 5$, Anna first chooses 1, with remainder 1 modulo 6, and Bob can only get remainders 2 and 0. Anna creates the sums $A_i = 6(i-1)+1$ and wins by creating $1111 \equiv 1 \pmod 6$.

Case 5: If $b = 6$, Anna first chooses 1, with remainder 1 modulo 7, and Bob can only get remainders 2 and 0. Anna creates the sums $A_i = 7(i-1)+1$ and wins by creating $3333 \equiv 1 \pmod 7$. Bob can't win getting a smaller multiple of 1111 since $1111 \equiv 5 \pmod 7$ and $2222 \equiv 3 \pmod 7$.

Case 6: If $b = 7$, Anna first chooses 7, with remainder 7 modulo 8, and Bob can only get remainders 0 and 6. Anna creates the sums $A_i = 8(i-1)+7$ and wins by creating $1111 \equiv 7 \pmod 8$.

Case 7: If $b = 8$, Anna first chooses 8 with remainder 8 modulo 9, and Bob can only get remainders 0 and 7. Anna creates the sums $A_i = 9(i-1)+8$ and wins by creating $2222 \equiv 8 \pmod 9$. Bob can't win getting by getting first to 1111 since $1111 \equiv 4 \pmod 9$. □

Strategy 2. Anna can also win by choosing $a = 2$.

Case 1: If $b = 1$, Anna chooses $1 \pmod 3$, Bob can only get remainders 2 and 0, hence Anna wins by creating $1111 \equiv 1 \pmod 3$.

Case 2: If $b = 3$, Anna chooses $2 \pmod 5$, Bob can only get remainders 4 and 0, hence Anna wins by creating $1111 \equiv 1 \pmod 5$.

Case 3: If $b = 4$, Anna chooses $2 \pmod 6$, Bob can only get remainders 4 and 0, hence Anna wins by creating $2222 \equiv 2 \pmod 6$. Bob can't win with $1111 \equiv 1 \pmod 6$.

Case 4: If $b = 5$, Anna chooses $5 \pmod 7$, Bob can only get remainders 0 and 3, hence Anna wins by creating $1111 \equiv 5 \pmod 7$.

Case 5: If $b = 6$, Anna chooses $6 \pmod 8$, Bob can only get remainders 0 and 4, hence Anna wins by creating $2222 \equiv 6 \pmod 8$. Bob can't win with $1111 \equiv 7 \pmod 8$.

Case 6: If $b = 7$, Anna chooses $7 \pmod 9$, Bob can only get remainders 0 and 5, hence Anna wins by creating $4444 \equiv 7 \pmod 9$ ($1111 \equiv 4 \pmod 9$, $2222 \equiv 8 \pmod 9$, and $3333 \equiv 3 \pmod 9$).

Case 7: If $b = 8$, Anna chooses $2 \pmod{10}$, Bob can only get remainders 4 and 0, hence Anna wins by creating $2222 \equiv 2 \pmod{10}$. Bob can't win with $1111 \equiv 1 \pmod{10}$. □

Remarks

If Anna chooses $a = 3,4,5,6,7$ or 8, then Bob can win by choosing $b = 8,7,6,5,4$ or 3, respectively, by creating running sums $B_i = 11i$, including.

C6 Given a set $\{x_1, x_2, \ldots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$ we say that its *ascending power sum* is

$$x_1 + x_2^2 + x_3^3 + \cdots + x_k^k.$$

The ascending power sum of the empty set is considered to be equal to 0.
Determine the number of subsets of $\{1, 2, \ldots, 10\}$ whose ascending power sum is a multiple of 3.

(Cyprus)

Solution. Let us write a_n, b_n, c_n for the number of subsets of $\{1, 2, \ldots, n\}$ with an odd number of elements whose ascending power sum is congruent to $0 \bmod 3, 1 \bmod 3, 2 \bmod 3$, respectively. Let a'_n, b'_n, c'_n be the corresponding number of subsets with an even number of elements.

For any natural number m, the sequence of its powers $m, m^2, m^3, m^4, \ldots$ modulo 3 is congruent to

- $0, 0, 0, 0, \ldots$ if $m \equiv 0 \bmod 3$.
- $1, 1, 1, 1, \ldots$ if $m \equiv 1 \bmod 3$.
- $2, 1, 2, 1, \ldots$ if $m \equiv 2 \bmod 3$.

So the numbers $a_n, b_n, c_n, a'_n, b'_n, c'_n$ satisfy the following recurrence relations for every $n \in \mathbb{N}_0$:

$$\begin{array}{lll}
a_{3n+1} = a_{3n} + c'_{3n} & a_{3n+2} = a_{3n+1} + b'_{3n+1} & a_{3n+3} = a_{3n+2} + a'_{3n+2} \\
b_{3n+1} = b_{3n} + a'_{3n} & b_{3n+2} = b_{3n+1} + c'_{3n+1} & b_{3n+3} = b_{3n+2} + b'_{3n+2} \\
c_{3n+1} = c_{3n} + b'_{3n} & c_{3n+2} = c_{3n+1} + a'_{3n+1} & c_{3n+3} = c_{3n+2} + c'_{3n+2} \\
a'_{3n+1} = a'_{3n} + c_{3n} & a'_{3n+2} = a'_{3n+1} + c_{3n+1} & a'_{3n+3} = a'_{3n+2} + a_{3n+2} \\
b'_{3n+1} = b'_{3n} + a_{3n} & b'_{3n+2} = b'_{3n+1} + a_{3n+1} & b'_{3n+3} = b'_{3n+2} + b_{3n+2} \\
c'_{3n+1} = c'_{3n} + b_{3n} & c'_{3n+2} = c'_{3n+1} + b_{3n+1} & c'_{3n+3} = c'_{3n+2} + c_{3n+2}
\end{array}$$

For example, to count a'_{3n+2}, that is the number of subsets of $\{1, 2, \ldots, 3n+2\}$ with an even number of elements whose ascending power sum is congruent to $0 \bmod 3$, we consider whether $3n+2$ is contained in the subset. If it is not, we have a'_{3n+1} such subsets. Contrarily, if $3n+2$ is contained in the subset, then it is the last element which means that it has to be raised to an even power in the ascending power sum, thus gives a contribution of $1 \bmod 3$ in that sum. So we need a contribution of $2 \bmod 3$ from the initial part of the sum obtained from an odd number of elements. Hence, there are c_{3n+1} such subsets.
Obviously $b_1 = a'_1 = 1$ and $a_1 = c_1 = b'_1 = c'_1 = 0$. Using the above recurrence relations we can now build the table below.

n	1	2	3	4	5	6	7	8	9	10
a_n	0	0	1	3	5	11	22	43	86	171
b_n	1	1	1	2	5	10	21	42	85	171
c_n	0	1	2	3	6	11	21	43	85	170
a'_n	1	1	1	3	6	11	22	43	86	171
b'_n	0	0	1	2	5	10	21	43	85	171
c'_n	0	1	2	3	5	11	21	42	85	170

The required number is $a_{10} + a'_{10} = 342$. □

Harder Formulation: One could make the problem harder by asking about the number of subsets of $\{1, 2, \ldots, 2025\}$ with ascending power sum a multiple of 3.

Solution. From the recurrence relations we see that $a_{3n} = a'_{3n}, b_{3n} = b'_{3n}, c_{3n} = c'_{3n}$ for every $n \geqslant 1$. These (together with the recurrence relations) give $a_{3n+1} = a'_{3n+1}, b_{3n+1} = b'_{3n+1}, c_{3n+1} = c'_{3n+1}$ and then $a_{3n+2} = b'_{3n+2}, b_{3n+2} = c'_{3n+2}, c_{3n+2} = a'_{3n+2}$ for every $n \geqslant 1$. The last three equalities are also easily seen to hold for $n = 0$ as well.
These, for $n \geqslant 1$, give

$$\begin{aligned} a_{3n+3} &= a_{3n+2} + a'_{3n+2} \\ &= (a_{3n+1} + b'_{3n+1}) + (a'_{3n+1} + c_{3n+1}) \\ &= (a_{3n+1} + b_{3n+1} + c_{3n+1}) + a_{3n+1} \\ &= 2^{3n} + a_{3n+1} \end{aligned}$$

Similarly for $n \geqslant 1$ we get

$$\begin{array}{ccc} a_{3n+1} = 2^{3n-2} + c_{3n-1} & a_{3n+2} = 2^{3n-1} + a_{3n} & a_{3n+3} = 2^{3n} + a_{3n+1} \\ b_{3n+1} = 2^{3n-2} + a_{3n-1} & b_{3n+2} = 2^{3n-1} + b_{3n} & b_{3n+3} = 2^{3n} + b_{3n+1} \\ c_{3n+1} = 2^{3n-2} + b_{3n-1} & c_{3n+2} = 2^{3n-1} + c_{3n} & c_{3n+3} = 2^{3n} + c_{3n+1} \ . \end{array}$$

We then have

$$a_{2025} = 2^{2022} + a_{2023} = 2^{2022} + 2^{2020} + c_{2021} = 2^{2022} + 2^{2020} + 2^{2018} + c_{2019} = \cdots$$

The general term of this sequence of equal terms is

$$2^{2022} + 2^{2020} + \cdots + 2^{2024-2k} + x_{2024-2k} \,,$$

where for $k = 1, 2, 3, \ldots$ the letter x takes values from the alphabet $\{a, b, c\}$ as follows:

$$a, c, c, c, b, b, b, a, a, a, \ldots$$

which repeats with a period of 9. In particular, for $k = 1011 = 9 \cdot 112 + 3$ we get $x = c$ and so

$$a_{2025} = 2^{2022} + 2^{2020} + \cdots + 2^4 + 2^2 + c_3 = 4 \cdot \frac{4^{1011} - 1}{3} + 2 = \frac{2^{2024} + 2}{3} \ .$$

So there are

$$a_{2025} + a'_{2025} = 2a_{2025} = \frac{(2^{2025} + 4)}{3}$$

such subsets. $\square$

C7 Let n be a positive integer. The integers from 1 to n are written in the cells of an $n \times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by r_x the number of pairs (a, b) of numbers in row x, such that $a > b$, but the column position of a is less than the column position of b. Similarly, denote by c_y the number of pairs (a, b) of numbers in column y, such that $a > b$, but the row position of a is less than the row position of b. Determine the largest possible value of the sum

$$r_1 + r_2 + \cdots + r_n + c_1 + c_2 + \cdots + c_n.$$

(Bulgaria)

Answer. $\dfrac{n(n-1)(2n-1)}{3}$

Solution. Suppose x is in position i in some row/column. Then after it there could be at most $\min(n-i, x-1)$ smaller numbers. Having in mind that this bound is separately for the row of x and for the column of x, as well as that i is different for the different appearances of x (as no row/column has a number appearing more than once, by the problem condition), we deduce that the contribution of x to the overall sum is at most

$$2\sum_{i=1}^{n} \min(n-i, x-1) = 2\sum_{i=n-x+1}^{n} (n-i) + 2\sum_{i=1}^{n-x} (x-1) = 2\sum_{i=0}^{x-1} i + 2(n-x)(x-1)$$
$$= x(x-1) + 2(n-x)(x-1) = (2n-x)(x-1).$$

Summing through $x = 1, \ldots, n$ now gives the following upper bound for the sum:

$$\sum_{x=1}^{n} (2n-x)(x-1) = (2n+1)\sum_{x=1}^{n} x - \sum_{x=1}^{n} x^2 - \sum_{x=1}^{n} 2n$$
$$= \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)(2n+1)}{6} - 2n^2 = \frac{n(n-1)(2n-1)}{3}.$$

Equality holds e.g. for the table in which the s-th row is $s, s-1, \ldots, 1, n, n-1, \ldots, s+1$, since for each x in position i in a row or column there are indeed exactly $\min(n-i, x-1)$ smaller numbers after it. $\square$

Remarks

It is very likely the case that there are no other possible examples attaining the bound, but proving this seems cumbersome and unnecessary to be required from the contestants.

Geometry

G1 Let I_A and ω be the excenter opposite to A and circumcircle, respectively of the triangle $\triangle ABC$. Let B_1 be the midpoint of $arc\widehat{AC}$ on circle ω which doesn't contain point B. Let C_1 be the midpoint of $arc\widehat{AB}$ on circle ω which doesn't contain point C. Let B_2 and C_2 be the points of intersection of the line B_1C_1 with the sides AC and AB, respectively. Let line parallel to AB which passes through B_2 intersect the line BC_1 at point C'. Let line parallel to AC which passes through C_2 intersect the line CB_1 at point B'. Prove that the circumcircles of the triangles $\triangle BC_1C_2$, $\triangle CB_1B_2$ and $\triangle B'I_AC'$ share a common point.

(Albania)

Solution 1. Lets write with $\angle CAB = 2\alpha, \angle ABC = 2\beta$ and $\angle BCA = 2\gamma$. And let I be the incenter of triangle $\triangle ABC$. It is well known that I is on the line BB_1 and since quadrilateral BCB_1C_1 is cyclic we have,

$$\angle IB_1B_2 = \angle BB_1C_1 = \angle BCC_1 = \angle C_1CB_2 = \angle ICB_2,$$

hence the quadrilateral ICB_1B_2 is cyclic. Similarly the quadrilateral IBC_1C_2 is cyclic too. So, I is a common point of the circumcircles of the triangles $\triangle BC_1C_2$ and $\triangle CB_1B_2$. Now if we prove that the quadrilateral $IB'I_AC'$ is cyclic then the problem is finished since we would have I is part of circumcircle of the triangle $\triangle B'I_AC'$ and hence we will have I as a common point of these three circumcircles.

First we are going to show that I is the intersection of lines B_2C' and C_2B'. Since the quadrilaterals ICB_1B_2, IBC_1C_2 and BCB_1C_1 are cyclic and it is also very well known that $C_1I = C_1B$ hence triangle $\triangle BC_1I$ is isosceles, so we have,

$$\angle C_1B_2I = \angle B_1CI = \angle B_1CC_1 = \angle B_1BC_1 = \angle IBC_1 = \angle C_1IB = \angle C_1C_2B,$$

which means B_2I is parallel to C_2B so B_2I is parallel to AB, which means I is on the line B_2C'. Similarly I is on the line C_2B'. So, I is the intersection of the lines B_2C' and C_2B'.

Because the quadrilaterals AB_1CB and ICB_1B_2 are cyclic and since I is the incenter of triangle $\triangle ABC$ we have,

$$\angle C_1B_2C' = \angle B_1CI = \angle B_1CA + \angle C_1CA = \angle B_1BA + \angle C_1CA = \beta + \gamma.$$

Now because the pentagon AB_1CBC_1 is cyclic and since I is the incenter of the triangle $\triangle ABC$ we have,

$$\angle C'C_1B_2 = \angle BC_1B_1 = \angle BC_1C + \angle B_1C_1C = \angle BAC + \angle B_1BC = 2\alpha + \beta.$$

Now from the last relations and because $\alpha + \beta + \gamma = 90^\circ$, then from the triangle $\triangle C_1B_2C'$ and knowing I is the incenter of the triangle $\triangle ABC$ we have,

$$\angle IC'B = \angle B_2C'C_1 = 180^\circ - \angle C'C_1B_2 - \angle C_1B_2C' = 180^\circ - 2\alpha - \beta - \beta - \gamma = \gamma = \angle ICB,$$

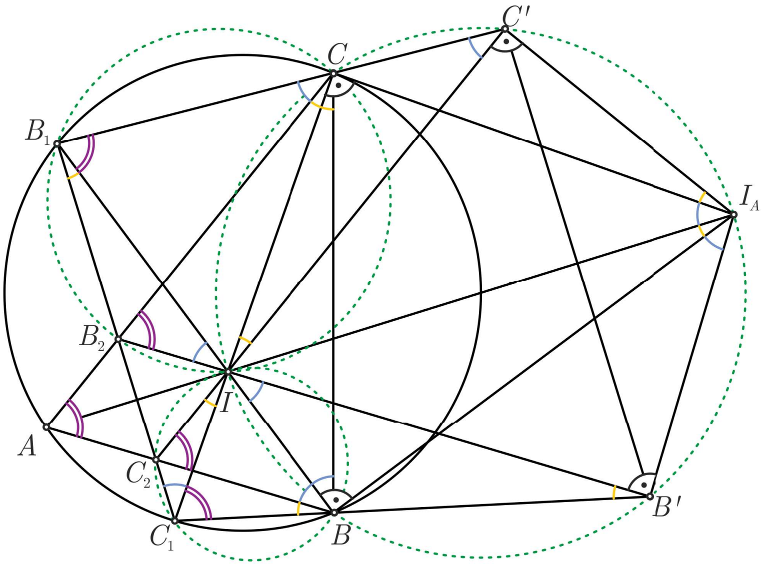
hence the quadrilateral $IBC'C$ is cyclic. Similarly the quadrilateral $ICB'B$ is cyclic.

Now since I_A is the excenter opposite to A of the triangle $\triangle ABC$ we have that BI_A is the angle bisector of the angle that is formed from the lines AB and BC, hence it is clear that $\angle I_ABI = 90^\circ$. Similarly $\angle I_ACI = 90^\circ$, so we have,

$$\angle I_ACI + \angle I_ABI = 180^\circ,$$

hence the quadrilateral I_ABIC is cyclic.

Now because the quadrialterals $IBC'C, ICB'B$ and I_ABIC are cyclic, we have that the hexagon $IBC'I_AB'C$ is cyclic. Which means I is part of the circumcircle of the triangle $\triangle B'I_AC'$, hence as desired.



Solution 2. Let I be the incenter of $\triangle ABC$, and let us denote its angles with $\angle CAB = 2\alpha, \angle ABC = 2\beta$ and $\angle BCA = 2\gamma$. First notice that BB_1, CC_1, and AI_A pass through I.
Since $\angle IC_1C_2 \equiv \angle CC_1B_1 = \beta = \angle B_1BA \equiv \angle C_2BI$, the circumcircle of $\triangle BC_2C_1$ passes through I. Now $\angle BC_2I = \angle BC_1I \equiv \angle BC_1C = \angle BAC = 2\alpha$, implies that $C_2I \parallel AC$, hence B', I, and C_2 are collinear.
By reason of symmetry I lies on the circumcircle of $\triangle CB_1B_2$ and on line $C'B_2$.
Now it suffices to prove that I lies on the circumcircle of $\triangle B'C'I_A$. Since

$$\angle II_AB \equiv \angle AI_AB = \gamma = \angle ACC_1 = \angle ABC_1 = \angle IC'B,$$

we conclude that $IBC'I_A$ is cyclic, hence $\angle I_AC'I = \angle IAB'I = 90°$.
By reasons of symmetry $\angle IB'I_A = 90°$, hence $IC'I_AB'$ is cyclic, i.e. the point I also lies on the circumcircle of $\triangle B'C'I_A$. □

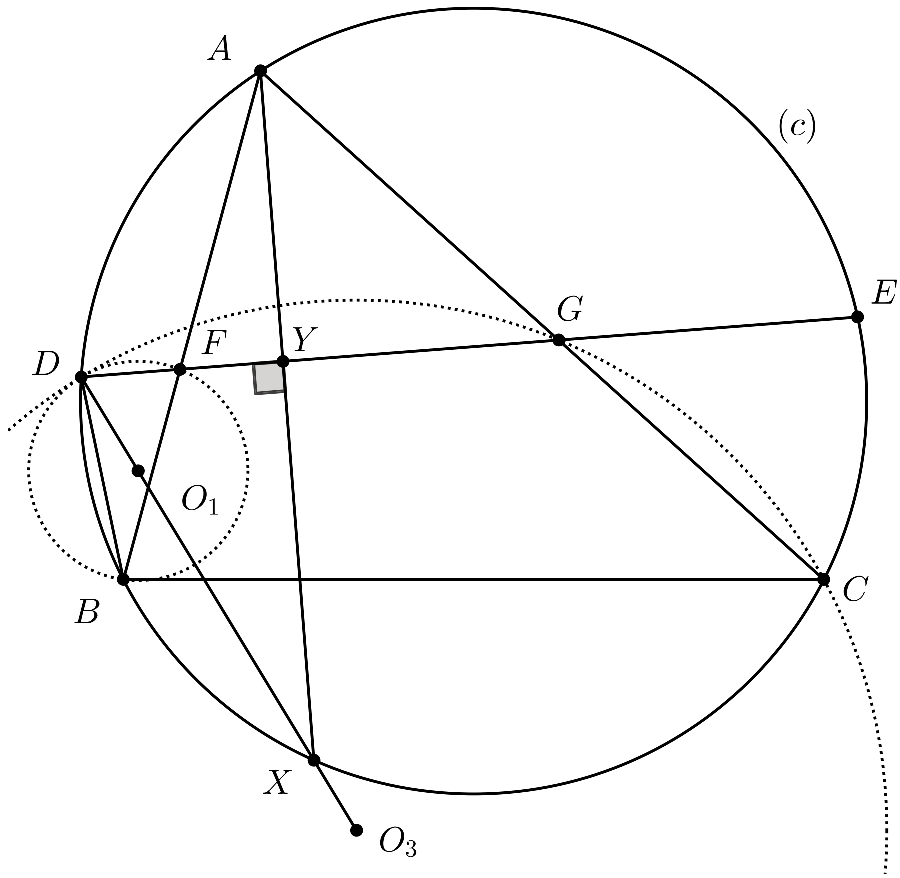
Remarks

After proving that I lies on $B'B_2$ and $C'C_2$, we can prove that $IBB'I_AC'C$ is cyclic in many ways, since all angles can be easily calculated using α, β, and γ.

G2 Let ABC be a triangle with circumcircle (c) and let D and E be points on the arc AB not containing C and on the arc AC not containing B respectively. Let F and G be the intersections of DE with AB and AC respectively. Let O_1, O_2, O_3 and O_4 be the centers of the circumcircles of triangles DFB, EFB, DGC and EGC respectively, and let ℓ be the perpendicular from A to DE.
Prove that O_1O_3, O_2O_4 and ℓ concur at a point on (c).

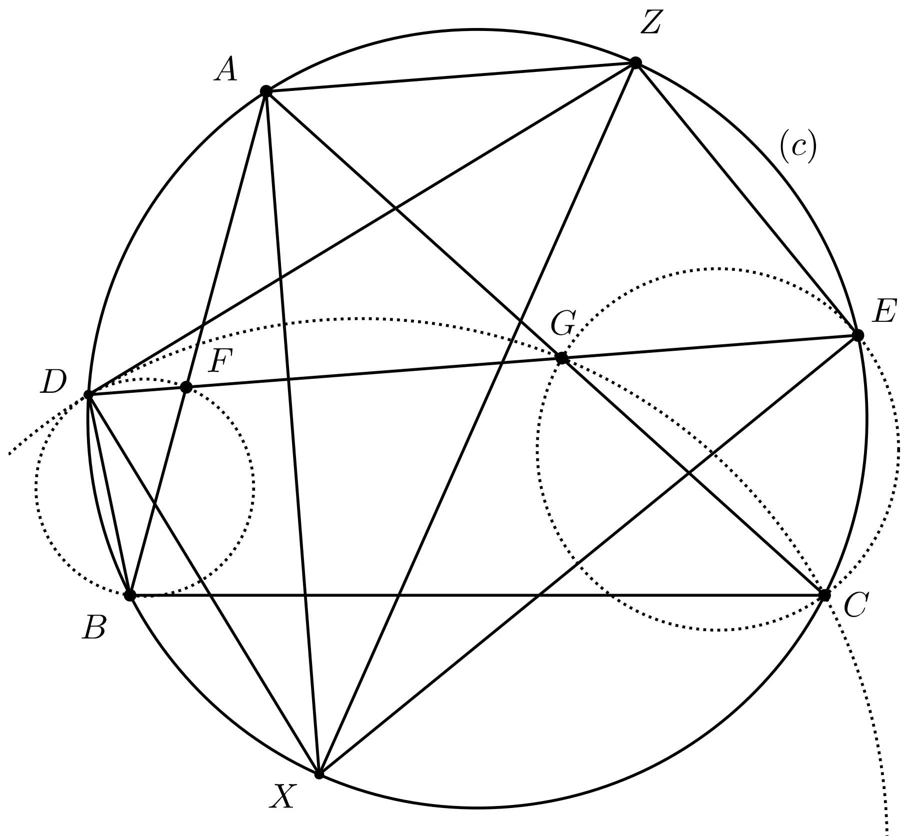
(Cyprus)

Solution 1. Let $\vartheta = \angle DBA$. Then $\angle DO_1F = 2\vartheta$ and so $\angle O_1DF = 90^\circ - \vartheta$. Similarly, $\angle O_3DF = 90^\circ - \vartheta$ and so O_1, O_3, D are collinear. Let $X \neq D$ be the other point of intersection of O_1O_3 with (c) and let Y be the point of intersection of AX with DE. Then $\angle DXY = \angle DBA = \vartheta$ and $\angle XDY = 90^\circ - \vartheta$, therefore $AX \perp DE$, i.e. AX is ℓ.



Similarly, O_2, O_4, E are collinear and defining $X' \neq E$ to be the other point of intersection of O_2O_4 with DE we get that AX' is also ℓ. Thus $X = X'$ and O_1O_3, O_2O_4 and ℓ concur at X.

Solution 2. As in Solution 1, $\angle O_1DF = \angle O_3DF$ which shows that O_1, O_3, D are collinear and furthermore, the circumcircles of triangles DFB and DGC are internally tangent. Let ℓ_1 be the common tangent at D. Analogously O_2, O_4, E are collinear and the circumcircles of triangles EFB and EGC are internally tangent. Let ℓ_2 be the common tangent at E.



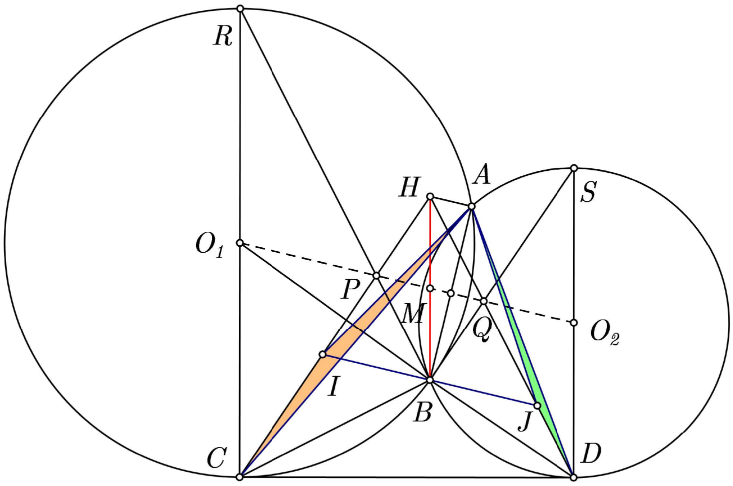
Let X be the point of intersection of O_1O_3 with O_2O_4 and let Z be the point of intersection of ℓ_1 and ℓ_2. Since $\angle XDZ = XEZ = 90^\circ$, then X, D, Z, E are concyclic on a circle with diameter ZX.
We have

$$\begin{aligned}\angle DZE &= 180^\circ - \angle EDZ - \angle ZED \\ &= 180^\circ - \angle DBF - \angle ECG \\ &= 180^\circ - \angle DXA - \angle EXA = \angle DAE\end{aligned}$$

which shows that Z belongs on (c). Therefore X also belongs on (c) and ZX is a diameter of (c).
Furthermore $\angle AZD = \angle ABD = \angle EDZ$ which shows that DE is parallel to AZ and therefore it is perpendicular to AX, i.e. AX is the line ℓ.

G3 Let $(O_1), (O_2)$ be two circles that intersect at A, B with the common tangent line CD (that closer to B than A) with $C \in (O_1), D \in (O_2)$ and $AC > AD$. Construct H so that triangle HCD has B as its orthocenter. Draw diameter CR of (O_1) and diameter DS of (O_2). Lines BR, HC meet at P and lines BS, HD meet at Q. Take I, J such that P, Q are midpoints of segments HI, HJ respectively. Prove that the circumradius of triangles AIC and AJD are equal.

(Saudi Arabia)



Solution. Since CR is the diameter of (O_1), we have $\angle CBR = 90°$, thus $BC \perp BR$. Note that $BC \perp HD$ implies that $BR \parallel HD$ or $BP \parallel HQ$.

Similarly, $BQ \parallel HP$ so quadrilateral $HPBQ$ is a parallelogram. Thus BH intersects PQ at the midpoint M of each segment. Now let O_1P intersects CR at M', then by Thales' theorem, we have

$$\frac{M'B}{O_1R} = \frac{PB}{PR} = \frac{PH}{PC} = \frac{M'H}{O_1P},$$

but $O_1P = O_1R$ so $M'H = M'B$. Thus $M \equiv M'$ and the points O_1, P, M are collinear. Similarly, M, Q, O_2 are collinear. Hence, O_1, P, M, Q, O_2 are collinear.

Since O_1O_2 is the perpendicular bisector of the segment AB, then $O_1O_2 \perp AB$. Note that PM, QM are also the midline of triangles HIB, HJB, then $PM \parallel IB, QM \parallel JB$, and P, M, Q are collinear, so I, B, J are also collinear and $IB = 2PM = 2QM = JB$.

Therefore, AB is the perpendicular bisector of IJ, from which we deduce $AI = AJ$.

We have $\angle BAC = \angle BCD, \angle BAD = \angle BDC$ so

$$\angle CAD = \angle BAC + \angle BAD = \angle BCD + \angle BDC = 180° - \angle CBD.$$

On the other hand, since B is the orthocenter of triangle HCD then $\angle CHD = 180° - \angle CBD$. Thus $\angle CAD = \angle CHD$ so $CDAH$ is cyclic. This implies that $\angle ACI = \angle ADJ$, combining with $AI = AJ$, by applying the sine law in two triangles AIC and AJD, we can conclude that the circumradius of these triangles are equal. □

Remarks

The collinearity of O_1, P, and M can be alternatively proven without phantom point in the following way. Since $HB \parallel RC$, we get $\triangle PHB \sim \triangle PCR$, and as M,O_1 are midpoints of HB, CR, respectively, by SAS criterion we get that $\triangle PHM \sim \triangle PCO_1$ and thus $\angle HPM = \angle CPO_1$. Since H, P, and C are collinear, so are M, P, and O_1.

G4 In triangle ABC, where $AB < AC$, the angle bisector of $\angle A$ intersects side BC at point L, and the circumcircle Ω - at point W. On the circumcircle of triangle $\triangle WLC$, a point $P \neq W$ is chosen such that $BP = CP$. The tangent from point A to Ω intersects BC at point T. On line BW, a point Q is chosen such that $TB = TQ$. Prove that points Q, A, P, W lie on the same circle.

(Mykhailo Shtandenko - Ukraine)

Solution 1. Since $BW = WC$, the condition $BP = CP$ is equivalent to $WP \perp BC$. On the ray AT beyond point T, mark a point R such that $TR = TB = TQ$. Then T is the circumcenter of triangle RQB, hence $\angle RTB = 2\angle RQB$. But we know $TA = TL$, so

$$\angle RAW = \angle TAL = \frac{180^\circ - \angle ATL}{2} = \frac{\angle RTB}{2} = \angle RQB = \angle RQW,$$

which implies that points R, Q, A, W lie on the same circle.

Note that P is the orthocenter of triangle BLW. Indeed, let us denote the orthocenter of triangle BLW by P'. Then:

$$\angle LP'W = 90^\circ - \angle BWP = \angle LBW = \angle LCW,$$

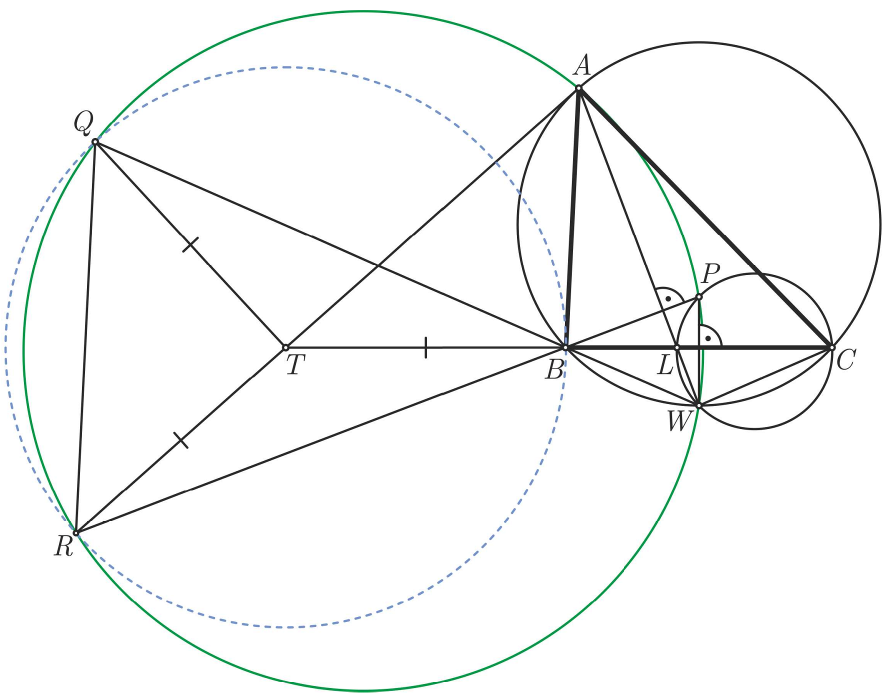
so point P' lies on the circumcircle of triangle WLC, and $WP' \perp BC$ because P' is the orthocenter of triangle BLW. Hence, $P' = P$. Therefore, P is truly the orthocenter of triangle BLW, which implies $BP \perp AL$.

From the conditions $TB = TR$ and $TA = TL$, it follows that $BR \perp AL$. Indeed, BR is parallel to the external angle bisector of $\angle RTB$, which is the internal angle bisector of $\angle ATL$, and it is perpendicular to AL.

Combining $BP \perp AL$ and $BR \perp AL$, we conclude that points R, B, P lie on the same line perpendicular to AL. Then:

$$\angle RAW = \angle TAL = \angle TLA = \angle BLA = \angle BPW = \angle RPW,$$

hence points R, A, P, W lie on the same circle, meaning that points Q, R, A, P, W are concyclic, as desired. □



Solution 2. Let us denote the angles in $\triangle ABC$ by $\angle BAC = \alpha$, $\angle BCA = \beta$, and $\angle ACB = \gamma$ and let line BP meet line AT at R. By definition $BW = CW$ (AW is angle bisector) and $BP = CP$, implying that $BWCP$ is a kite, hence $PW \perp BC$. Since $\angle BTA = \beta - \gamma$ and

$$\angle TBR = \angle CBP = \angle PCB \equiv \angle PCL = \angle PWL = 90^\circ - \angle WLC = \frac{\alpha+\beta+\gamma}{2} - (\gamma + \frac{\alpha}{2}) = \frac{\beta-\gamma}{2},$$

$\triangle BRT$ is isosceles. This implies that $TR = TB = TQ$, hence T is the circumcenter of $\triangle BQR$.

Now

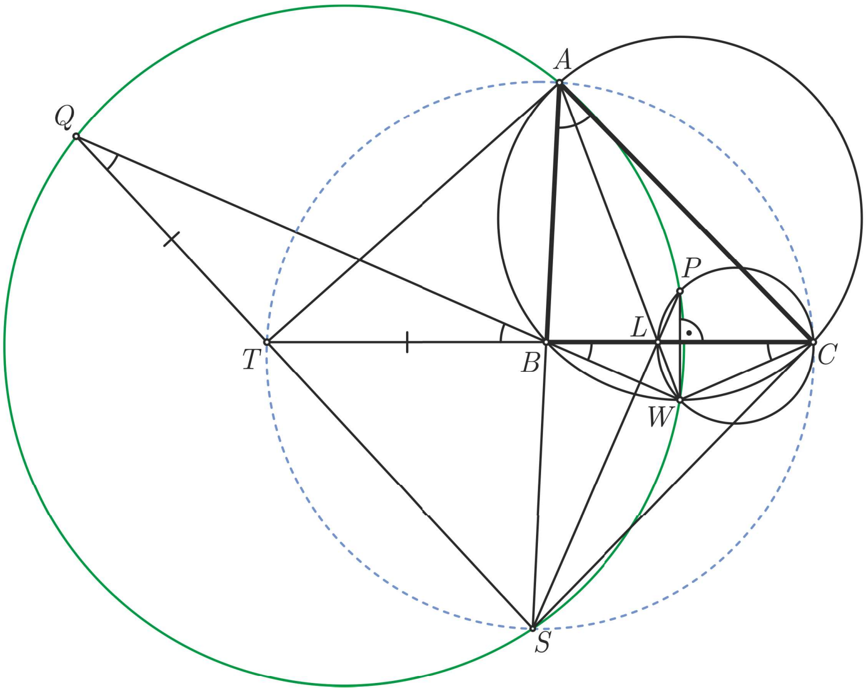
$$\angle PRQ \equiv \angle BRQ = 90^\circ - \angle QBT = 90^\circ - \angle WBC = \angle PWB \equiv \angle PWQ,$$

implies that P, Q, R, and W lie on a circle.

Similarly

$$\angle RQW \equiv \angle RQB = 90^\circ - \angle TBR = \gamma + \frac{\alpha}{2} = \angle TAW \equiv \angle RAW,$$

implies that A lies on the same circle, finishing the proof. □



Solution 3. If $\angle BAC = \alpha$, since $BT = QT$ we have

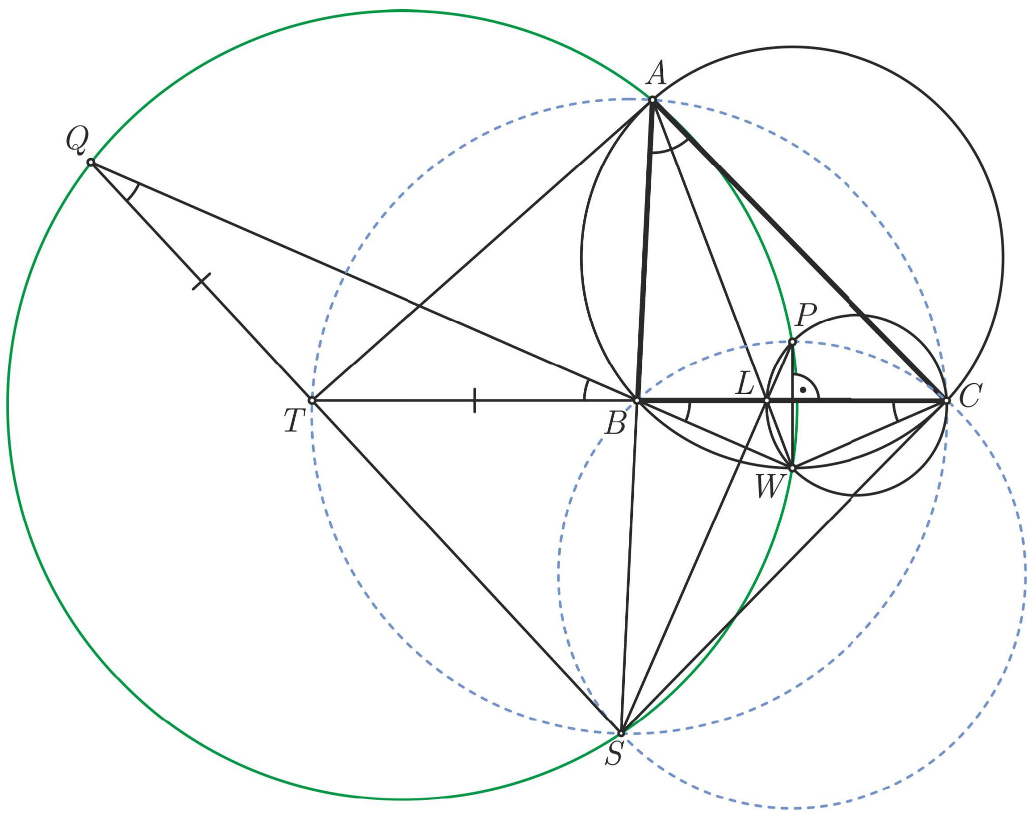
$$\angle TQB = \angle QBT = \angle WBC = \angle WAC = \angle BAW = \angle BCW = \frac{\alpha}{2}.$$

Let QT meet AB at S. Since $\angle SQW \equiv \angle TQB = \angle BAW \equiv \angle SAW$, the point S lies on the circumcircle of $\triangle AQW$.

Since $\angle STC = 2\angle TQB = \alpha = \angle SAC$, $ATSC$ is cyclic, hence $\angle TCS = \angle TAS = \angle ACB$. This implies that L is the incenter of $\triangle ASC$, hence $\angle BLS = \angle 90^\circ - \frac{\alpha}{2}$.

On the other hand $BW = CW$ and $BP = CP$, implies that $BWCP$ is a kite, hence $WP \perp BC$. Now $\angle PLC = \angle PWC = 90^\circ - \angle BCW = 90^\circ - \frac{\alpha}{2}$, implies that S, L, and P are collinear.

Finally $\angle SPW \equiv \angle LPW = \angle LCW = \frac{\alpha}{2} = \angle SAW$, implies that P lies on the circumcircle of $AQSW$, as desired. □



Solution 4. Let us denote the angles in $\triangle ABC$ by $\angle BAC = \alpha$, $\angle BCA = \beta$, and $\angle ACB = \gamma$. Since $BT = QT$ we have

$$\angle TQB = \angle QBT = \angle WBC = \angle WAC = \angle BAW = \angle BCW = \frac{\alpha}{2}.$$

Let PL meet AB at S. Since $\angle SPC \equiv \angle LPC = 180^\circ - \angle CWL = \angle SBC$, $BSCP$ is cyclic. Using power of the point L we get $\overline{SL} \cdot \overline{LP} = \overline{BL} \cdot \overline{LC} = \overline{AL} \cdot \overline{LW}$, implying that P lies on the circumcircle of $\triangle ASW$.

From $BW = CW$ and $BP = CP$, the quadrilateral $BWCP$ is a kite, hence $WP \perp BC$. Now

$$\angle CSB = 180^\circ - \angle BPC = 180^\circ - 2\angle WPC = 180^\circ - 2\angle WLC = 180^\circ - \alpha - 2\gamma$$

and $\angle CTA = \angle CBA - \angle TAB = \beta - \gamma$, implies that $ATSC$ is cyclic.

From $\triangle BQT \sim \triangle BCW$, and power of point B we have $\overline{QB} \cdot \overline{BW} = \overline{TB} \cdot \overline{BC} = \overline{AB} \cdot \overline{BS}$, hence Q also lies on the circumcircle of $\triangle ASW$. We conclude that points A, Q, S, W, and P lie on a circle. □

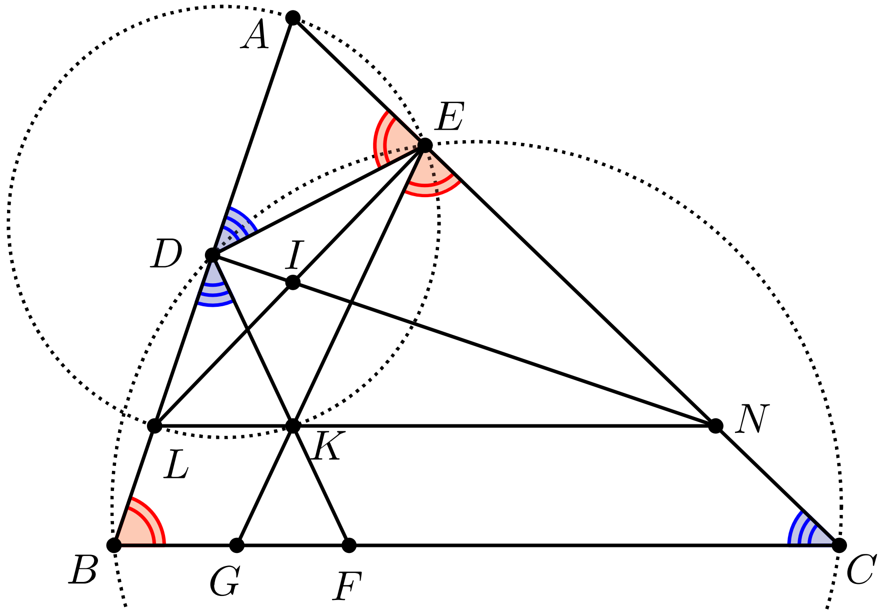
Remarks

- In Solution 4, we can prove that $SAPW$ is cyclic alternatively without power of a point: $\angle SPW \equiv \angle LPW = \angle LCW \equiv \angle BCW = \angle BAW \equiv \angle SAW$.
- In Solution 4, we can prove that $SQAW$ is cyclic alternatively without power of a point: From $(ATSC)$ we get that $\angle STC = \angle SAC = \alpha$. Since $\angle CTQ \equiv \angle BTQ = 180^\circ - \alpha$, we get that points Q, T, S are collinear. Finally, $\angle SQW \equiv \angle TQB = \frac{\alpha}{2} = \angle BAW \equiv \angle SAW$.

G5 Let ABC be a triangle and let D, E be interior points of the sides AB and AC respectively such that B, D, E, C are concyclic. Let F, G be interior points of the side BC such that $\angle BDF = \angle ADE$ and $\angle GEC = \angle AED$. Prove that DF, EG and the perpendicular from A to BC are concurrent.

(Cyprus)

Solution 1. Let ℓ_1 be the perpendicular of AB at the point D and ℓ_2 the perpendicular of AC at the point E. Let I be the point of intersection of ℓ_1 with ℓ_2 and let K be the point of intersection of DF with EG. By the given conditions we have that I is the incenter of triangle DKE. It is enough to show that AI is perpendicular to BC.



Suppose that ℓ_1 intersects AC at N, and ℓ_2 intersects AB at L. Then I is the orthocenter of triangle ALN. Let K' be the foot of the perpendicular from A on LN. Then $DK'E$ is the orthic triangle of triangle ALN with I its incenter. So $K = K'$ and A, I, K are collinear.
We have $\angle AKL = 90^\circ = \angle AEL$, therefore A, L, K, E are concyclic and so

$$\angle AKL = \angle KEC = \angle AED = \angle ABC$$

with the last equality following since B, D, E, C are concyclic. Thus KL is parallel to BC and so AI is perpendicular to BC as required. □

Solution 2. Let K be the point of intersection of DF and EG and let I be the incenter of triangle DEK. Since $\angle BDF = \angle ADE$ and $\angle KDI = \angle EDI$, then ID is perpendicular to AB. Similarly, IE is perpendicular to AC and therefore A, D, I, E are concyclic. Thus

$$\angle DAI = \angle DEI = 90^\circ = \angle DEA = 90^\circ - \angle ABC\,.$$

It follows that AI is perpendicular to BC.
Since I is the incenter of triangle DEK, then IK is bisector of $\angle DKE$ and therefore bisector of $\angle GKF$.
Since $\angle KGF = \angle KFG = \angle BAC$, then $GK = KF$ and so IK is not only a bisector but also an altitude of triangle GKF.
We now have that AI and IK are both perpendicular to BC, therefore AK is perpendicular to BC as required. □

Solution 3. Let M be the foot of the altitude from A to BC. Then

$$BM - MC = \frac{BM^2 - MC^2}{a} = \frac{(c^2 - AM^2) - (b^2 - AM^2)}{a} = \frac{c^2 - b^2}{a}\,.$$

With K defined as before, it is enough to show that KM is perpendicular to BC. Since KGF is isosceles, it is enough to show that M is the midpoint of GF. This is equivalent to showing that $BF - GC = BM - MC = (c^2 - b^2)/a$.

The triangles ABC, AED, FBD, GEC are all similar. Let $BD = x$ and $CE = y$. Then $AD = c - x$ and $AE = b - y$. Thus

$$\frac{c-x}{b-y} = \frac{AD}{AE} = \frac{b}{c} \Longrightarrow c^2 - cx = b^2 - by\,.$$

It follows that

$$BF - GC = \frac{(AB)(BD)}{(BC)} - \frac{(AC)(EC)}{(BC)} = \frac{cx - by}{a} = \frac{c^2 - b^2}{a}$$

as required. □

Remarks

After we prove that $IK \perp BC$, we can alternatively finish by proving that A, I, and K are collinear. We present two ways to do this:

Let $\angle ABC = \beta$. Since I is incenter of $\triangle DEK$, we get

$$\angle DIK = 90^\circ + \tfrac{1}{2}\angle DEK = 90^\circ + \tfrac{1}{2}(180^\circ - 2\beta) = 180^\circ - \beta.$$

Since $ADIE$ is cyclic, $\angle AID = \angle AED = \beta$, hence $\angle AID + \angle DIK = \beta + 180^\circ - \beta = 180^\circ$.

Since DI is an internal angle bisectors in $\triangle KDE$ and $DA \perp DI$, we get that DA is an external angle bisector in $\triangle KDE$. Similarly, EA is another external bisector in the same triangle, so their intersection, A is the K-excenter of $\triangle KDE$, hence A lies on the internal angle bisector KI.

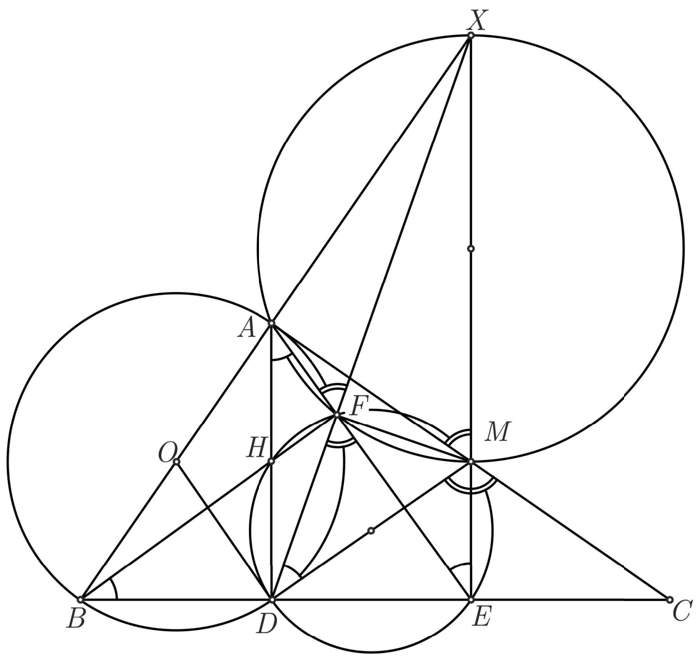
G6 Let $\triangle ABC$ be right-angled at A and let D be the foot of altitude from A to BC and let E be the midpoint of DC. The circumcircle of $\triangle ABD$ intersects AE again at point F. Let X be the intersection of AB and DF. Prove that $XD = XC$.

(Albania)

Solution 1. Since E is the midpoint of DC, it is sufficient to show that $XE \perp DC$ which is equivalent to proving that $XE \parallel AD$.
Let H be the intersection of BF and AD. Since $\angle ADB = 90^\circ \Rightarrow \angle AFB = 90^\circ \Rightarrow H$ is the orthocenter of $\triangle ABE \Rightarrow EH \perp AB \Rightarrow EH \parallel AC$. Since E is the midpoint of DC we have that H is also the midpoint of AD.
Apply (unoriented) Menelaus theorem in $\triangle ABH$ for points D, F, X and in $\triangle BHD$ for points A, F, E. We get that

$$\frac{AX}{XB}\frac{BF}{FH}\frac{HD}{DA} = 1 = \frac{DE}{EB}\frac{BF}{FH}\frac{HA}{AD}.$$

Since we have that $AH = HD$ the ratios cancel and we get that $\frac{AX}{XB} = \frac{DE}{EB}$ which implies that $AD \parallel XE$ by Thales. □



Solution 2. Since E is the midpoint of CD, we need to prove that $XE \perp CD \equiv BC$. But $AD \perp BC$, so we need $AD \parallel XE$. Let M be the midpoint of AC. As midsegment in $\triangle ADC$, $ME \parallel AD$, so we need to prove that the points X, M, E are collinear. By Thales' Theorem in the right triangle ADC, we get $MA = MD = MC$.

Claim 1. MD is tangent to $(ABDF)$ ...(1)

Proof 1. Let O be the midpoint of AB and therefore center of $(ABDF)$. Then, $OA = OD$, so by criterion SSS, we get $\triangle OAM \cong \triangle ODM$, thus $\angle ODM = \angle OAM = 90^\circ$. ◇

Proof 2. Since $MA = MD$, we get $\angle MDA = \angle MAD = 90^\circ - \angle DAB = \angle DBA$, so the angle between the line MD and the chord DA is equal to the inscribed angle over DA. ◇

$\angle MEF \overset{ME \| AD}{=} \angle FAD \overset{(1)}{=} \angle MDF$, so $DEMF$ is cyclic. ...(2)

$\angle MFX \overset{(2)}{=} \angle MED = 90^\circ = \angle MAX$, so $MFAX$ is cyclic. ...(3)

Claim 2. X, M, E are collinear.

Proof 1. $\angle XMA \overset{(3)}{=} \angle XFA = \angle DFE \overset{(2)}{=} \angle DME = \angle CME$ and since A, M, C are collinear, so are X, M, E. ◇

Proof 2. $\angle XMF \overset{(3)}{=} \angle FAB \overset{(ABDF)}{=} \angle FDE \overset{(2)}{=} 180^\circ - \angle FME$, so $\angle XME = 180^\circ$. ◇

Proof 3. $\angle AMX \overset{(3)}{=} \angle AFX = \angle DFE = \angle BFE - \angle BFD = 90^\circ - \angle BAD = \angle ABD$ and thus $\angle MXA = 90^\circ - \angle AMX = 90^\circ - \angle ABD$, so $XM \perp BD \equiv BC$. Combining with $ME \perp DC \equiv BC$ we get $XM \parallel ME$ ◇

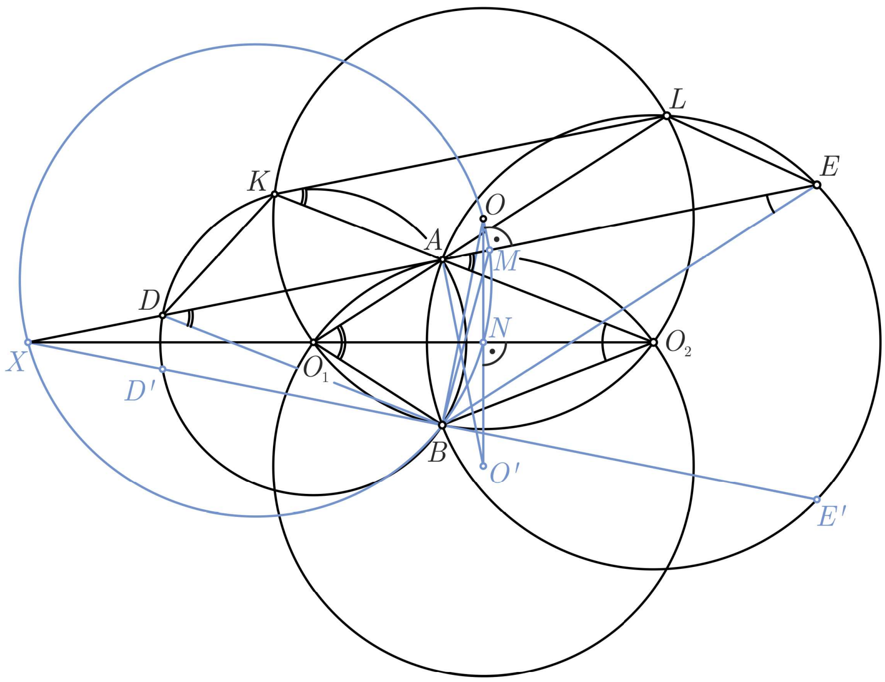
Proof 4. $\angle MAB + \angle MEB = 180^\circ$, so $MABE$ is cyclic. The three pairwise radical axes of the circles $(MABE)$, $(ABDF)$ and $(DFME)$ are AB, DF and ME. Then $X = AB \cap DF$ is their radical center, so $X \in ME$. ◇

Now $XE \parallel AD$ and E being the midpoint of CD implies that $\triangle DCX$ is isosceles as desired. □

G7 Two circles ω_1 and ω_2 with centers O_1 and O_2, respectively, intersect at points A and B. The tangent at point A to the circumcircle of $\triangle AO_1O_2$ intersects ω_1 and ω_2 at points $D \neq A$ and $E \neq A$, respectively. Prove that the center of the circumcircle of $\triangle BO_1O_2$ is equidistant from points D and E.

(Vadym Solomka - Ukraine)

Solution 1. Let the circumcircle of $\triangle BO_1O_2$ intersect ω_1 and ω_2 at the points $K \neq B$ and $L \neq B$, respectively. Since $O_1B = O_1K$, we have that O_2O_1 is the bisector of $\angle BO_2K$, and since O_2O_1 is the bisector of $\angle BO_2A$, it follows that the points O_2, A, and K lie on the same line. Similarly, the points O_1, A, and L lie on the same line. Note that $\angle O_2KL = \angle LO_1O_2 = \angle O_2AE = \angle KAD$. Hence, $KL \parallel DE$. On the other hand, $\angle AKL = \angle AO_1O_2 = \angle ADB = \angle BKA$. We have that KA and KL are angle bisectors of $\angle BKL$ and $\angle BLK$, respectively, so BA is the angle bisector of $\triangle KBL$. From the last, $\angle KDA = \angle KBA = \angle ABL = \angle AEL$. Hence, $DKLE$ is the isosceles trapezoid and the perpendicular bisectors of the segments KL and DE are the same. Obviously, the circumcenter of $\triangle BO_1O_2$ is equidistant from points K and L, and therefore it is also equidistant from points D and E, as we wanted. □



Solution 2. Let O be the circumcenter of $\triangle BO_1O_2$ and let M be the midpoint of DE. Then, proving $OD = OE$ is equivalent to proving $OM \perp DE$.

Since O_1O_2 is the side bisector of AB, we get that A and B are images of each other under reflection w.r.t. line O_1O_2. Let D', E', O' be images of D, E, O, respectively, under the reflection w.r.t. O_1O_2. Then, $DE \cap D'E' = X$ lies on O_1O_2. Also, O' is the circumcenter of $\triangle AO_1O_2$ and $O'A \perp DE$, thus $OB \perp D'E'$. Let $OO' \cap O_1O_2 = N$, then N is midpoint of O_1O_2 and $ON \perp O_1O_2$. So, $\angle ONX \equiv \angle ONO_1 = 90^\circ = \angle OBD' \equiv \angle OBX$ and therefore $ONBX$ is a cyclic quadrilateral with diameter OX. We want to prove that M lies on its circle.

Next, $\angle BO_1O_2 = \frac{1}{2}\angle BO_1A = \angle BDA \equiv \angle BDE$ and similarly $\angle BO_2O_1 = \angle BED$, so by criterion AA, $\triangle BO_1O_2 \sim \triangle BDE$. Since N, M are midpoints of O_1O_2, DE, respectively, by criterion SAS we get $\triangle BO_1N \sim \triangle BDM$. Therefore, $\angle BNX \equiv \angle BNO_1 = \angle BMD \equiv \angle BMX$, so M lies on the circle $(BNX) \equiv (BNOX)$ with diameter OX and therefore $\angle OMD \equiv \angle OMX = 90^\circ$. □

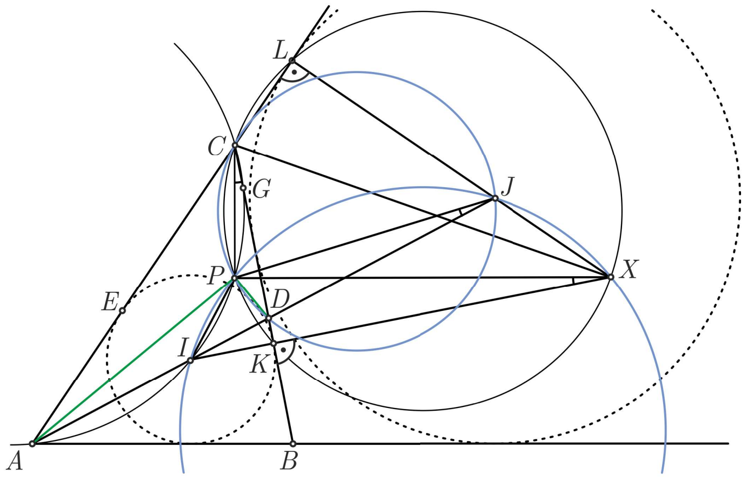
G8 Let ABC be a triangle with $AC > AB$ have incenter I and A-excenter J. Assume that its incircle touches BC at point K, the A-excircle touches AC at point L, and AI intersects BC at point D. If the intersection of the circumcircles of $\triangle AIC$ and $\triangle KCL$ is P, prove that $\angle APD = 90^\circ$.

(Turkmenistan)

Solution 1. Let $IK \cap JL = X$. Since $AC > AB$, J is between X and L. Since $\angle XKC = \angle CLX = 90^\circ$, X lies on the circumcircle of $CPKL$. Hence $\angle XPC = 90^\circ$ and we calculate

$$\angle IPX = 360^\circ - \angle XPC - \angle CPI = 360^\circ - 90^\circ - (180^\circ - \angle IAC) = 90^\circ + \tfrac{1}{2}\angle BAC = \angle AJX,$$

implying that $IPJX$ is cyclic. We continue angle chasing by observing that $\angle PJI = \angle PXI = \angle PXK = \angle PCK = \angle PCD$, so $DPCJ$ is also cyclic.
Moreover we have $\angle IPJ = 180^\circ - \angle JXI = 180^\circ - \angle LXK = 180^\circ - \angle ACB = 2\angle BCJ = 2\angle DCJ = 2\angle DPJ$, so DP bisects $\angle IPJ$.
To finish note that by internal external bisector theorem on $\triangle ABD$ we have $\frac{AI}{ID} = \frac{AJ}{JD}$, and since DP bisects $\angle IPJ$, we can conclude that PA is the external bisector, hence $\angle APD = 90^\circ$. $\square$



Solution 2. Let G be the second intersection of the circumcircle of $\triangle AIC$ and BC and E be the point of tangency of the incircle and AC. We have $\angle JAP = \angle IAP = \angle IGP$. Furthermore, $IE = IK$ and $\angle IGK = 180^\circ - \angle CGI = \angle IAE$ so $\triangle IAE \cong \triangle IGK$ which gives $IA = IG$ and $IE = GK$.
We will now prove that $\triangle JAP \sim \triangle IGP$. From the above angle equality, we only need to prove that $AP : AJ = GP : GI$ or equivalently $AP : GP = AJ : GI$. Now since $AI = GI$, $AG = GK$, and $IE \parallel JL$ we have $AJ : GI = AJ : AI = AL : AE = AL : GK$.
Since $\angle GKP = \angle CKP = \angle CLP = \angle ALP$ and $\angle PGK = 180^\circ - \angle CGP = \angle PAC = \angle PAL$, $\triangle KGP \sim \triangle LAP$, hence $AL : GK = AP : GP$. Combining this with the previous equality we get $AP : GP = AJ : GI$ i.e. $\triangle JAP \sim \triangle IGP$.
Finally, this implies $\angle PJD = \angle PJA = \angle PIG = \angle PCG = \angle PCD$, so $DJCP$ is cyclic and

$$\angle DPA = 360^\circ - APC - CPD = 180^\circ - AIC + DJC = 180^\circ - (90^\circ + \frac{1}{2}ABC) + \frac{1}{2}ABC = 90^\circ,$$

as desired. $\square$

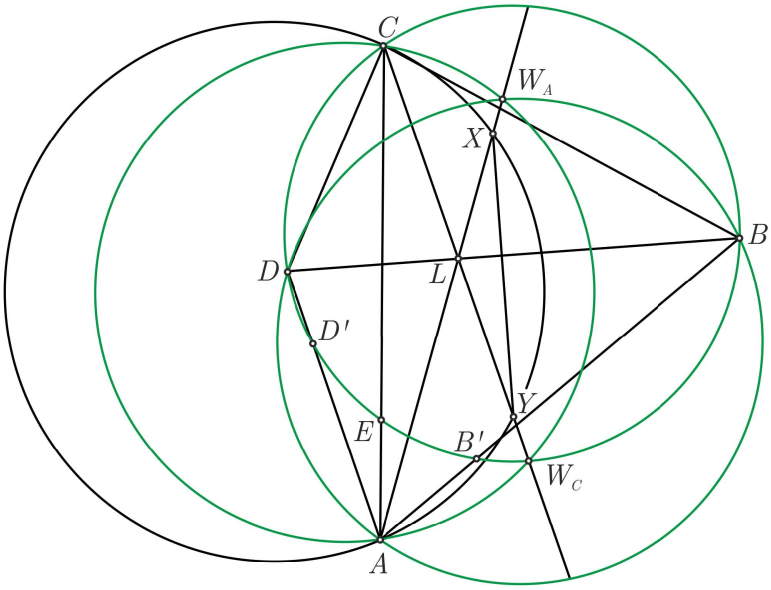
Remarks

In Solution 1, after proving that $IPJX$ and $DPCJ$ ae cyclic, we can complete the proof by angle chasing: $\angle CPA = \angle CIA = 90^\circ + \frac{\beta}{2}$, and $\angle DPC = 180^\circ - \angle CJD = 180^\circ - \angle CJA = 180^\circ - \frac{\beta}{2}$ implies $\angle APD = 360^\circ - (\angle CPA + \angle DPC) = 90^\circ$.

G9 Points X and Y are chosen inside quadrilateral $ABCD$ on the internal bisectors of angles A and C respectively such that points A, X, Y, C are lying on one circle and $XY \perp BD$. Prove that either angles of the quadrilateral $ABCD$ can be divided into two groups with equal sums, or sides of the quadrilateral can be divided into two groups with equal products.

(Mykhailo Shtandenko - Ukraine)

Solution 1. Let ω_A be the circumcircle of the $\triangle ABD$ and W_A be its midpoint of the arc BD, not containing point A. Analogously define ω_C and W_C with circumcircle BCD. Then $W_AW_C \perp BD$ as a perpendicular bisector, thus $W_AW_C \parallel XY$ and $\angle(CW_C, W_cW_A) = \angle(CY, YX) = \angle(CA, AX) = \angle(CA, AW_A)$, hence points A, C, W_A, W_C are concyclic. Denote this circle as ω. Now we have two cases: either circles $\omega, \omega_A, \omega_C$ are pairwise different, or they are all the same circle. In the latter case quadrilateral $ABCD$ is cyclic and opposite angles sum up to $180°$, hence the first condition is satisfied. For the first case consider the radical axes of three aforementioned circles. They will be AW_A, CW_C, BD, thus they are concurrent. Denote their point of intersection as L. Then, by angle bisector theorem, $\frac{AD}{AB} = \frac{DL}{BL} = \frac{DC}{BC}$, hence $AB \cdot CD = BC \cdot AD$ and the second condition from problem statement is satisfied, as desired. $\square$



Solution 2. By angle chasing, we have $\frac{1}{2}\angle BCD - \angle BCA = \angle BCY - \angle BCA = \angle ACY = \angle AXY = 90° - \angle(AC, BD) = \angle XAB + \angle ABD - 90° = \frac{1}{2}\angle DAB + \angle ABD - 90°$. Similarly, $\frac{1}{2}\angle DAB - \angle CAB = \frac{1}{2}\angle BCD + \angle DBC - 90°$. Adding these two and canceling, we get

$$180° - \angle CBD = \angle BCD + \angle DBC - \angle BCA = \angle DAB + \angle ABD - \angle CAB = 180° - \angle BDA - \angle CAB,$$

which gives $\angle CDB + \angle BCA = \angle BDA + \angle CAB$. Now, let the circumcircle of $\triangle BCD$ intersect AC again at E. If $E \equiv A$, then $ABCD$ is cyclic and the first condition is met.
Otherwise, $\angle EDA = \angle BDA - \angle BDE = \angle BDA - \angle BCE = \angle BDA - \angle BCA = \angle CDB - \angle CAB = \angle CEB - \angle EAB = \angle ABE \neq 0$. Let AB and AD intersect the circumcircle of $\triangle BCD$ again at B' and D' respectively. We have $\angle B'CE = \angle ABE = \angle EDA = \angle ECD'$ so $EB' = ED'$. Finally, $\triangle AEB' \sim \triangle ABC$ and $\triangle AD'E \sim \triangle ACD$, implying

$$\frac{AB}{AC} = \frac{AE}{EB'} = \frac{AE}{ED'} = \frac{AD}{DC},$$

hence $AB \cdot CD = AD \cdot BC$, as desired. $\square$

Number Theory

N1 Determine all numbers of the form

$$20252025\ldots 2025$$

(where the block 2025 is repeated one or more times) that are perfect squares of positive integers.

(Serbia)

Solution. Let us prove that the only such number is $2025 = 45^2$.
First, observe that the considered number with n blocks of 2025 is equal to

$$2025(1 + 10^4 + 10^8 + \ldots + 10^{4(n-1)}),$$

for some $n \in \mathbb{N}$. Since 2025 is a perfect square, it must hold that $1 + 10^4 + 10^8 + \ldots + 10^{4(n-1)} = x^2$, for some $x \in \mathbb{N}$. Multiplying both sides by $10^4 - 1$ gives $10^{4n} - 1 = 9999x^2 = 11 \cdot 101 \cdot (3x)^2$. Applying the difference of squares twice gives

$$(10^n - 1)(10^n + 1)(10^{2n} + 1) = 11 \cdot 101 \cdot (3x)^2. \qquad (\star)$$

Lemma 1. The equation $10^{2n} + 1 = m^2$ has no solutions in the set of positive integers.

Proof. The number $10^{2n} + 1$ cannot be a perfect square because $(10^n)^2 < 10^{2n} + 1 < (10^n + 1)^2$ for all $n \in \mathbb{N}$. ◇

Lemma 2. The equation $10^n + 1 = m^2$ has no solutions in the set of positive integers.

Proof. The equation becomes $2^n 5^n = (m-1)(m+1)$. By the Euclidean algorithm, the GCD of the two factors on the right-hand side is 2; (alternative argument: those factors are two consecutive even numbers, so one is divisible by 2 but not by 4, while the other is). Clearly, both cannot be divisible by 5. So we are to distribute the numbers 2, 2^{n-1}, and 5^n in two pots: $m-1$ and $m+1$. Since $m - 1 < m + 1$, the number 5^n must be a factor of $m + 1$. The options are:

1°: $m - 1 = 1$, so $m = 2$, but this is not a solution.

2°: $m - 1 = 2$, so $m = 3$, but this is not a solution.

3°: $m - 1 = 2^{n-1}$ and $m + 1 = 2 \cdot 5^n$. Subtracting gives $2 = 2 \cdot 5^n - 2^{n-1}$. Note that $n = 1$ is not a solution, and for $n \geqslant 2$ we have $1 = 5^n - 2^{n-2}$. Then the right-hand side is divisible by 3 (since it's congruent modulo 3 to $(-1)^n - (-1)^{n-2}$, and n and $n-2$ have the same parity, so this is 0), while the left-hand side is not, hence there is no solution (here we can also invoke Mihailescu's theorem). ◇

Lemma 3. The only solution to the equation $10^n - 1 = m^2$ in the set of positive integers is $(n, m) = (1, 3)$.

Proof. Consider the equation modulo 4. For $n \geqslant 2$, $10^n \equiv 0 \pmod 4$, so the right-hand side gives remainder 3 modulo 4. On the other hand, $n = 1$ gives $m = 3$ as a solution. ◇

Using the Euclidean algorithm, we see that $10^n - 1$ and $10^n + 1$ are coprime (their difference is 2 and they are odd, so GCD is 1). Additionaly, $10^{2n} + 1$ is coprime with both $10^n - 1$ and $10^n + 1$ since it is coprime with their product $10^{2n} - 1$, for the same reason as before. Hence, all three factors on the left-hand side of $(\star)$ are pairwise coprime. Since the numbers 11 and 101 are prime, at least one of the factors on the left-hand side of $(\star)$ must be a perfect square. By Lemmas 1, 2, and 3, this is only possible when $10^n - 1 = 9$, from which we get that the only solution is $(n, x) = (1, 1)$.

Therefore, the only solution is the number 2025. □

Remarks

Lemma 2 (short proof): The left side is congruent to 2 modulo 3, which isn't a quadratic residue.

N2 Find all positive integers n, $n \geq 2$, such that there exist infinitely many perfect squares which can be expressed as a sum of n consecutive positive integers. *(Croatia)*

Solution 1.

Claim 1. For $n = 2k+1$, where k is positive integer, infinitely many perfect squares of the form $(nl)^2$, where l is positive integer, can be obtained as a sum of the following n numbers

$$\frac{(nl)^2}{n} + i, \qquad i = -\frac{n-1}{2}, ..., -1, 0, 1, ..., \frac{n-1}{2}.$$

Proof. It is obvious that these numbers are n consecutive positive integers which sum up to $(nl)^2$. ◇

Claim 2. For $n = 2^{2j}k + 2^{2j-1}$, where j is positive integer and k nonnegative integer, infinitely many perfect squares of the form $\left(nl + \frac{n}{2^j}\right)^2$, where l is positive integer, can be obtained as a sum of the following n numbers

$$\frac{\left(nl + \frac{n}{2^j}\right)^2 + i}{n}, \qquad i = -\frac{(n-1)n}{2}, ..., -\frac{3n}{2}, -\frac{n}{2}, \frac{n}{2}, \frac{3n}{2}, ..., \frac{(n-1)n}{2}.$$

Proof. Again, it is obvious that the neighbouring numbers differ by 1 and all the numbers are positive and sum up to $\left(nl + \frac{n}{2^j}\right)^2$.
To prove that these numbers are n consecutive positive integers it is enough to prove that one of them is integer. For instance,

$$\frac{\left(nl + \frac{n}{2^j}\right)^2 + \frac{n}{2}}{n} = \frac{n^2l^2 + \frac{n^2 l}{2^{j-1}} + \frac{n^2}{2^{2j}} + \frac{n}{2}}{n} = nl^2 + \frac{nl}{2^{j-1}} + \frac{n}{2^{2j}} + \frac{1}{2}$$

is integer since, for nonnegative integer k,

$$\frac{nl}{2^{j-1}} + \frac{n}{2^{2j}} + \frac{1}{2} = (2^{j+1}k + 2^j)l + k + \frac{1}{2} + \frac{1}{2} = (2^{j+1}k + 2^j)l + k + 1$$

is integer. ◇

Claim 3. For $n = 2^{2j+1}k + 2^{2j}$, where j is positive integer and k nonnegative integer, there do not exist n consecutive positive integers which sum up to a perfect square.

Proof. Since $0+1+2+3+\ldots+2^{2j}-1 = (2^{2j}-1)\cdot 2^{2j-1} \equiv 2^{2j-1} \pmod{2^{2j}}$, the sum of 2^{2j} consecutive positive integers has residue 2^{2j-1} modulo 2^{2j} and, more generally, for positive integer j and nonnegative integer k, the sum of $n = 2^{2j+1}k + 2^{2j}$ consecutive positive integers has residue 2^{2j-1} modulo 2^{2j}. On the other hand, 2^{2j-1} is not a quadratic residue modulo 2^{2j}, which settles the claim. ◇

Finally, we can conclude that all positive integers n satisfying the given conditions are odd numbers $n \geq 3$ and the numbers of the form $n = 2^{2j}k + 2^{2j-1}$, where j is positive integer and k nonnegative integer. □

Solution 2. For the sum we have

$$a^2 = (m+1)+(m+2)+\cdots+(m+n) = \frac{(m+1)+(m+n)}{2} + \frac{(m+2)+(m+n-1)}{2} + \cdots + \frac{(m+n)+(m+1)}{2} = \frac{n(2m+n+1)}{2}.$$

Equivalently, $2a^2 = n(2m+n+1)$, and thus $2m = \frac{2}{n}a^2 - n - 1$.
If n is odd, there are infinitely many a such that $n \mid a^2$, which gives us infinitely many positive integers m and hence solutions to the equation.
If n is even and $v_2(n)$ is odd, we can take $a = 2^{\frac{v_2(n)-1}{2}}s$ for infinitely many s, which again yields infinitely many solutions.
Finally, in the case of n being even and $v_2(n)$ even, the right-hand side of $2m = \frac{2}{n}a^2 - n - 1$ is odd or not an integer, which means that we have no solutions for m in this case. □

Solution 3. The problem statement is equivalent to finding infinitely many $(x, m) \in \mathbb{N}$ such that:

$$x^2 = \frac{n(n-1)}{2} + mn \tag{3}$$

which you get by a simple sum of consecutive integers starting from the positive integer m. Next, we will discuss two different cases regarding the parity of n.

Case 1: n is odd

For any $k \in \mathbb{N}$ set:

$$m = (k^2 - 1)n + \tfrac{n+1}{2}\,.$$

As $2|n+1$, m is an integer. We can directly check that

$$\frac{n(n-1)}{2} + mn = n\Big(\frac{n-1}{2} + (k^2-1)n + \frac{n+1}{2}\Big) = (nk)^2\,.$$

We have found a solution for any $k \in \mathbb{N}$, hence producing infinitely many different solutions. (Notice that $(np)^2 \neq (nq)^2$ for $p \neq q$, $p, q \in \mathbb{N}$ and it is easy to check that $m \in \mathbb{N}$.)

Case 2: n is even

We can rewrite (3) as:

$$x^2 = \frac{n}{2}(n - 1 + 2m) \tag{4}$$

As n is even, notice that $n - 1 + 2m$ is odd and $\frac{n}{2} \in \mathbb{N}$. Looking at r - the highest degree of 2 dividing x, from (4) we get $2r = v_2(x^2) = v_2(\frac{n}{2})$ for some $r \in \mathbb{N}_0$. Hence, for n to be a possible solution we must have $v_2(n) = 2r + 1$, so:

$$n = 2^{2r+1}t\,,$$

where t is an odd positive integer.

Take $m = \frac{t(k^2 - 2^{2r+1}) + 1}{2}$ for any odd integer k such that $k^2 > 2^{2r+1}$. Because k is odd, $k^2 - 2^{2r+1}$ is odd and t is also odd, thus $t(k^2 - 2^{2r+1}) + 1$ is even, so $m \in \mathbb{N}$. It is easy to check that

$$\frac{n(n-1)}{2} + mn = 2^{2r+1}t\Big(\frac{2^{2r+1}t - 1}{2} + \frac{t(k^2 - 2^{2r+1}) + 1}{2}\Big) = (2^r tk)^2\,,$$

and that this indeed produces a different solution which satisfies the problem conditions, for any odd integer k such that $k^2 > 2^{2r+1}$.
Hence, the solutions are all odd n, and even n whose exponent of two, denoted $v_2(n)$, is odd. $\square$

Remarks

The form $n = 2^{2j}k + 2^{2j-1}$, where j is positive integer and k nonnegative integer as in the first solution isn't very natural. Better options would be $n = t \cdot 2^{2j-1}$ for positive odd integer t or simply n such that the power of two in n, denoted $v_2(n)$, is odd.

N3 A set S of positive integers is called *nice* if $\frac{a+b}{a-b}$ is an integer for any different $a, b \in S$. Show that for any positive integer n, there exist infinitely many nice sets with n elements.

(Turkey)

Solution. For $n = 1$, any set with only one integer is nice. $S_2 = \{a, a+1\}$ for any positive integer a is a nice set with 2 elements. The lemma below immediately finishes the problem with induction on n.

Lemma. For any $n \geq 2$, if S_n is a nice set with n elements, then for any $k \in \mathbb{Z}^+$,

$$S_{n+1} = \{kM\} \cup \{kM + a | a \in S_n\}$$

is a nice set with $n+1$ elements where

$$M = \operatorname{lcm}(S_n \cup \{a - b | a, b \in S_n, a \neq b\}).$$

Proof. Take two different elements u, v from S_{n+1}. They are either in the form $(u, v) = (kM+a, kM+b)$, where a, b are two different elements from S_n, or in the form $(u, v) = (kM, kM+a)$, where a is an element of S_n. In the first case,

$$\frac{u+v}{u-v} = \frac{2KM + a + b}{a - b} = \frac{2KM}{a-b} + \frac{a+b}{a-b} \in \mathbb{Z},$$

since $a - b | M$. In the second case,

$$\frac{u+v}{u-v} = \frac{2KM + a}{a} = \frac{2KM}{a} + 1 \in \mathbb{Z},$$

since $a | M$. ◇

By induction we have a good set for every $n \in \mathbb{N}$ and since the lemma is true for any $k \in \mathbb{Z}^+$ we have infinitely many such sets for each n. □

Remarks

The infinitude of sets satisfying the problem conditions is trivial, so the only step that has some idea behind it is the translation of elements. Coming up with the element we're supposed to add to the set upon induction (e.g. the translation distance) isn't too difficult either (a geometric intuition could help here, as the solution is basically "homothety + translation"), and the constraints we're supposed to impose on it come naturally.

N4 Let m be a positive integer. Suppose that there are at least $k \geq 3$ distinct positive integers n with the following property: the sum of n and its largest divisor, different from n, equals m. Prove that $m > 3^{2^{k-2}}$.

(Bulgaria)

Solution. Write $n = sp$, where p is the smallest prime divisor of n and correspondingly s is its largest divisor, different from n. The equation we work with is $sp + s = m$, i.e. $s(p+1) = m$. Suppose (s_i, p_i) are distinct pairs such that $s_i(p_i + 1) = m$ for $i = 1, 2, \ldots, k$ and $2 \leq p_1 < p_2 < \ldots < p_k$ are distinct primes. Note that s_i has no prime divisor smaller than p_i.

Now consider $s_i(p_i + 1) = s_{i+1}(p_{i+1} + 1)$ for $i = 2, 3, \ldots, k-1$. Since s_{i+1} has no prime divisor smaller than p_{i+1} and $p_{i+1} > p_i + 1$ as p_i and p_{i+1} are both odd (by $p_i > 2$ for $i \geq 2$), it follows that $p_i + 1$ and s_{i+1} are relatively prime. In particular, $p_i + 1$ must divide $p_{i+1} + 1$. Hence

$$\frac{p_{i+1} + 1}{p_i + 1} = \frac{s_i}{s_{i+1}}$$

are the same integer and since s_i has no prime divisor less than p_i, it follows that this common integer is at least p_i. In particular, we have proven that $p_{i+1} > p_i^2$ for $i = 2, 3, \ldots, k$. Applying this for every i, we conclude that $p_k > p_2^{2^{k-2}} \geq 3^{2^{k-2}}$, as desired.

Remarks

Feel free to put a concrete value of k if you want, e.g. $k = 1000$.

N5 Determine all positive integers n such that

$$\frac{1!\cdot 2!\cdot 3!\cdots 2024!}{n!}$$

is a perfect square. *(Serbia)*

Solution. Let $1!\cdot 2!\cdot 3!\cdots 2024! = n!\cdot x^2$, for some positive integer x. Note that

$$\begin{aligned} A &= 1!\cdot 2!\cdot 3!\cdots 2024! = (1!\cdot 2!)\cdot(3!\cdot 4!)\cdots(2023!\cdot 2024!) \\ &= (1!\cdot 1!\cdot 2)\cdot(3!\cdot 3!\cdot 4)\cdots(2023!\cdot 2023!\cdot 2024) = (1!)^2\cdot(3!)^2\cdot(5!)^2\cdots(2023!)^2\cdot 2\cdot 4\cdot 6\cdots 2024 \\ &= (1!\cdot 3!\cdot 5!\cdots 2023!)^2\cdot 2^{1012}\cdot 1\cdot 2\cdot 3\cdots 1012 = (1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})^2\cdot 1012!, \end{aligned}$$

from which we find one solution $(n,x) = (1012, 1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})$. Let us prove that there are no other solutions.

Suppose the contrary, that there exists a solution $n \neq 1012$ to this equation. Then the number $\frac{A}{n!}$ must be a perfect square of a positive integer.

If $n \leqslant 1008$, then

$$\frac{A}{n!} = (1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})^2\cdot 1012\cdot 1011\cdot 1010\cdot 1009\cdots(n+1).$$

It is easy to check that 1009 is a prime number, and its exponent in the right-hand side is odd, hence this product is not a perfect square. For $n = 1009$, we have

$$\frac{A}{n!} = (1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})^2\cdot 1012\cdot 1011\cdot 1010,$$

and for $n = 1010$ we get

$$\frac{A}{n!} = (1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})^2\cdot 1012\cdot 1011\,,$$

so these products are not perfect squares either, since the exponent of the number 3 is odd. Also, $n = 1011$ is not a solution because 1012 is not a perfect square.

Finally, if $n \geqslant 1013$, then

$$\frac{A}{n!} = \frac{(1!\cdot 3!\cdot 5!\cdots 2023!\cdot 2^{506})^2}{1013\cdot 1014\cdots n}.$$

The number 1013 is prime, and its exponent in $\frac{A}{n!}$ is odd for $n < 2026$, so it cannot be a perfect square. On the other hand, $\frac{A}{n!}$ is not even an integer for $n \geqslant 2027$, since 2027 is a prime number that does not divide A. The number $n = 2026$ is not a solution either because 1019 is a prime number whose exponent in $\frac{A}{n!}$ is odd. $\square$

Remarks

The main idea of this problem is to look at the p-adic valuation of the given expression for appropriate values of p. Seeing how all of the factorials contribute to a certain valuation should readily suggest the consideration of primes $p = 1009$ and $p = 1013$. There are many ways of excluding 1009, 1010, and 1011 (one alternative is considering $p = 337$ and $p = 43$ for example). It is easy to get the idea of bounding n by large primes, but the caveat is that you have to find those large primes by hand.

Understanding there's a solution for $n = 1012$ might not be too obvious. If somehow, prior to doing the aforementioned checks, you find out that $n = 1012$ is a solution to the equation, as the solution starts, the problem becomes more solvable. Possible alternative statement would be:

"Prove that there is exactly one $n \in \mathbb{N}$ such that ... is a perfect square".

N6 Find all prime numbers $p < 2025$ for which

$$p^2+2p-7=n^3\,,$$

for some integer n.

(Albania)

Solution. Since $p \geq 2$, the left-hand side is positive, hence n must be positive. Let us rewrite the given equation as

$$(p+1)^2 = n^3 + 8 = (n+2)(n^2 - 2n + 4)\,.$$

Since $\gcd(n+2, n^2-2n+4) = \gcd(n+2, (n+2)(n-4)+12) = \gcd(n+2, 12)$, we can conclude that n^2-2n+4, $2(n^2-2n+4)$, $3(n^2-2n+4)$, or $6(n^2-2n+4)$ is a perfect square.

Case 1. If $n^2 - 2n + 4 = m^2$, then $m^2 = (n-1)^2 + 3$, i.e. $(m+n-1)(m-n+1) = 3$ with only solution $m+n-1 = 3$, $m-n+1 = 3$, i.e. $m = 2$, and $n = 2$.

Case 2. If $2(n^2-2n+4) = m^2$, then $m = 2u$ is even, hence $n = 2v$ is even. We have $2(v^2-v+1) = u^2$. Again, u is even; however, $v^2 - v + 1$ is odd, giving us a contradiction.

Case 3. If $3(n^2-2n+4) = m^2$, then $3 \mid n+2$, implying that $n = 3y+1$ for some integer y. We have

$$m^2 = 3((9y^2 + 6y + 1) - 2(3y+1) + 4) = 9(3y^2 + 1).$$

Now $3 \mid m$, hence $m = 3x$ for some integer x and $x^2 = 3y^2 + 1$. Here $p + 1 = 3x\sqrt{y+1} > \sqrt{3y}^3$, so we are interested for the solutions where $y < 60$. These are

$$(x, y) \in \{(1,0), (2,1), (7,4), (26,15), (97,56)\}\,,$$

of which only for $y = 1$ and $y = 15$, the number $y + 1$ is a perfect square. This gives us the solutions $(p, n) \in \{(2,1), (311,46)\}$.

Case 4. If $6(n^2-2n+4) = m^2$, then $m = 2u$ is even, hence $n = 2v$ is even. We have $6(v^2-v+1) = u^2$. Again, u is even; however, $v^2 - v + 1$ is odd, giving us a contradiction.

Finally, we conclude that the only prime numbers satisfying the desired condition are 2, 3 and 311.

Remarks

The condition p to be prime is unnecessary.
By the PSC opinion, solving The Pell equation in **Case 3** is outside the scope of JBMO.