APMO 2026 – Problems and Solutions

Problem 1. Find all quadruples of positive integers (a, p, m, o) with $o \geq 2$ such that

$$a! + p! = m^o + 26.$$

Answer: $(a, p, m, o) \in \{(4, 3, 2, 2), (3, 4, 2, 2), (5, 3, 10, 2), (3, 5, 10, 2)\}$.

Solution.

Without loss of generality, assume that $a \geq p$. First, observe that if $p \geq 4$, then $m^o = a! + p! - 26 \equiv 2 \pmod 4$, which forces $o = 1$, a contradiction. Hence, it suffices to consider the remaining cases $p = 1, 2, 3$.

Case $p = 1$**.** Then $a! = m^o + 25$. Clearly $a \geq 5$, which implies that $5 \mid m$ and m is odd. If $o = 2$, or more generally if o is even, then $m^o + 25 \equiv 2 \pmod 4$, which is impossible. Thus assume $o \geq 3$. Then $a! = m^o + 25 \equiv 5^2 \pmod{5^3}$, which yields $10 \leq a \leq 14$. Note that $5^3 \mid m^o = a! - 25$. However, a direct check shows that $5^3 \nmid a! - 25 \quad$ for all $10 \leq a \leq 14$, which is again a contradiction.

Case $p = 2$**.** Then $a! = m^o + 24$. Clearly $a \geq 5$. If $a = 5$, then $m^o = 120 - 24 = 96$, which is impossible. If $a \geq 6$, then $m^o = a! - 24 \equiv 3 \pmod 9$, forcing $o = 1$, again a contradiction.

Case $p = 3$**.** Then $a! = m^o + 20$. Clearly $a \geq 4$. If $a = 4$, then $m^o = 4! - 20 = 4$, so $m = o = 2$. If $a = 5$, then $m^o = 5! - 20 = 100$, giving $m = 10$ and $o = 2$. Now suppose that $a \geq 6$. Then $2^4 \mid a! = m^o + 20$, which implies that m is even. However, if $4 \mid m$ or $o \geq 3$, then $m^o + 20 \equiv 4 \pmod 8$, and if $o = 2$ and $m \equiv 2 \pmod 4$, then $m^o + 20 = m^2 + 20 \equiv 4 + 20 \equiv 8 \pmod{16}$, contradiction.

Therefore, the solutions are

$$(a, p, m, o) \in \{(4, 3, 2, 2), (3, 4, 2, 2), (5, 3, 10, 2), (3, 5, 10, 2)\}.$$

Problem 2. Let $n \geq 2$ be an integer. Let $A_1, A_2, \cdots, A_{2^n}$ be the 2^n subsets of an n-element set in some order. Prove that

$$|A_1 \cap A_2| + |A_2 \cap A_3| + \cdots + |A_{2^n-1} \cap A_{2^n}| + |A_{2^n} \cap A_1| \geq 2^{n-2}.$$

Solution 1.

Indices are reduced modulo 2^n throughout. Set, $a_i := |A_i \cap A_{i+1}|$ and $c_i = |A_{i-1}| + 2|A_i| + |A_{i+1}| - 2n$. Note that $a_i = |A_i| + |A_{i+1}| - |A_i \cup A_{i+1}| \geq |A_i| + |A_{i+1}| - n$. It follows that $a_i + a_{i-1} \geq c_i$. In particular, if $c_i \geq 1$ then

$$a_{i-1} + a_i \geq \frac{1}{2}(c_i + 1). \tag{1}$$

Notice that, because $a_i, a_{i-1} \geq 0$, (1) is trivially satisfied whenever $c_i < 0$. We now verify that (1) also holds in the case $c_i = 0$, and hence in all cases. Indeed, if either $a_i > |A_i| + |A_{i+1}| - n$ or $a_{i-1} > |A_i| + |A_{i+1}| - n$ then $a_i + a_{i-1} > c_i$, thus $a_i + a_{i-1} \geq 1$ and (1) holds. Otherwise, we must have $|A_{i-1} \cup A_i| = |A_i \cup A_{i+1}| = n$, which implies that both A_{i-1} and A_{i+1} contain the complement of A_i. Since they are distinct one of them strictly contain the complement of A_i, therefore has a non-empty intersection with A_i. Consequently, $a_i + a_{i-1} \geq 1$, which shows that (1) holds in this case as well.

Consequently,

$$\begin{aligned}
\sum_{i=1}^{2^n} a_i &= \frac{1}{2}\sum_{i=1}^{2^n}(a_{i-1} + a_i) \geq \frac{1}{4}\sum_{i=1}^{2^n}(c_i + 1) = \frac{1}{4}\sum_{i=1}^{2^n} c_i + 2^{n-2} \\
&= \sum_{i=1}^{2^n} |A_i| - 2^n n + 2^{n-2} = \sum_{k=0}^{n} k\binom{n}{k} - 2^{n-1}n + 2^{n-2} = 2^{n-2}.
\end{aligned}$$

The last equality follows from the well-know identity $\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}n$.

Solution 2.

Let indices of the A_i be taken modulo 2^n. Consider any element x of the n-element set. The number of i for which x is in both A_i and A_{i+1} equals the number of i for which x is in neither of those sets (since there are equal numbers of sets containing and not containing x, and equal numbers of transitions in each direction, from containing to not containing x and vice versa, as we go through the cycle of sets). Thus the required sum equals

$$\frac{1}{2}\sum_{i=1}^{2^n}\Big(|A_i \cap A_{i+1}| + |\overline{A_i} \cap \overline{A_{i+1}}|\Big). \tag{2}$$

A term of this sum is zero only for two sets that are complements, so no two consecutive terms of it can be zero, so the sum is at least 2^{n-2}.

Comments. It is possible to construct examples for which equality holds..

Problem 3 . Let $\mathbb{R}_+$ denote the set of positive real numbers. Determine all the functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}_+$,

$$|x - y| < |y - z| \text{ if and only if } |f(x) - f(y)| < |f(y) - f(z)|. \tag{1}$$

Answer: All functions of the form $f(x) = ax + b$ for fixed real numbers a, b with $a \neq 0$.

Solution.

It is clear that (1) holds for all functions f of the form $f(x) = ax + b$ with $a \neq 0$. We shall show that these are the only ones that satisfy (1). Suppose (1) holds for f.

Note that, from (1) it's obvious that $|x - y| > |y - z|$ if and only if $|f(x) - f(y)| > |f(y) - f(z)|$. It follows that

$$|x - y| = |z - y| \text{ implies } |f(x) - f(y)| = |f(z) - f(y)|. \tag{2}$$

Claim 1. *f is one-to-one.*

Proof. Suppose that $f(y) = f(z)$ for some $y \neq z$. By choosing $x = y$, (1) implies $|f(x) - f(y)| < |f(y) - f(z)| = 0$, a contradiction.

□

Claim 2. *For all $x, z \in \mathbb{R}_+$, we have*

$$f\left(\frac{x+z}{2}\right) = \frac{f(x) + f(z)}{2}. \tag{3}$$

Proof. Since (3) holds trivially for $x = z$, we assume that $x \neq z$. By Claim 1, $f(x) \neq f(z)$. Set $y = \frac{x+z}{2}$, then $|y - z| = |x - z|$. It follows from (2) that $|f(y) - f(x)| = |f(y) - f(z)|$, which implies $f(y) = \frac{f(x)+f(z)}{2}$ (since $f(x) \neq f(z)$). □

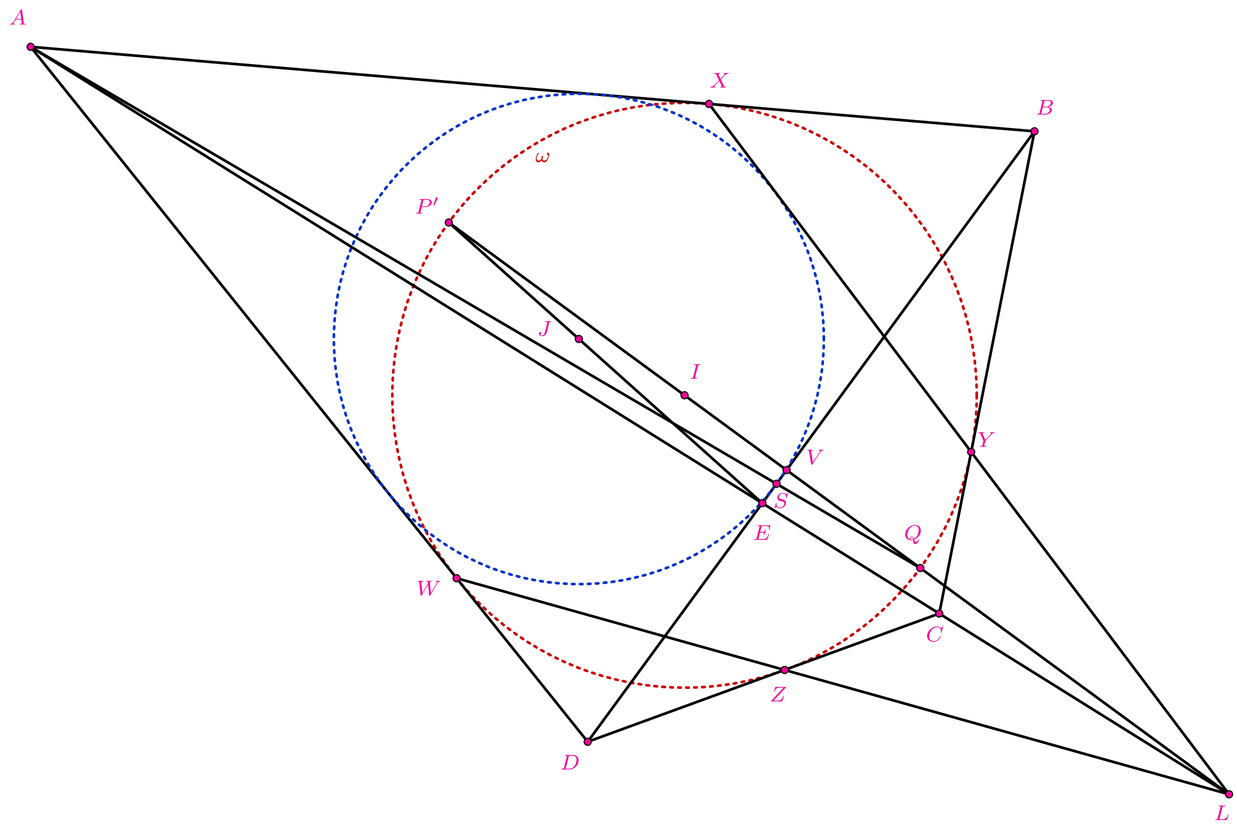
Claim 3. *f is monotone.*

Proof. Assume f is not monotone. Then there exist $y < x < z$ such that $f(x) - f(y)$ and $f(z) - f(x)$ have different signs. Hence $|f(y) - f(z)| < \max(|f(x) - f(y)|, |f(z) - f(x)|)$, whereas (1) gives $|f(y) - f(z)| > \max(|f(x) - f(y)|, |f(z) - f(x)|)$, a contradiction. □

Note that for any constants $a \neq 0, b \in \mathbb{R}$, the function f satisfies (1) if and only if $af + b$ does. Thus, without loss of generality, we may assume that $f(1) = 1$ and $f(2) = 2$. We will prove that this forces $f(x) = x$ for all $x \in \mathbb{R}_+$. Given $f(1) = 1, f(2) = 2$, (3) yields $f(3) = 3$, $f\left(\frac{3}{2}\right) = \frac{3}{2}$, $f\left(\frac{1}{2}\right) = \frac{1}{2}$. By a straightforward induction argument, it follows that $f(n) = n$ for all positive integers n, and more generally $f\left(\frac{n}{2^k}\right) = \frac{n}{2^k}$ for all positive integers n and non-negative integers k. Since the numbers of the form $\frac{n}{2^k} = \frac{n}{2^k}$ is a dense subset of $\mathbb{R}_+$, the monotonicity of f implies that $f(x) = x$ for all $x \in \mathbb{R}_+$.

Problem 4. Let $ABCD$ be a quadrilateral with an incircle ω of centre I. The diagonals AC and BD intersect at E. Let J be the incentre of triangle ABD. The extension of the ray EJ intersects ω at P. Prove that $PI \perp BD$.

Solution.



Draw the diameter $P'Q$ of ω such that $P'Q \perp BD$ and $P'A < QA$. We claim that P' actually coincide with P, in other words, E, J, P' are collinear.

Let S be the contact point of the incircle (J) of the triangle ABD with BD. Since A is the center of the external homothety of the circles (I), (J), the points A, Q, S are collinear.

Let X, Y, Z, W be the contact points of AB, BC, CD, DA with (I) respectively. Let L be the intersection point of XY and ZW. By Pascal's theorem (for $\begin{pmatrix} W & Y & Z \\ X & Z & Y \end{pmatrix}$ and $\begin{pmatrix} W & X & Y \\ X & W & Z \end{pmatrix}$), L lies on AC. It follows that BD is the polar of L with respect to the circle (I). Therefore, if we denote by V the intersection point of IL and BD then P', V, Q, L form a harmonic range. We deduce that

$$\overline{IV} \cdot \overline{IL} = IP'^2 = IQ^2 \quad \text{and} \quad \overline{VI} \cdot \overline{VL} = \overline{VP'} \cdot \overline{VQ} = \mathcal{P}_{V/(I)}.$$

It follows that $\dfrac{\overline{VI}}{\overline{VP'}} = \dfrac{\overline{VQ}}{\overline{VL}}$, and hence $\dfrac{\overline{P'I}}{\overline{P'V}} = \dfrac{\overline{LQ}}{\overline{LV}}$. Consequently,

$$(IV, P'\infty) = (QV, L\infty).$$

Since inversion conserves cross-ratios, we obtain $(QV, L\infty) = (QL, VI)$. Hence $(IV, P'\infty) = (QL, VI)$, and therefore $J(IV, P'\infty) = A(QL, VI)$, which gives

$$J(IV, P'S) = A(SE, VJ).$$

On the other hand, we have

$$A(SE, VJ) = J(SE, VA) = J(AV, ES) = J(IV, ES).$$

Combining these equalities yields $J(IV, P'S) = J(IV, ES)$, which implies that the points E, J, P' are collinear, as desired.

Problem 5. Let n be a positive integer. There exist $n(n+1)$ rooms arranged in a $(n+1) \times n$ grid. For each two adjacent rooms, there is a door between them. Find the number of ways to choose a subset of doors and lock them such that there are two rooms S and G satisfying the following properties:

(i) S is in the first row and G is in the $(n+1)$-th row.

(ii) One can reach G from S only through unlocked doors.

Answer: 2^{2n^2-2}.

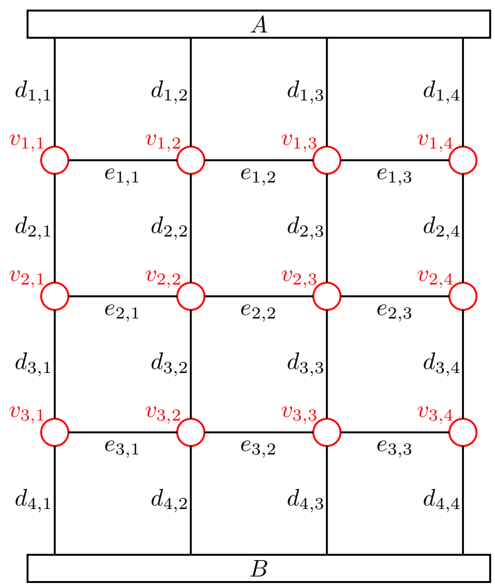
Solution.

Without loss of generality, we may assume that every door between two rooms in the first row and the $(n+1)$-th row does not be locked. Consider the following graph $\mathcal{G}$:

- The vertex set of $\mathcal{G}$ consists of $n(n-1)+2$ points A, B, $v_{i,j}$ $(1 \leqq i \leqq n-1, 1 \leqq j \leqq n)$.
- The edge set of $\mathcal{G}$ consists of $n^2 + (n-1)^2$ edges $d_{i,j}$ $(1 \leqq i, j \leqq n)$, $e_{i,j}$ $(1 \leqq i, j \leqq n-1)$.
- $d_{1,j}$ joins A and $v_{1,j}$ for all $1 \leqq j \leqq n$, and $d_{n,j}$ joins $v_{n-1,j}$ and B for all $1 \leqq j \leqq n$.
- $d_{i,j}$ joins $v_{i-1,j}$ and $v_{i,j}$ for all integers i, j with $2 \leqq i \leqq n-1$ and $1 \leqq j \leqq n$.
- $e_{i,j}$ joins $v_{i,j}$ and $v_{i,j+1}$ for all integers i, j with $1 \leqq i, j \leqq n-1$.

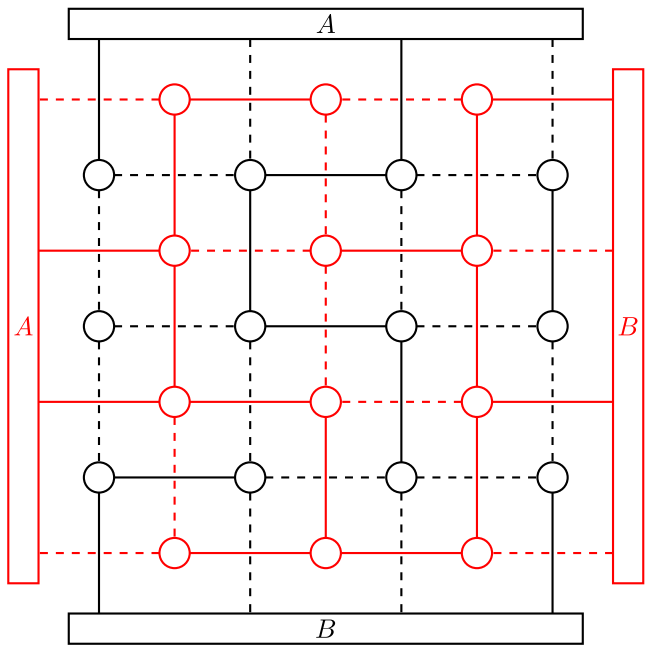
For example, the following figure illustrates $\mathcal{G}$ for $n = 4$.

$\mathcal{G} =$



Rooms and doors between them correspond to vertices and edges of the graph as follows. Rooms on the first row correspond to A, and that on the $(n+1)$-th row correspond to B. For all integers i, j with $2 \leqq i \leqq n$ and $1 \leqq j \leqq n$, the room in row i and column j corresponds to $v_{i-1,j}$. Each door between two rooms corresponds to the edge between the corresponding vertex. The problem can be reframed as determining the number of spanning subgraphs $\mathcal{H}$ which has a path from A to B.

Let $S(\mathcal{G})$ be the set of spanning subgraphs of $\mathcal{G}$. Define $f : S(\mathcal{G}) \to S(\mathcal{G})$ such that $d_{i,j} \in f(\mathcal{H})$ if and only if $d_{j,i} \notin \mathcal{H}$ and $e_{i,j} \in f(\mathcal{H})$ if and only if $e_{j,i} \notin \mathcal{H}$. For example, the following figure illustrates $\mathcal{H}$ and $f(\mathcal{H})$ for $n = 4$. Black lines correspond to $\mathcal{H}$ and red lines correspond to $f(\mathcal{H})$.



We say that $\mathcal{H} \in S(\mathcal{G})$ is *achievable* if there is a path from A to B. Now, we prove two lemmas.

Lemma 1. *For each $\mathcal{H} \in S(\mathcal{G})$, we have $f(f(\mathcal{H})) = \mathcal{H}$. In particular, f is a bijection from $S(\mathcal{G})$ to $S(\mathcal{G})$.*

Proof. For all integers i, j with $1 \leqq i, j \leqq n$, we have $d_{i,j} \in \mathcal{H}$ if and only if $d_{j,i} \notin f(\mathcal{H})$, which is equivalent to $d_{i,j} \in f(f(\mathcal{H}))$. Similarly, $e_{i,j} \in \mathcal{H}$ if and only if $e_{i,j} \in f(f(\mathcal{H}))$ for all integers i, j with $1 \leqq i, j \leqq n-1$. Since $\mathcal{H}$ and $f(f(\mathcal{H}))$ are spanning subgraphs of $\mathcal{G}$, we obtain $\mathcal{H} = f(f(\mathcal{H}))$. □

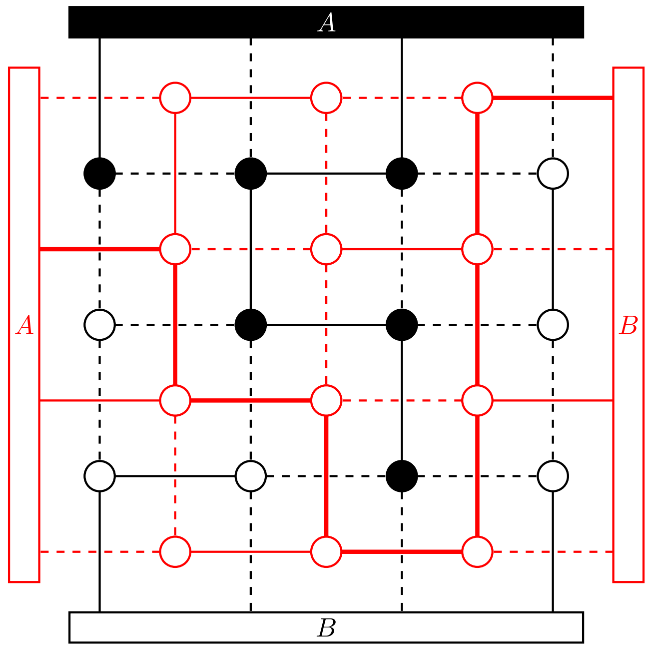
Lemma 2. *$\mathcal{H}$ is achievable if and only if $f(\mathcal{H})$ is not achievable.*

Proof. Let us denote $\overline{d_{i,j}} = d_{j,i}$ and $\overline{e_{i,j}} = e_{j,i}$. Suppose $\mathcal{H}$ is achievable. Then there are edges $p_1, p_2, \ldots, p_k$ such that $(p_1, p_2, \ldots, p_k)$ is a path from A to B. If there is a path from A to B in $f(\mathcal{H})$, then the path must pass through one of $\overline{p_1}, \overline{p_2}, \ldots, \overline{p_k}$. This contradicts the fact that $\overline{p_1}, \overline{p_2}, \ldots, \overline{p_k}$ are not contained in $f(\mathcal{H})$.

Conversely, assume that $\mathcal{H}$ is not achievable. Let V be the vertex set of $\mathcal{G}$ and let V_A be the connected component of $\mathcal{H}$ containing A. Denote by E the subset of the edge set of $\mathcal{G}$ such that $e \in E$ if one of the two endpoints of e is in V_A and the other is in $V \setminus V_A$. Then every edge $e \in E$ is not an edge of $\mathcal{H}$. Define a set $\overline{E}$ of edges of $\mathcal{G}$ by

$$\overline{E} = \{\overline{d_{i,j}}; d_{i,j} \in E\} \cup \{\overline{e_{i,j}}; e_{i,j} \in E\}.$$

Then $\overline{E}$ is a subset of the edge set of $f(\mathcal{H})$, and we can find a path from A to B in $\overline{E}$ by connecting these edges. Hence, $f(\mathcal{H})$ is achievable. In the following figure, black dots represent vertices in V_A and white dots represent vertices in $V \setminus V_A$. The thick red line is a path from A to B in $f(\mathcal{H})$.



□

Using these lemmas, we see that

$$\begin{aligned}
\#\{\mathcal{H} \in S(\mathcal{G}) \mid \mathcal{H} \text{ is achievable}\} &= \#\{\mathcal{H} \in S(\mathcal{G}) \mid f(\mathcal{H}) \text{ is achievable}\} \\
&= \#S(\mathcal{G}) - \#\{\mathcal{H} \in S(\mathcal{G}) \mid f(\mathcal{H}) \text{ is not achievable}\} \\
&= 2^{2n^2-2n+1} - \#\{\mathcal{H} \in S(\mathcal{G}) \mid \mathcal{H} \text{ is achievable}\}
\end{aligned}$$

Therefore, $\#\{\mathcal{H} \in S(\mathcal{G}) \mid \mathcal{H} \text{ is achievable}\} = 2^{2n^2-2n}$ and considering the state of doors in the first row and the $(n+1)$-th row, we obtain that the answer is $2^{2n^2-2n} \cdot 2^{2n-2} = 2^{2n^2-2}$.

Remark. The converse direction in the proof of Lemma 2 can be explained in a more geometric way as follows. Place a unit square with center at each vertex in V_A (the set A it self is considered as n vertices). Then we get a connected polynomio P. Since $\mathcal{H}$ is not achievable P does not contain B. The boundary of P acts as a connected path from the leftmost square to the rightmost square. This path (which corresponds to the thick red line) connects A to B in the graph $f(\mathcal{H})$.
