39$^{\text{th}}$ Balkan Mathematical Olympiad

Shortlisted Problems with Solutions

May 04 – 09, 2022

Agros, Cyprus

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until BMO 2023

Contributing countries

The Organising Committee and the Problem Selection Committee of the BMO 2022 wish to thank the following countries for contributing problem proposals:

- Greece
- North Macedonia
- Romania
- Serbia
- United Kingdom

Problem Selection Committee

Chairman:	Demetres Christofides
Members:	Demetris Karantanos
	Theoklitos Parayiou

Contents

PROBLEMS

ALGEBRA

A1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x(x + f(y))) = (x + y)f(x)$$

for all $x, y \in \mathbb{R}$.

A2. Let $k > 1$ be a real number, $n \geqslant 3$ be an integer, and $x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n > 0$ be real numbers. Prove the inequality:

$$\frac{x_1 + kx_2}{x_2 + x_3} + \frac{x_2 + kx_3}{x_3 + x_4} + \cdots + \frac{x_{n-1} + kx_n}{x_n + x_1} + \frac{x_n + kx_1}{x_1 + x_2} \geqslant \frac{n(k+1)}{2}\,.$$

A3. Let a, b, c, d be non-negative real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 3\,.$$

Prove that

$$3(ab + ac + ad + bc + bd + cd) + \frac{4}{a+b+c+d} \leqslant 5\,.$$

A4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$f(f(x)) + f(f(y)) = f(x + y)f(xy)$$

for all $x, y \in \mathbb{R}$.

A5. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(yf(x)^3 + x) = x^3 f(y) + f(x)$$

for all $x, y > 0$.

A6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying the following property:

If $A, B, C, D \in \mathbb{R}^2$ are the vertices of a square with sides of length 1, then

$$f(A) + f(B) + f(C) + f(D) = 0\,.$$

Show that $f(x) = 0$ for all $x \in \mathbb{R}^2$.

COMBINATORICS

C1. There are 100 positive integer numbers written on a board. At each step, Alex composes 50 fractions using each number written on the board exactly once, brings these fractions to their irreducible form, and then replaces the 100 numbers on the board with the new numerators and denominators to create 100 new numbers. Find the smallest positive integer n such that regardless of the values of the initial 100 numbers, after n steps Alex can arrange to have on the board only pairwise coprime numbers.

C2. Alice is drawing a shape on a piece of paper. She starts by placing her pencil at the origin, and then draws line segments of length 1, alternating between vertical and horizontal segments. Eventually, her pencil returns to the origin, forming a closed, non-self-intersecting shape. Show that the area of this shape is even if and only if its perimeter is a multiple of eight.

C3. Find the largest positive integer k for which there exists a convex polyhedron P with the following properties:

(a) P has exactly 2022 edges.

(b) The degrees of the vertices of P don't differ by more than 1.

(c) It is possible to colour the edges of P with k colours such that for every colour c, and every pair of vertices (v_1, v_2) of P, there is a monochromatic path between v_1 and v_2 in the colour c.

C4. Let $n \geqslant 3$ be an odd positive integer, and consider an $n \times n$ grid containing n^2 cells. Dionysus colours each cell either red or blue. A frog can hop directly between two cells if they have the same colour and share at least one vertex. Xanthias views the colouring, and wants to place frogs on k of the cells so that any cell can be reached by a frog in a finite number of hops. Find the least value of k such that Xanthias can always be successful regardless of the colouring chosen by Dionysus.

C5. A cube of side length 2021 is given. In how many ways we can place a $1 \times 1 \times 1$ cubelet on the border of this cube in such a way that the newly formed solid can be completely filled using $k \times 1 \times 1, 1 \times k \times 1$ and $1 \times 1 \times k$ cuboids, for some $k \in \mathbb{N} \setminus \{1\}$?

GEOMETRY

G1. Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let τ_A, τ_B be the tangents to ω at A and B, which meet at X. Now, let Y be the foot of the perpendicular from O onto CX, and let the line through C parallel to AB meet τ_A at Z. Prove that YZ bisects AC.

G2. Let ABC be a triangle with $AB > AC$ with incenter I. The internal bisector of angle BAC intersects the BC at the point D. Let M the midpoint of the segment AD, and let F be the second intersection point of MB with the circumcircle of the triangle BIC. Prove that AF is perpendicular to FC.

G3. Let ABC a triangle and let ω be its circumcircle. Let E be the midpoint of the minor arc BC of ω, and M the midpoint of (BC). Let V be the other point of intersection of AM with ω, F the point of intersection of AE with BC, X the other point of intersection of the circumcircle of FEM with ω, X' the reflection of V with respect to M, A' the foot of the perpendicular from A to BC and S the other point of intrsection of XA' with ω. If $Z \in \omega$ with $Z \neq X$ is such that $AX = AZ$, then prove that S, X' and Z are collinear.

G4. Let ABC be a triangle and let the tangent at B to its circumcircle meet the internal bisector of angle A at P. The line through P parallel to AC meets AB at Q. Assume that Q lies in the interior of segment AB and let the line through Q parallel to BC meet AC at X and PC at Y. Prove that PX is tangent to the circumcircle of triangle XYC.

G5. Let ABC be a triangle with circumcircle ω, circumcenter O, and orthocenter H. Let K be the midpoint of AH. The perpendicular to OK at K intersects AB and AC at P and Q, respectively. The lines BK and CK intersect ω again at X and Y, respectively. Prove that the second intersection of the circumcircles of triangles KPY and KQX lies on ω.

G6. Let ABC be a triangle with $AB < AC$ and let D be the other intersection point of the angle bisector of A with the circumcircle of triangle ABC. Let E and F be points on the sides AB and AC respectively, such that $AE = AF$ and let P be the point of intersection of AD and EF. Let M be the midpoint of BC. Prove that AM and the circumcircles of triangles AEF and PMD pass through a common point.

NUMBER THEORY

N1. Let n be a positive integer. What is the smallest sum of digits of $5^n + 6^n + 2022^n$?

N2. Let a, b, n be positive integers such that:

(i) $a^{2021} \mid n$ and $b^{2021} \mid n$

(ii) $2022 \mid a - b$ and $a > b$.

Prove that there is a subset of the divisors of the number n having sum of elements divisible by 2022 but not by 2022^2.

N3. For every natural number x, let $P(x)$ be the product of the digits of the number x. Is there a natural number n such that the numbers $P(n)$ and $P(n^2)$ are non-zero squares of natural numbers, where the number of digits of the number n is equal to

(a) 2021

(b) 2022

N4. A hare and a tortoise run in the same direction, at constant but different speeds, around the base of a tall square tower. They start together at the same vertex, and the run ends when both return to the initial vertex simultaneously for the first time. Suppose the hare runs with speed 1, and the tortoise with speed less than 1. For what rational numbers x is it true that, if the tortoise runs with speed x, the fraction of the entire run for which the tortoise can see the hare is also x?

SOLUTIONS

ALGEBRA

A1. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x(x+f(y)))=(x+y)f(x)$$

for all $x,y\in\mathbb{R}$.

Proposed by ....

Solution 1. By putting $x=y=0$, we get $f(0)=0$. Also, by putting $x\in\mathbb{R}$ and $y=0$, we get $f(x^2)=xf(x)$ for all x, and therefore $xf(x)=-xf(-x)$ for all x, which implies that f is an odd function.

Obviously, the constant function $f(x)=0$ for all $x\in\mathbb{R}$, is a solution.

Assume that f is not identically zero. So there is some $a\in\mathbb{R}$ such that $f(a)\neq 0$. Thus, $a\neq 0$. Suppose that there is another $0\neq b\in\mathbb{R}$, such that $f(b)=0$. For $x=a, y=b$ we get

$$af(a)=f(a^2)=f(a(a+f(b)))=(a+b)f(a)=af(a)+bf(a)\,.$$

Since $f(a)\neq 0$ we get $b=0$.

Finally, by putting $x\in\mathbb{R}$ and $y=-x$, we get $f(x^2+xf(-x))=f(x^2-xf(x))=0$, for all $x\in\mathbb{R}$. Thus, $x^2-xf(x)=0$, i.e. $x^2=xf(x)=f(x^2)$, for all $x\in\mathbb{R}$. Hence $f(x)=x$ for all $x\in\mathbb{R}^+$ and since f is odd we have $f(x)=x$ for all $x\in\mathbb{R}$. This last function obviously satisfies the equation.

Solution 2. As in Solution 1 we have that $f(x^2)=xf(x)$ for all $x\in\mathbb{R}$ and that if f is not identically zero then $f(x)=0 \implies x=0$.

Now for $x=-f(y)$ we get

$$(y-f(y))f(-f(y))=f(0)=0$$

and therefore $y=f(y)$ or $f(-f(y))=0$ for each $y\in\mathbb{R}$. I.e. $f(y)=y$ or $f(y)=0$ for each $y\in\mathbb{R}$. Since $f(y)=0 \implies y=0$ then $f(y)=y$ for each $y\in\mathbb{R}$.

A2. Let $k>1$ be a real number, $n \geqslant 3$ be an integer, and $x_1 \geqslant x_2 \geqslant x_3 \geqslant \cdots \geqslant x_n > 0$ be real numbers. Prove the inequality:

$$\frac{x_1+kx_2}{x_2+x_3}+\frac{x_2+kx_3}{x_3+x_4}+\cdots+\frac{x_{n-1}+kx_n}{x_n+x_1}+\frac{x_n+kx_1}{x_1+x_2} \geqslant \frac{n(k+1)}{2}\,.$$

Proposed by ....

Solution 1. Writing $x_{n+1}=x_1$, by AM-GM we have

$$\frac{x_1+x_2}{x_2+x_3}+\frac{x_2+x_3}{x_3+x_4}+\cdots+\frac{x_{n-1}+x_n}{x_n+x_1}+\frac{x_n+x_1}{x_1+x_2} \geqslant n\sqrt{\prod_{i=1}^{n}\frac{x_i+x_{i+1}}{x_{i+1}+x_{i+1}}} = n\,.$$

So it is enough to prove that

$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\cdots+\frac{x_n}{x_n+x_1} \geqslant \frac{n}{2}\,.$$

Letting $a_i = x_{i+1}/x_i$ for $i=1,2,\ldots,n$, it is enough to prove that

$$\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n} \geqslant \frac{n}{2}\,.$$

Note that $a_1,\ldots,a_{n-1} \leqslant 1$ and $a_1a_2\cdots a_n = 1$.

Equivalently, it is enough to prove that if $m \geqslant 2$ is an integer and $a_1,\ldots,a_m \leqslant 1$ are real numbers then

$$\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_m} \geqslant \frac{m+1}{2} - \frac{a_1a_2\cdots a_m}{1+a_1a_2\cdots a_m}\,.$$

We proceed by induction on m. In fact the statement is true even for $m=1$ so we assume that it is true for $m=k$ and proceed with the inductive step. Letting $a=a_1\cdots a_k$ and $b=a_{k+1}$ it is enough to prove that

$$\frac{1}{1+b}-\frac{a}{1+a} \geqslant \frac{1}{2}-\frac{ab}{1+ab}\,.$$

We have

$$\frac{a}{1+a}-\frac{ab}{1+ab} = \frac{a(1-b)}{(1+a)(1+ab)} \leqslant \frac{1-b}{1+b} = \frac{1}{1+b}-\frac{1}{2}$$

so the result follows.

Solution 2. Since $\frac{1}{1+x} = \frac{1}{2}+\frac{1}{2}\cdot\frac{1-x}{1+x}$, with the notation of Solution 1 it is enough to prove that

$$\frac{1-a_1}{1+a_1}+\cdots+\frac{1-a_n}{1+a_n} \geqslant 0\,.$$

Letting $f(x)=\frac{1-x}{1+x}$ one can check that

$$f(x)+f(y)-f(xy) = \frac{(1-x)(1-y)(1-xy)}{(1+x)(1+y)(1+xy)}\,.$$

Thus $f(x)+f(y) \geqslant f(xy)$ for $x,y \leqslant 1$. So inductively

$$\frac{1-a_1}{1+a_1}+\cdots+\frac{1-a_{n-1}}{1+a_{n-1}} \geqslant \frac{1-a_1\cdots a_{n-1}}{1+a_1\cdots a_{n-1}} = \frac{1-1/a_n}{1+1/a_n} = -\frac{1-a_n}{1+a_n}$$

and the result follows.

A3. Let a, b, c, d be non-negative real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=3\,.$$

Prove that

$$3(ab+ac+ad+bc+bd+cd)+\frac{4}{a+b+c+d}\leqslant 5\,.$$

Proposed by ....

Solution. Let $S=a+b+c+d$. By AM-HM (or Cauchy-Schwarz) we have

$$S+4=(a+1)+(b+1)+(c+1)+(d+1)\geqslant \frac{16}{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}}=\frac{16}{3}$$

giving $S\geqslant \frac{4}{3}$.

Multiplying the given equality by $(a+1)(b+1)(c+1)(d+1)$ we get

$$\sum abc+2\sum ab+3S+4=3\left(abcd+\sum abc+\sum ab+S+1\right)$$

giving

$$3abcd+2\sum abc+\sum ab=1\,.$$

In particular $ab+ac+ad+bc+bd+cd\leqslant 1$. So we may assume that $S<2$ as otherwise the inequality is immediate.

The given equality transforms to

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}+\frac{d}{d+1}=1\,,$$

and so by Cauchy-Schwarz

$$\sum a(a+1)\sum\frac{a}{a+1}\geqslant S^2\,.$$

Thus

$$S^2\leqslant a^2+b^2+c^2+d^2+S=S^2-2\sum ab+S\,.$$

So $\sum ab\leqslant S/2$ and it is enough to prove that

$$\frac{3S}{2}+\frac{4}{S}\leqslant 5\,.$$

This is equivalent to $3S^2-10S+8\leqslant 0$ which in turn is equivalent to $(S-2)(3S-4)\leqslant 0$. Since $S\geqslant 4/3$ and we are also assuming that $S<2$, then the inequality is true and the result follows.

Remark. From the above Solution it follows that we have equality in the cases that $a=b=c=d=\frac{4}{3}$ and in the case that two of the variables are equal to 1 and the other two are equal to 0.

A4. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)\neq 0$ and

$$f(f(x))+f(f(y))=f(x+y)f(xy)$$

for all $x,y\in\mathbb{R}$.

Proposed by ....

Solution 1. We show that $f(x)=2$ for every $x\in\mathbb{R}$.

Let $f(0)=c\neq 0$. Then for $x=y=0$ we get $f(c)=\frac{c^2}{2}$. Now for $y=0$ we get

$$f(f(x))=cf(x)-\frac{c^2}{2}\,. \tag{1}$$

The given equation can now be rewritten as

$$c(f(x)+f(y))=c^2+f(x+y)f(xy)\,. \tag{2}$$

For $x=1, y=-1$ in (2) we obtain

$$cf(1)+cf(-1)=c^2+cf(-1)\,.$$

Since $c\neq 0$, then $f(1)=c$. For $y=1$ in (2) we have

$$cf(x)=f(x)f(x+1) \tag{3}$$

for all $x\in\mathbb{R}$. Now if $f(x)\neq 0$ for all x, then $f(x+1)=c$ for all x, so f is constant. Plugging into the given equation we get $f(x)=2$ for all $x\in\mathbb{R}$.

On the other hand, if there exists x_0 such that $f(x_0)=0$, then $f(f(x_0))=f(0)=c$, so for $x=x_0$ in (1) we get

$$cf(x_0)-\frac{c^2}{2}=f(f(x_0))=c \implies -\frac{c^2}{2}=c \implies c=-2\,.$$

From here we get $f(-2)=f(c)=\frac{c^2}{2}=2$, hence $f(f(-2))=f(2)$. For $x=-2$ in (1) we get

$$f(2)=f(f(-2))=cf(-2)-\frac{c^2}{2}=-2f(-2)-2=-6\,.$$

We also have, for $x=1$ in (3), that $f(2)=c=-2$, a contradiction. So there are no more functions.

Solution 2.

For $x=-1, y=t$ and for $x=-1, y=1-t$ we get

$$f(f(-1))+f(f(t))=f(t-1)f(-t)=f(-t)f(t-1)=f(f(-1))+f(f(1-t))\,.$$

It follows that $f(f(t))=f(f(1-t))$ for every $t\in\mathbb{R}$. Now for $x=t, y=1-t$ we get

$$2f(f(t))=f(f(t))+f(f(1-t))=f(1)f(t-t^2)$$

and therefore

$$\frac{f(1)}{2}\left[f(x-x^2)+f(y-y^2)\right]=f(x+y)f(xy)\,. \tag{1}$$

For $x = y = 0$ in (1) we get $f(1) = f(0)$. Now for $y = 0$ in (1) we get

$$f(x - x^2) + f(0) = 2f(x)$$

for every $x \in \mathbb{R}$. Now (1) gives

$$\frac{f(0)}{2}\left[2f(x) + 2f(y) - 2f(0)\right] = f(x+y)f(xy)$$

thus

$$f(x) + f(y) = f(0) + \frac{f(x+y)f(xy)}{f(0)} \tag{2}$$

for every $x, y \in \mathbb{R}$.

For $x = y = 1$ in (2), and since $f(1) = f(0)$ we get $f(2) = f(0)$ as well.

We claim that $f(x) \neq 0$ for each x so assume for contradiction that $f(a) = 0$ for some $a \in \mathbb{R}$. For $x = y = a/2$ in (2) we get $f(a/2) = f(0)/2$. Now for $x = a/2, y = 2$ in (2) we get $3f(0)/2 = f(0)$, a contradiction as $f(0) \neq 0$.

Now for $y = 1$ in (2), since $f(1) = f(0)$ and $f(x) \neq 0$ we get $f(x+1) = f(0)$. Thus f is constant and from the original equation we get $f(x) = 2$ for each $x \in \mathbb{R}$.

Alternative Formulation 1. Same problem but without the restriction $f(0) \neq 0$.

Solution. Pick an $\alpha \in (0, 4)$ and a set $A \subseteq [\alpha, 2\sqrt{\alpha}]$ not containing both of α and $2\sqrt{\alpha}$. Then any function satisfying $f(x) = 0$ for $x \notin A$ and $f(x) \notin A$ for $x \in A$ is a solution.

To see this, note that $f(f(x)) = 0$ for every $x \in \mathbb{R}$. Furthermore, given $x, y \in \mathbb{R}$ we must have $s = x + y \notin A$ or $p = xy \notin A$. If this is not the case then the quadratic $t^2 - st + p$ has discriminant $s^2 - 4p \leqslant (2\sqrt{\alpha})^2 - 4\alpha = 0$ where the equality is not possible as A does not contain both of α and $2\sqrt{\alpha}$. This is a contradiction as the quadratic has the real numbers x and y as roots. From these it follows that any such function is indeed a solution.

We now show that every additional solution of the functional equation is of the above form. Assuming $f(0) = 0$ and putting $y = 0$ in the original equation we get $f(f(x)) = 0$ for each $x \in \mathbb{R}$. Therefore $f(x + y) = 0$ or $f(xy) = 0$ for each $x, y \in \mathbb{R}$.

Let $A = \{x \in \mathbb{R} : f(x) \neq 0\}$ and suppose that it is non-empty. Given $b \notin (0, 4)$, the discriminant of the quadratic $x^2 - bx + b$ is $\Delta = b^2 - 4b > 0$. So there are real numbers x_1, x_2 with $x_1 + x_2 = b = x_1x_2$ giving $f(b) = 0$ and so $b \notin A$. Now let $\alpha = \inf A$. If $\alpha \in A$, then for any $b \geqslant 2\sqrt{\alpha}$ the quadratic $t^2 - bt + \alpha$ has discriminant $b^2 - 4\alpha > 0$ so it has real roots. This gives $f(\alpha)f(b) = 0$ and so $f(b) = 0$. If $\alpha \notin A$, then for any $b > 2\sqrt{\alpha}$ we can find $\alpha' \in A$ with $b \geqslant 2\sqrt{\alpha'}$. A similar argument as before shows $b \notin A$ concluding the proof.

Alternative Formulation 2. Same problem as in Alternative Formulation 1 but in the case of $f(0) = 0$ instead of asking for all of the solutions ask for the determination of infinitely many solutions.

Alternative Formulation 3. Same problem, without the restriction that $f(0) \neq 0$ but with the restriction that f is monotonic.

Solution. If $f(0) = 0$ then as in the Solution of the Alternative Formulation 1 we have $f(a) = 0$ for $a \notin (0, 4)$. Since f is monotonic it must then be identically equal to 0 which is a valid solution.

Note: The monotonicity can be used to give slightly shorter solutions in the case $f(0) \neq 0$.

A5. Find all functions $f:(0,\infty)\to(0,\infty)$ such that

$$f(yf(x)^3+x)=x^3f(y)+f(x)$$

for all $x,y>0$.

Proposed by ....

Solution. Setting $y=\frac{t}{f(x)^3}$ we get

$$f(x+t)=x^3f\left(\frac{t}{f(x)^3}\right)+f(x) \tag{1}$$

for every $x,t>0$.

From (1) it is immediate that f is increasing.

Claim. $f(1)=1$

Proof of Claim. Let $c=f(1)$. If $c<1$, taking $x=1$ and $y=\frac{1}{1-c^3}$ we have $y-yc^3=1$, so $yf(1)^3+1=y$ and $f(yf(1)^3+1)=f(y)=1^3f(y)$. Thus $f(1)=0$, a contradiction. Assume now for contradiction that $c>1$. We claim that

$$f(1+c^3+\cdots+c^{3n})=(n+1)c$$

for every $n\in\mathbb{N}$. We proceed by induction, the case $n=0$ being trivial. The inductive step follows easily by taking $x=1, t=c^3+c^6+\cdots+c^{3(k+1)}$ in (1).

Now taking $x=1+c^3+\cdots+c^{3n-3}, t=c^{3n}$ in (1) we get

$$(n+1)c=f(1+c^3+\cdots+c^{3n})=(1+c^3+\cdots+c^{3n-3})f\left(\frac{c^{3n}}{(n+1)^3}\right)+nc$$

giving

$$f\left(\frac{c^{3n}}{(n+1)^3}\right)=\frac{c}{(1+c^3+\cdots+c^{3n})^3}<c=f(1)\implies\frac{c^{3n}}{(n+1)^3}<1\,.$$

But this leads to a contradiction if n is large enough. □

Now for $x=1$ we get $f(y+1)=f(y)+1$ and since $f(1)=1$ inductively we get $f(n)=n$ for every $n\in\mathbb{N}$. For $m,n\in\mathbb{N}$, setting $x=n, y=q=m/n$ we get

$$mn^2+n=f\left(qn^3+n\right)=f(yf(x)^3+x)=x^3f(y)+f(x)=n^3f\left(q\right)+n\implies f(q)=q\,.$$

Since f is strictly increasing with $f(q)=q$ for every $q\in\mathbb{Q}^{>0}$ we deduce that $f(x)=x$ for every $x>0$. It is easily checked that this satisfies the functional equation.

Alternative Formulation. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(yf(x)^3+x)=x^3f(y)+f(x)$$

Solution. For $x=y=0$ we get $f(0)=0$. Let $a\neq 0$ and assume $f(a)=0$. For $x=a$ we get $a^3f(y)=0$ for every $y\in\mathbb{R}$ and therefore $f(x)=0$ for every $x\in\mathbb{R}$ which is obviously a solution.

So we may assume that $f(x)\neq 0$ for every $x\neq 0$. Let $c=f(1)$. If $c\neq 1$, taking $x=1$ and $y=\frac{1}{1-c^3}$ we have $y-yc^3=1$, so $yf(1)^3+1=y$ and $f(yf(1)^3+1)=f(y)=1^3f(y)$. Thus $f(1)=0$, a contradiction.

Thus $f(1)=1$ which for $x=1$ gives $f(y+1)=f(y)+1$ for every $y\in\mathbb{R}$. From here we easily get $f(n)=n$ for every $n\in\mathbb{Z}$.

For $x=-1$ in the original equation we get

$$f(-y-1)=-f(y)-1=-(f(y)+1)=-f(y+1)$$

for every $y\in\mathbb{R}$ and therefore f is an odd function.

For $x\neq 0$ and $y=\pm\frac{t}{f(x)^3}$ we have

$$f(t+x)=x^3 f\left(\frac{t}{f(x)^3}\right)+f(x)$$
$$f(-t+x)=-x^3 f\left(\frac{t}{f(x)^3}\right)+f(x)$$

for every $x\neq 0$ and every $t\in\mathbb{R}$. Adding those two we get

$$2f(x)=f(x+t)+f(x-t)$$

for every $x\neq 0$ and every $t\in\mathbb{R}$. Since f is odd the above also holds for every $x,t\in\mathbb{R}$. Given $a,b\in\mathbb{R}$ and putting $x=a,t=b$ and then $x=b,t=a$ in the above we have

$$f(a+b)=2f(a)-f(a-b)=2f(a)+f(b-a)=2f(a)+2f(a)-f(a+b)$$

giving $f(a+b)=f(a)+f(b)$ for every $a,b\in\mathbb{R}$. I.e. f is Cauchy. In particular from the original equation we get $f(yf(x)^3)=x^3f(y)$ for every $x,y\in\mathbb{R}$. For $y=1$ we get $f(f(x)^3)=x^3$ for every $x\in\mathbb{R}$. In particular f is surjective.

Putting $y=f(x)^3$ in the last equation we get $f(f(x)^6)=x^3f(f(x)^3)=x^6$. By surjectivity we get that $f(a)\geqslant 0$ for every $a\geqslant 0$. So for $x\geqslant y$ we have $f(x)-f(y)=f(x-y)\geqslant 0$. I.e. f is monotone and since it is Cauchy it is also linear which gives $f(x)=x$ for every $x\in\mathbb{R}$. This clearly satisfies the functional equation.

A6. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function satisfying the following property:

If $A,B,C,D\in\mathbb{R}^2$ are the vertices of a square with sides of length 1, then

$$f(A)+f(B)+f(C)+f(D)=0\,.$$

Show that $f(x)=0$ for all $x\in\mathbb{R}^2$.

Proposed by ....

Solution. Let V be the set of all such functions. It is obvious that if $f,g\in V$ then $f+g\in V$. It is also obvious that if $f\in V$ and $a\in\mathbb{R}^2$ then for $g:\mathbb{R}^2\to\mathbb{R}$ defined by $g(x)=f(x+a)$ we have $g\in V$.

Clearly, we can consider the elements of $\mathbb{R}^2$ as vectors. We denote the inner product of two vectors $a=(a_1,a_2)$ and $b=(b_1,b_2)$ by $a\cdot b=a_1b_1+a_2b_2$. A vector a is unit vector if $a\cdot a=1$. Two vectors a,b are orthogonal if and only if $a\cdot b=0$. So a square $ABCD$ whose vertices are given by the vectors a,b,c,d is a unit square if and only if the vectors $b-a,c-b,d-c,a-d$ are unit vectors with consecutive pairs of them being orthogonal.

We say that $f\in V$ is good for a unit vector v if either $f(x+v)=-f(x)$ for every $x\in\mathbb{R}^2$ or $f(x+v^\perp)=-f(x)$ for some unit vector $v^\perp$ orthogonal to v and every $x\in\mathbb{R}^2$. We also say that f is good for a set of unit vectors if it is good for every vector in the set.

Note that if $f,g\in V$ are good for S then so is $f+g$. Furthermore, if $f\in V$ is good for S and $a\in\mathbb{R}^2$ then $g(x)=f(x+a)$ is also good for S.

Claim 1. If $V\neq\{0\}$, then given any finite set S of unit vectors, there is a $g\in V\setminus\{0\}$ which is good for S.

Proof of Claim 1. We proceed by induction on $|S|$. So assume that $f\in V\setminus\{0\}$ is good for S and that we are trying to find a $g\in V\setminus\{0\}$ which is good for $S\cup\{v\}$ for some unit vector v.

Let $v^\perp$ be a unit vector orthogonal to v. If $f(x+v^\perp)=-f(x)$ for all $x\in\mathbb{R}^2$, we are done. Otherwise, define $g:\mathbb{R}^2\to\mathbb{R}$ by

$$g(x)=f(x)+f(x+v^\perp)\,.$$

From the observations above g is good for S. Furthermore, since $x,x+v,x+v+v^\perp,x+v^\perp$ form the vertices of a unit square we have

$$g(x)+g(x+v)=f(x)+f(x+v^\perp)+f(x+v)+f(x+v+v^\perp)=0\,.$$

Hence g is good for v as well and therefore it is good for $S\cup\{v\}$. $\square$.

We now say that $f\in V$ is excellent for a unit vector v if $f(x+12v)=f(x)$ for every $x\in\mathbb{R}$. We also say that f is excellent for a set of unit vectors if it is excellent for every vector in the set.

Note that if $f,g\in V$ are excellent for S then so is $f+g$. Furthermore, if $f\in V$ is excellent for S and $a\in\mathbb{R}^2$ then $g(x)=f(x+a)$ is also excellent for S.

Claim 2: If $V\neq\{0\}$, then given any finite set S of unit vectors, there is a $g\in V\setminus\{0\}$ which is excellent for S.

Proof of Claim 2. We proceed by induction on $|S|$. So assume that $f\in V\setminus\{0\}$ is excellent for S and that we are trying to find a $g\in V\setminus\{0\}$ which is excellent for $S\cup\{v\}$ for some unit vector v.

The proof of Claim 1 shows that we may assume that f is good for v. (The construction in the proof of the Claim preserves the excellence.) If $f(x+v) = -f(x)$ for every $x \in \mathbb{R}^2$, then $f(x+12v) = f(x)$ for every $x \in \mathbb{R}^2$ and we are done. So we may suppose that $f(x+v^\perp) = -f(x)$ for every $x \in \mathbb{R}^2$ where $v^\perp$ is a unit vector orthogonal to v.

Let $u = \frac{3v+4v^\perp}{5}$ and let $u^\perp = \frac{4v-3v^\perp}{5}$. Then u and $u^\perp$ are unit vectors. The proof of Claim 1 shows that we may assum that f is good for u.

If $f(x+u) = -f(x)$ for every $x \in \mathbb{R}^2$, then

$$f(x) = -f(x+5u) = -f(x+3v+4v^\perp) = -f(x+3v)\,.$$

So $f(x+12v) = f(x)$ for every $x \in \mathbb{R}^2$ and we are done.

If $f(x+u^\perp) = -f(x)$ for every $x \in \mathbb{R}^2$, then

$$f(x) = -f(x+5u^\perp) = -f(x+4v-3v^\perp) = f(x+4v)\,.$$

So again $f(x+12v) = f(x)$ for every $x \in \mathbb{R}^2$ and we are done. $\square$

From Claim 1 we have a unit vector v and an $f \in V \setminus \{0\}$ such that $f(x+v) = -f(x)$ for every $x \in \mathbb{R}^2$. Pick unit vectors u, w such that $12u, 12w, v$ form a triangle. I.e. $12u + 12w + v = 0$.

From the proof of Claim 2 we may also assume that f is excellent for u, w. (While preserving the property $f(x+v) = -f(x)$ for every $x \in \mathbb{R}^2$.)

Now given any $x \in \mathbb{R}^2$ we have

$$f(x) = f(x+12u+12w+v) = f(x+12w+v) = f(x+v) = -f(x)$$

thus $f(x) = 0$. Since this holds for every $x \in \mathbb{R}^2$, this contradicts the fact that $f \in V \setminus \{0\}$ and therefore our assumption that $V \neq \{0\}$.

COMBINATORICS

C1. There are 100 positive integer numbers written on a board. At each step, Alex composes 50 fractions using each number written on the board exactly once, brings these fractions to their irreducible form, and then replaces the 100 numbers on the board with the new numerators and denominators to create 100 new numbers. Find the smallest positive integer n such that regardless of the values of the initial 100 numbers, after n steps Alex can arrange to have on the board only pairwise coprime numbers.

Proposed by ....

Solution. Equivalently, we have a graph on 100 vertices and a positive integer written on each vertex. At each step we pick a perfect matching (i.e. a set of disjoint edges covering all vertices) and for each edge of the matching we divide the numbers in its endpoints with their highest common divisor.

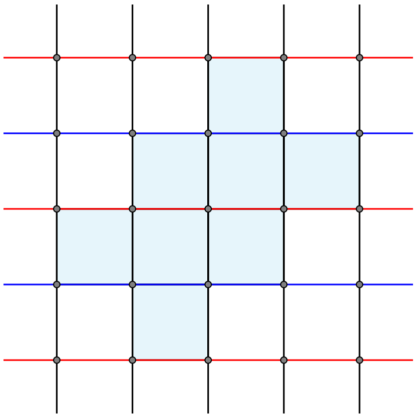
If initially the numbers on the vertices are $p_1, p_2, \ldots, p_{99}$ and $p_1 p_2 \cdots p_{99}$, where $p_1, \ldots, p_{99}$ are distinct prime numbers, then we need at least 99 steps. This is because the vertex having the number $p_1 p_2 \cdots p_{99}$ needs to be matched with every other vertex and we need 99 steps for this.

We show that 99 steps are enough. For this it is enough to show that K_{100} has a 1-factorisation. I.e. we can decompose the edges of the complete graph on 100 vertices into 99 perfect matchings. In general it is a known fact that K_{2n} has a 1-factorisation. For one way to achieve this, write $x, x_1, \ldots, x_{2n-1}$ for the vertices, and for the i-th matching ($1 \leqslant i \leqslant 2n-1$) consider all edges of the form $x_r x_s$ with $1 \leqslant r < s \leqslant 2n-1$ and $r + s \equiv i \bmod 2n-1$ together with the edge $x x_t$ where $2t \equiv i \bmod 2n-1$.

C2. Alice is drawing a shape on a piece of paper. She starts by placing her pencil at the origin, and then draws line segments of length 1, alternating between vertical and horizontal segments. Eventually, her pencil returns to the origin, forming a closed, non-self-intersecting shape. Show that the area of this shape is even if and only if its perimeter is a multiple of eight.

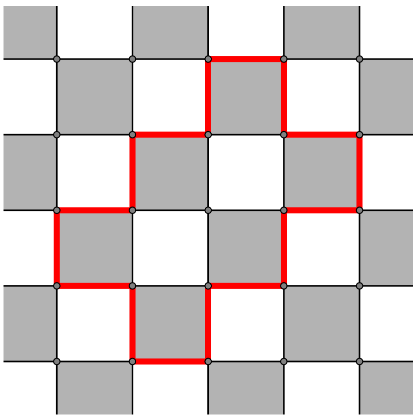
Proposed by ....

Solution 1. Colour the horizontal segments in every other line of the grid alternately red and blue as shown below:



Let there be r red segments on the perimeter and s red segments in the interior of the shape. By considering the possibilities starting from a red segment, we see that every fourth segment on the perimeter of the shape will be red, therefore we have $P = 4r$. Also, every square has exactly one red edge, thus $A = r + 2s$. So $A \equiv r \bmod 2$ from which the result follows.

Solution 2. Colour the square in the grid with a chessboard colouring. The alternation of vertical and horizontal segments means that all squares with an edge on the perimeter and lying within the shape are of the same colour, say black.



Any internal edge within the shape lies between a white and black square, so if the number of white squares within the shape is W, the number of edges of the chessboard lying inside the shape is $4W$. If the total number of squares in the shape is A, then $4A$ counts every edge on the perimeter once, and every internal edge twice, so the perimeter has length $P = 4A - 8W$, which is a multiple of 8 if and only if A is even.

Solution 3. We have as many horizontal perimeter edges as vertical, so it is enough to show that the area is even if and only if the number of vertical perimeter edges is a multiple of 4. In each horizontal strip oof height 1, pair the vertical perimeter edges in order from left to right. (We can do so because there must be an even number of vertical perimeter edges in every such strip.) Let us assume that we have P such pairs. So we need to show that the area is even if and only if P is even.

As in Solution 2, every such pair of perimeter edges encloses a consecutive set of squares of the shape with the first and last of these squares being, without loss of generality, black. So each such pair accounts for an odd number of squares inside the shape and therefore the area is even if and only if P is even as required.

Solution 4. By Green's Theorem the area of the shape is equal to

$$\int_C x \, \mathrm{d}y$$

where C is the boundary of the shape traversed anticlockwise. We partition C into its line segments of length 1. Each line segment contributes 0 to the integral if it is horizontal and $\pm a$ to the integral if it is on the vertical line $x = a$. Every two consecutive vertical line segments contribute $\pm a \pm (a \pm 1) \equiv 1 \bmod 2$.

So the area is even if and only if the number of vertical line segments is 0 mod 4 which happens if and only if the perimeter is 0 mod 8. (Since there is an equal even number of horizontal and vertical line segments.)

C3. Find the largest positive integer k for which there exists a convex polyhedron P with the following properties:

(a) P has exactly 2022 edges.

(b) The degrees of the vertices of P don't differ by more than 1.

(c) It is possible to colour the edges of P with k colours such that for every colour c, and every pair of vertices (v_1, v_2) of P, there is a monochromatic path between v_1 and v_2 in the colour c.

Proposed by ....

Solution 1. We divide the solution in two steps, first we prove that $k < 3$, and then give an inductive construction of P for $k = 2$.

Let P have V vertices, E edges and F faces. Suppose the contrary, that $k > 2$. We have k disjoint trees on V vertices, so $E \geqslant 3(V-1)$. This is a contradiction as for every polyhedron we have $E \leqslant 3V - 6$.

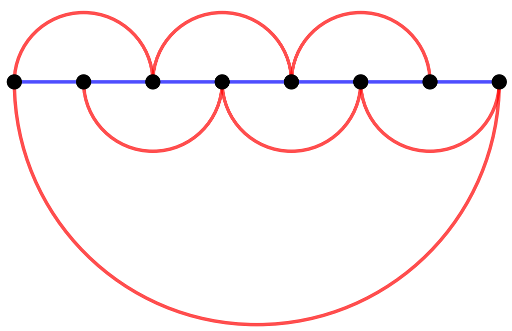
Now we take $k = 2$ and prove that for every positive integer n, we can find a convex polyhedron P_{6n} with exactly $6n$ edges that satisfies the condition of the problem.

For $n = 1$, let's consider the tetrahedron $ABCD$. One colouring that works is: AB, AD and CD are in one colour, and AC, BC and BD are in the other colour.

Suppose we have constructed P_{6n}, here is how to construct P_{6n+6}: Consider the triangular face $T = xyz$ most recently added to P_{6n}, glue on top of T a truncated pyramid whose larger base is T. We are effectively adding 3 new vertices, say x', y', z', 3 faces, and 6 edges, say $xx', yy', zz', x'y', y'z', z'x'$. We colour $x'y', x'z', yy'$ with the first colour, and all other new edges with the second colour. It is now easy to see that in any one of the colours, a tree on the vertices of P_{6n} in that colour, together with the newly added edges of that colour give a tree on the vertices of P_{6n+6} in that colour.

Note that $x'y'z'$ is the most recently added triangular face, so the construction can proceed inductively by glueing another truncated pyramid on top of it. It's easy to see (and prove inductively) that every vertex has degree 3 or 4, so condition (b) is satisfied and we are done.

Solution 2. For the construction we can also use Steinitz's Theorem on the characterization of convex polyhedra: A planar graph is the graph of a convex polyhedron if and only if it is 3-vertex connected. So we can take for example the graph on $\{v_1, v_2, \ldots, v_{2n}\}$ with edges $v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}$ in one colour and $v_1v_3, v_2v_4, \ldots, v_{2n-2}v_{2n}$ and $v_{2n}v_1$ in the other colour. It is easy to get a planar embedding of this graph as for example in the following figure for the case $n = 4$.



This graph is 3-connected: Suppose we remove the vertices v_i and v_j with $i < j$. If $i = 1, j = 2$ or $i = 2n - 1, j = 2n$ the remaining graph is obviously connnected. If $1 < i$ and $j = i + 1 < 2n$ the remaining graph is also connected having the spanning path $v_{i-1}v_{i-2}\cdots v_1v_{2n}v_{2n-1}\cdots v_{i+1}$. Otherwise $j > i + 1$ and we have the spanning path $v_1\cdots v_{i-1}v_{i+1}\cdots v_{j-1}v_{j+1}\cdots v_{2n}$.

This graph has $2n - 2$ edges so for $n = 1012$ it has the required number of edges. Furthermore it satisfies (b) as every vertex has degree 3 or 4.

Comments.

1. One can consider a slightly simplified version of the problem in which we specify that $k = 2$, i.e. we only consider colourings with 2 colours.
2. The goal of condition (b) (vertex degrees not differing by more than 1) is to avoid a simple solution with a pyramid with 1012 vertices. There can be other ways to introduce this type of constraint, for example requiring that there is pair of edges with distance at least 300 (in terms of edges between them). Another way could be that no vertex has a degree higher than 4. Another way can be that for every vertex (face) we can find another vertex (face) with same number of edges.

C4. Let $n \geqslant 3$ be an odd positive integer, and consider an $n \times n$ grid containing n^2 cells. Dionysus colours each cell either red or blue. A frog can hop directly between two cells if they have the same colour and share at least one vertex. Xanthias views the colouring, and wants to place frogs on k of the cells so that any cell can be reached by a frog in a finite number of hops. Find the least value of k such that Xanthias can always be successful regardless of the colouring chosen by Dionysus.

Proposed by ....

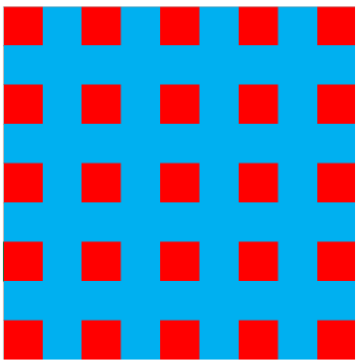
Solution. Let G be the graph whose vertices are all $(n+1)^2$ vertices of the grid and where two vertices are adjacent if and only if they are adjacent in the grid and moreover the two cells in either side of the corresponding edge have different colours.

The connnected components of G, excluding the isolated vertices, are precisely the boundaries between pairs of monochromatic regions each of which can be covered by a single frog. Each time we add one of these components in the grid, it creates exactly one new monochromatic region. So the number of frogs required is one more than the number of such components of G.

It is easy to check that every corner vertex of the grid has degree 0, every boundary vertex of the grid has degree 0 or 1 and every 'internal' vertex of the grid has degree $0, 2$ or 4. It is also easy to see that every component of G which is not an isolated vertex must contain at least four vertices unless it is the boundary of a single corner of the grid, in which case it contains only three vertices.

Writing N for the number of components which are not isolated vertices, we see that in total they contain at least $4N-4$ vertices. (As at most four of them contain 3 vertices and all others contain 4 vertices.) Since we also have at least 4 components which are isolated vertices, then $4N = (4N-4)+4 \leqslant (n+1)^2$. Thus $N \leqslant \frac{(n+1)^2}{4}$ and therefore the minimal number of frogs required is $\frac{(n+1)^2}{4}+1$.

This bound for $n = 2m+1$ is achieved by putting coordinates (x, y) with $x, y \in \{0, 1, \ldots, 2m\}$ in the cells and colouring red all cells both of whose coordinates are even, and blue all other cells. An example for $n = 9$ is shown below.



C5. A cube of side length 2021 is given. In how many ways we can place a $1\times1\times1$ cubelet on the border of this cube in such a way that the newly formed solid can be completely filled using $k\times1\times1, 1\times k\times1$ and $1\times1\times k$ cuboids, for some $k\in\mathbb{N}\setminus\{1\}$?

Proposed by ....

Solution 1. Suppose that for some $k>1$ and some placed cubelet there is a valid filling. In each unit cubelet (of the original cube) with coordinates (x,y,z) where $0\leqslant x,y,z\leqslant 2020$, we assign the complex number ω^{x+y+z} where $\omega=e^{\frac{2\pi i}{k}}$. We also assign the number ω^{a+b+c} in the additional cubelet in position (a,b,c).

Since $1+\omega+\omega^2+\cdots+\omega^{k-1}=\frac{\omega^k-1}{\omega-1}=0$, then the sum of numbers in any $1\times1\times k$ cuboid is equal to zero. So the sum of all assigned numbers is equal to

$$0=(1+\omega+\cdots+\omega^{2020})^3+\omega^{a+b+c}\,. \tag{1}$$

This gives

$$1=\left|-\omega^{a+b+c}\right|=\left|(1+\omega+\cdots+\omega^{2020})^3\right|=\left|\frac{1-\omega^{2021}}{1-\omega}\right|^3\,.$$

Thus $|1-\omega|=|1-\omega^{2021}|$ which means that 1 is equidistant from the numbers ω and ω^{2021}. Since $|\omega^{2021}|=1$, this happens if and only if $\omega^{2021}=\omega$ or $\omega^{2021}=\omega^{-1}$. Then $\omega^{2020}=1$ or $\omega^{2022}=1$ which gives $k|2020$ or $k|2022$. However $k|2021^3+1$. Since $2021^3+1\equiv 2 \bmod 2020$, if $k|2020$ then $k|2$. So in any case we have $k|2022$. Now (1) gives

$$\omega^{a+b+c}=-(1+\omega+\cdots+\omega^{2020})^3=-(-\omega^{2021})^3=\omega^{6063}=\omega^{-3}\,.$$

So $a+b+c\equiv -3 \bmod k$.

If we have a valid filling for k, then we have a valid filling for every prime factor of k. So we may assume that k is prime and therefore $k\in\{2,3,337\}$.

Assume without loss of generality that the additional cubelet is at the bottom of the cube, i.e. $c=-1$. Then $a+b\equiv -2 \bmod k$. By symmetry, if we have a valid filling for $(a,b,-1)$, then we have a valid filling for $(2022-a,b,-1)$. So we must also have $2022-a+b\equiv -2 \bmod k$ and so $b-a\equiv -2 \bmod k$. Since also $a+b\equiv -2 \bmod k$ we get $a\equiv b\equiv -1 \bmod k$ for $k\neq 2$ and $a\equiv b \bmod 2$ for $k=2$. We will now show that the above necessary conditions are also sufficient to have a valid filling.

We claim first that a square defined by coordinates (x,y) where $0\leqslant x,y\leqslant 2020$, with a removed cell (a,b) satisfying the above restrictions can be covered by $1\times k$ rectangles. This is because such a square can be covered by four rectangles of sizes $(a+1)\times b$, $(2020-a)\times(b+1)$, $(2021-a)\times(2020-b)$ and $a\times(2021-b)$, where each of these rectangles can be covered by $1\times k$ rectangles. This follows since $k\mid a+1, b+1, 2021-a, 2021-b$ if $a\equiv b\equiv -1 \bmod k$ and since $2|b, 2020-a, 2020-b, a$ if $a\equiv b\equiv 0 \bmod 2$.

Now we fill the cube with the added cubelet as follows: The lowest $k-1$ "layers" of the original cube of side 2021 are filled using the previous method together with a $1\times1\times k$ cuboid covering the holes in these layers and the additional cubelet. The remainder is the $2021\times2021\times(2022-k)$ cuboid which can be easily filled by $1\times1\times k$ cuboids because $k\mid 2022-k$.

To complete the solution, we need to count the number of ordered pairs (a,b) with $0\leqslant a,b\leqslant 2020$, such that $a\equiv b \bmod 2$, or $a\equiv b\equiv -1 \bmod 3$ or $a\equiv b\equiv -1 \bmod 337$. There are 1011^2 choices with $a\equiv b\equiv 0 \bmod 2$ and 1010^2 choices with $a\equiv b\equiv 1 \bmod 2$. If $a\equiv b\equiv -1 \bmod 3$ but $a\not\equiv b \bmod 2$ then one of a,b must be congruent to $2 \bmod 6$ and the other to $5 \bmod 6$. There are $2\times337\times336$ such choices. Finally, if $a\equiv b\equiv -1 \bmod 337$ but is not yet accounted for, then one of them is equal to $\{336, 1010, 1684\}$ and the other to $\{673, 1347\}$. (Note that in this case the second one is definitely not congruent to $-1 \bmod 3$.) There are $2 \times 3 \times 2$ such choices. In total we have 2268697 choices for the pair (a, b). Therefore, because of symmetry, the total number of ways is $6 \times 2268697 = 13612182$.

Solution 2. Since there are a total of $2021^3 + 1$ cubelets and each cuboid covers k of them, we must have $k | 2021^3 + 1$. We have

$$2021^3 = 2022 \cdot (2021^2 - 2021 + 1) = 2 \cdot 3 \cdot 337 \cdot 3 \cdot 7 \cdot 31 \cdot 6271$$

as a product of prime factors.

If we have a valid filling for k, then we have a valid filling for every prime factor of k. So we may assume that k is prime and therefore $k \in \{2, 3, 7, 31, 337\}$. (The case $k = 6271$ is easily seen to be impossible.)

We colour the cubelet at position (x, y, z) for $0 \leqslant x, y, z \leqslant 2020$ by the colour $x + y + z \bmod k$. So every $1 \times 1 \times k$ cuboid covers 1 cubelet of each colour.

If $k = 2$ then it is easy to see that we have one more cubelet of the form $0 \bmod 2$ than of the form $1 \bmod 2$. So assuming that the additional cubelet is in position $(a, b, -1)$, we must have $a + b \equiv 0 \bmod 2$ as in Solution 1.

If $k = 3$ then, since it is each to cover all cubelets with $1 \times 1 \times k$ cuboids except those of the form (x, y, z) with $0 \leqslant x, y, z \leqslant 2$, we see that there is one less cubelet of the form $0 \bmod 3$ than of the form $1, 2 \bmod 3$. So assuming that the additional cubelet is in position $(a, b, -1)$, we must have $a + b \equiv -2 \bmod 3$. Exploiting the symmetry as in Solution 1 we get $a \equiv b \equiv -1 \bmod 3$.

If $k = 7$ then note that (since $7 | 2016$) it is easy to cover all cubelets with $1 \times 1 \times k$ cuboids except those of the form (x, y, z) with $0 \leqslant x, y, z \leqslant 4$. Note that exactly 19 of these cubicles have the colour $0 \bmod 7$. (3 when $z = 0$, 3 when $z = 1$, 4 when $z = 2$, 5 when $z = 3$ and 4 when $z = 4$.) These are more than $\frac{5^3+1}{7} = 18$ so one of these will remain uncovered. So $k = 7$ is impossible.

If $k = 31$ then note that (since $31 | 2015$) it is easy to cover all cubelets with $1 \times 1 \times k$ cuboids except those of the form (x, y, z) with $0 \leqslant x, y, z \leqslant 5$. Note that only one of those cubelets has the colour $0 \bmod 31$. So even if the additional cubelet has the same colour it is still less than the $\frac{6^3+1}{31} = 7$ which are expected in a proper covering. So $k = 31$ is impossible.

If $k = 337$ then note that the square containing all cells of the form (x, y) with $0 \leqslant x \leqslant 2020$ and $0 \leqslant y \leqslant 2021$ can be covered with $1 \times k$ rectangles and so contains equal number of cells of each colour (thinking of $z = 0$). In the column with $y = 2021$, the cells have in order the colours $-1, 0, 1 \ldots, -3 \bmod k$. So the colour $-2 \bmod k$ appears one time less in that column and therefore one time more in the square of the form (x, y) with $0 \leqslant x, y \leqslant 2020$. In the 'layer' above the extra colour is $-1 \bmod k$, then $0 \bmod k$ and so on until $2020 - 2 \equiv -4 \bmod k$. So in the original cube all colours appear an equal number of times except $-3 \bmod k$ which appear one time less and must be the colour of the additional cubelet $(a, b, -1)$. Thus $a + b \equiv -2 \bmod k$. Exploiting the symmetry as in Solution 1 we get $a \equiv b \equiv -1 \bmod 337$.

So we get the same necessary conditions as in Solution 1. The sufficiency of these conditions is proved in a similar way as in Solution 1.

GEOMETRY

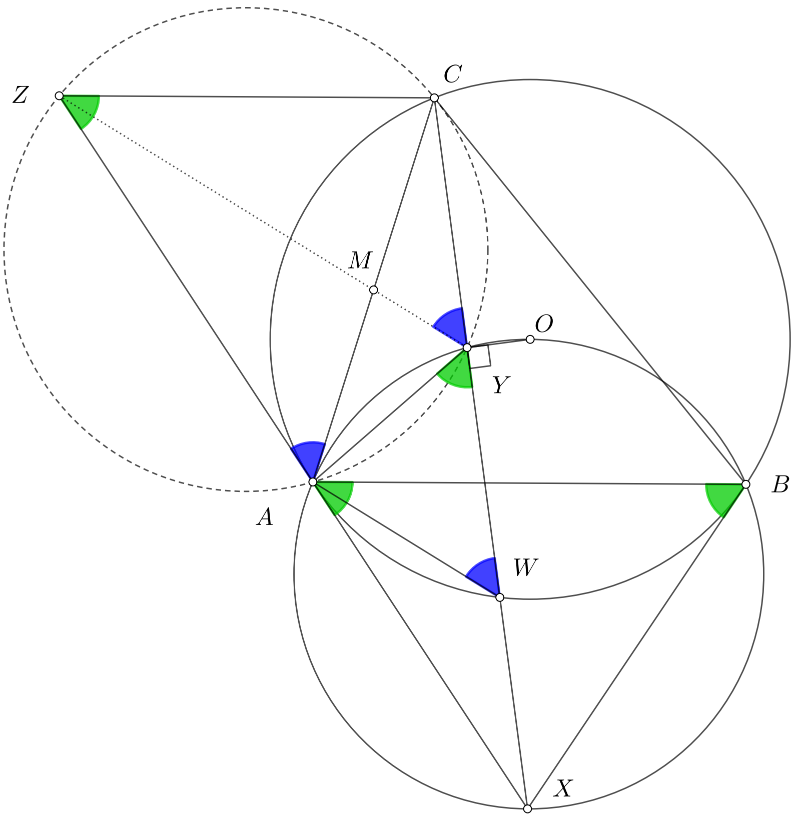
G1. Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let τ_A, τ_B be the tangents to ω at A and B, which meet at X. Now, let Y be the foot of the perpendicular from O onto CX, and let the line through C parallel to AB meet τ_A at Z. Prove that YZ bisects AC.

Proposed by ....

Solution 1. Firstly observe that $OAXB$ is cyclic, with diameter OX, and Y also lies on this circle since $OY \perp XC$. Hence:

$$\angle AZC = \angle XAB = \angle ABX = \angle AYX$$

and so $CYAZ$ is cyclic.



Let M be the intersection of YZ and AC and let CY intersect ω again at W. Using the new cyclic relation we get $\angle CYZ = \angle CAZ$ and then using that ZA is tangent to ω we get $\angle CAZ = \angle CWA$, so $\angle CYM = \angle CWA$. Therefore the triangles CWA and CYM are similar. But CW is a chord of ω, and Y is the foot of the perpendicular from O, hence Y is the midpoint of CW. It follows from the similarity relation that M is the midpoint of AC, as required.

Solution 2. Let M be the midpoint of AC. We have $\angle CAZ = \angle CBA$ and $\angle ZCA = \angle BAC$ so the triangles CAZ and ABC are similar. The line CYX is the C-symmedian of triangle ABC, and ZM is the corresponding median in triangle CAZ, hence by isogonality $\angle AZM = \angle ACY$. So

$$\angle ZMA = 180^\circ - \angle AZM - \angle MAZ = 180^\circ - \angle ACY - \angle CBA \quad (1)$$

Now observe $\angle OMC = \angle OYC = 90^\circ$, so $CMYO$ is cyclic. Thus:

$$\angle CYM = \angle COM = \frac{1}{2}\angle COA = \angle CBA.$$

This shows that

$$\angle YMC = 180^\circ - \angle MCY - \angle CYM = 180^\circ - \angle ACY - \angle CBA$$

Combining this with (1) we get that $\angle YMC = \angle ZMA$ and as A, C, M are collinear, it follows that Z, M, Y are collinear as required.

Solution 3. As in Solution 2 we have that CX is the A-symmedian of triangle ABC and that triangle ABC is similar to triangle CAZ.

Let f be the spiral similarity which maps AC onto AB and let g be the reflection on the perpendicular bisector of AB. Note that f is a rotation about A by an angle of $\angle CAB$ (clockwise in our figure) followed by a homothety centered at A by a factor of AB/AC. By the similarity of triangles ABC and CAZ we have that $g(f(Z)) = C$, so actually $f(Z)$ is the other point of intersection, say C', of CZ with ω.

As in Solution 1 we have that $CYAZ$ is cyclic. Therefore, letting W be the other point of intersection of CY with ω, we have $\angle WAB = \angle WCB = \angle CAY$. We also have $\angle ACY = \angle ABW$. It follows that $f(Y) = W$.

Let $W' = g(W)$. Then $W' \in \omega$ and since CW is the A-symmedian, then CW' passes through the midpoint N of AB. Now CW' and $C'W$ intersect on the perpendicular bisector of AB and therefore they intersect on N. It follows that $N = AB \cap C'W = Af(C) \cap f(Z)f(Y)$ is the image of $M = AC \cap ZY$ under f. Since N is the midpoint of AB, then M is the midpoint of AC.

G2. Let ABC be a triangle with $AB > AC$ with incenter I. The internal bisector of angle BAC intersects the BC at the point D. Let M the midpoint of the segment AD, and let F be the second intersection point of MB with the circumcircle of the triangle BIC. Prove that AF is perpendicular to FC.

Proposed by ....

Solution 1. Let r be the inradius and r_A the exradius of triangle ABC. Let also I_A be the A-excenter of triangle ABC. It is well-known that

$$\frac{r}{r_A} = \frac{DI}{DI_A} = \frac{AI}{AI_A}\,.$$

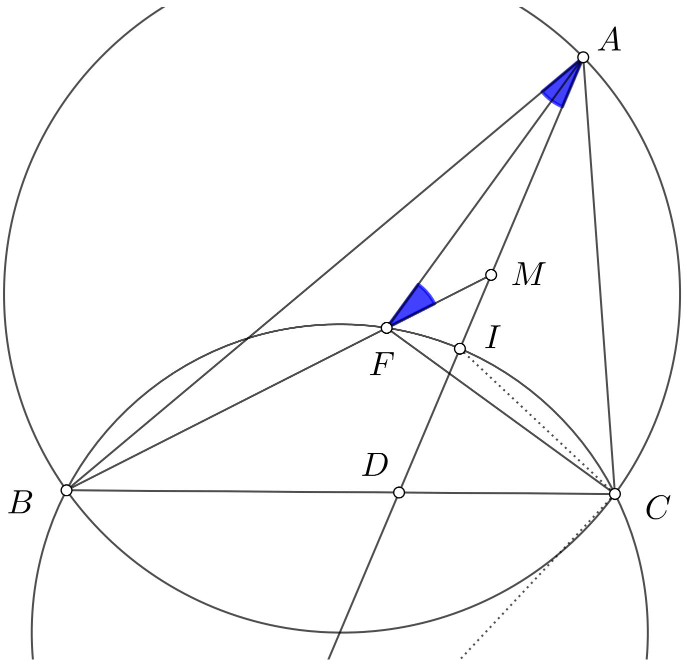
Since M is the midpoint of AD we then have

$$\frac{DI}{DI_A} = \frac{AI}{AI_A} \implies \frac{AM - MI}{MI_A - AM} = \frac{AM + MI}{MI_A + AM} \implies MA^2 = MI \cdot MI_A\,.$$

It is well-known that the A-excenter I_A belongs to the circumcircle of BIC. So by the power of the point M we also have $MI \cdot MI_A = MF \cdot MB$, therefore $MA^2 = MF \cdot MB$ which gives that MA is tangent to the circumcircle of triangle AFB. Therefore $\angle AFM = \angle BAM = \hat{A}/2$ and so

$$\begin{aligned}\angle AFC = \angle AFM + \angle MFC &= \frac{1}{2}\hat{A} + 180^\circ - \angle BFC = \frac{1}{2}\hat{A} + 180^\circ - \angle BIC \\ &= \frac{1}{2}\hat{A} + 180^\circ - \left(180^\circ - \frac{1}{2}\hat{B} - \frac{1}{2}\hat{C}\right) = \frac{1}{2}\left(\hat{A} + \hat{B} + \hat{C}\right) = 90^\circ\,.\end{aligned}$$

Therefore $AF \perp FC$ as required.



Solution 2. In the triangle CDA the lines CI and CI_A are the internal and external angle bisectors, therefore the quadruple $(A, D; I, I_A)$ is harmonic. Since M is the midpoint of AD it follows (by Newton's relation for harmonic quadrilaterals) that $MA^2 = MI \cdot MI_A$. The result now follows as in Solution 1.

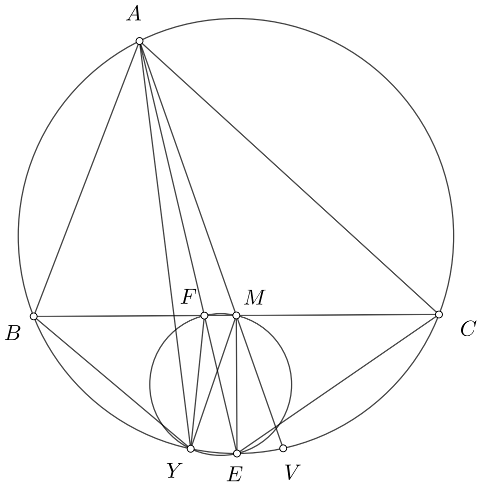
G3. Let ABC a triangle and let ω be its circumcircle. Let E be the midpoint of the minor arc BC of ω, and M the midpoint of (BC). Let V be the other point of intersection of AM with ω, F the point of intersection of AE with BC, X the other point of intersection of the circumcircle of FEM with ω, X' the reflection of V with respect to M, A' the foot of the perpendicular from A to BC and S the other point of intrsection of XA' with ω. If $Z \in \omega$ with $Z \neq X$ is such that $AX = AZ$, then prove that S, X' and Z are collinear.

Proposed by ....

Solution.

Claim 1. AX is the A-symmedian of $\triangle ABC$.

Prooof of Claim 1. Let $Y \in \omega$ such that AY is the A-symmedian of triangle ABC. We want to prove that $Y = X$.



We have that $\angle BAY = \angle CAM$ and $\angle BYA = \angle BCA = \angle MCA$, therefore the triangles ABY and AMC are similar. It follows that $(AY)(AM) = (AB)(AC)$.

Since AE is the bisector of $\angle BAC$, then $\angle BAF = \angle CAE$. We also have $\angle ABF = \angle ABC = \angle AEC$, therefore the triangles BAF and EAC are similar. It follows that $(AE)(AF) = (AB)(AC)$.

We get $(AY)(AM) = (AE)(AF)$ and since also $\angle YAF = \angle EAM$, then the triangles YAF and EAM are similar. So $\angle AFY = \angle AME$ and $\angle YFE = \angle EMV$. But as E is the midpoint of the arc YV, it follows that $\angle EMV = \angle YME$. So $\angle YFE = \angle YME$ from which it follows that the quadrilateral $YFME$ is cyclic. But since $Y \in \omega$, we finally get that $Y = X$. □

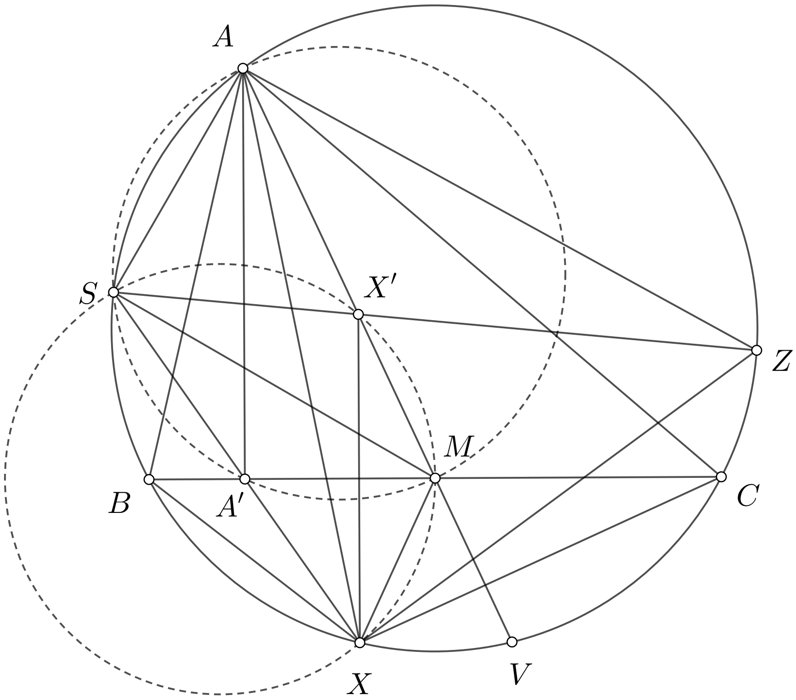
From Claim 1 we conclude that the triangles XBC and VCB are equal. Thus $MX = MV = MX'$. So the triangle $X'XV$ is a right-angled triangle and $X'X$ is perpendicular to XV and therefore also to BC. Thus X' is the reflection of X on BC.

Claim 2. The quadrilateral $ASA'M$ is cyclic.

Proof of Claim 2. We have $\angle ASA' = \angle ASX = \angle ABX$ But from Claim 1 we also have $\angle ABX = \angle AMC$. So $\angle ASA' = \angle AMC$ and the result follows. □

Claim 3. The quadrilateral $XSX'M$ is cyclic.

Proof of Claim 3. From Claim 2 we have $\angle XSM = \angle A'SM = \angle A'AM$. Since XX' is parallel to AA' we have $\angle A'AM = \angle XX'M$. So $\angle XSM = \angle XX'M$ and the result follows. □



Now from Claim 3 we have

$$\angle XSX' = \angle XMV = \angle XX'M + \angle X'XM = 2\angle XX'M = 2\angle AA'M\,.$$

So to conclude the proof it is enough to also show that $\angle XSZ = 2\angle AA'M$. From Claim 1 we have $\angle ACX = \angle MAB$ and therefore

$$\angle AZX = \angle ACX = \angle AMB = 90^\circ - \angle A'AM\,.$$

Since the triangle XAZ is isosceles, we deduce that

$$\angle XSZ = \angle XAZ = 180^\circ - 2\angle AZX = 2\angle A'AM$$

thus completing the proof.

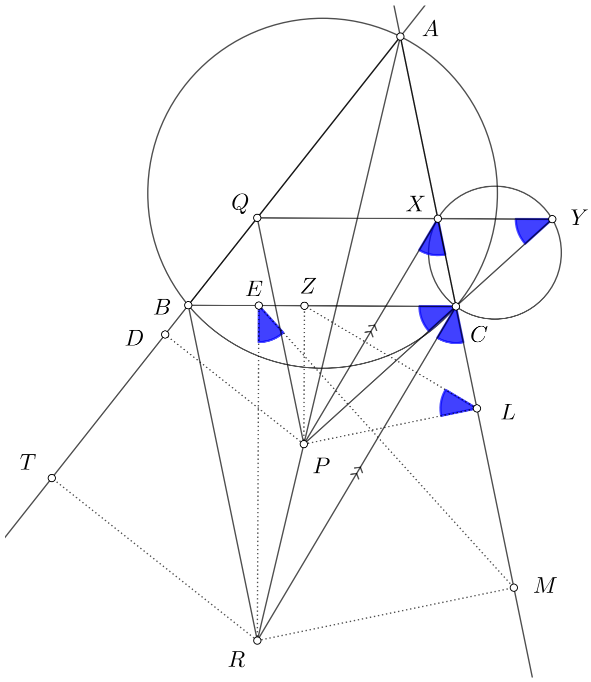
Notes.

1. Claim 1 is Problem 9.2/10.2 from the All Russian Mathematical Olympiad of 2009. There are various ways to prove this but we only added one proof here.
2. There are various other results which can be helpful towards a proof. These include the following:
   (a) If H is the orthocenter of the triangle ABC, then the quadrilateral $A'MX'H$ is cyclic.
   (b) The points H, M, S are collinear.
   (c) The quadrilateral $AX'HS$ is inscribed in a circle with diameter AH.
   (d) If P is the intersection of AS with BC and K is the intersection of AA' with SM then the quadrilateral $PSKA'$ is cyclic. (In fact K is the orthocenter of the triangle ABC.)
   (e) The quadrilateral $PX'MX$ is cyclic.

G4. Let ABC be a triangle and let the tangent at B to its circumcircle meet the internal bisector of angle A at P. The line through P parallel to AC meets AB at Q. Assume that Q lies in the interior of segment AB and let the line through Q parallel to BC meet AC at X and PC at Y. Prove that PX is tangent to the circumcircle of triangle XYC.

Proposed by ....

Solution. Let R be the point on AP such that BR is parallel to AC. Let D, Z, L be the projections of P on AB, BC, AC respectively and let T, E, M be the analogous projections of R.



Since BP is tangent to the circumcircle and $BR \parallel AC$, we have $\angle PBZ = \angle BAC = \angle TBR$. It follows that the right-angled triangles RTB and PZB are similar and therefore $\frac{PZ}{RT} = \frac{PB}{RB}$.

Analogously the triangles REB and PDB are also similar and therefore $\frac{PD}{RE} = \frac{PB}{RB}$.

Since P, R belong on the bisector of A, we have that $PD = PL$ and $RT = RM$ so from the results of the previous two paragraphs we get that $\frac{PL}{RE} = \frac{PZ}{RM}$.

The quadrilaterals $ZPLC$ and $ERMC$ are cyclic, therefore $\angle ZPL = 180^\circ - \angle ECL = \angle ERM$. Together with the previous result we get that the triangles ZPL and MRE are similar. Using this together with properties of cyclic qudrilaterals and the fact that $BC \parallel XY$ we get that

$$\angle XYC = \angle ZCP = \angle ZLP = \angle MER = \angle MCR$$

Since $BR \parallel AC \parallel QP$ and $QX \parallel BC$ we get

$$\frac{AP}{PR} = \frac{AQ}{QB} = \frac{AX}{XC}$$

which implies that $XP \parallel CR$. Thus $\angle PXC = \angle RCM = \angle XYC$. So the result follows.

Comment. Since R belongs on AP, BP is tangent to the circumcircle of ABC, and BR is parallel to AC, then R is the isogonal conjugate of P. This gives the equality $\angle ZCP = \angle MCR$ providing a shorter proof of the result.

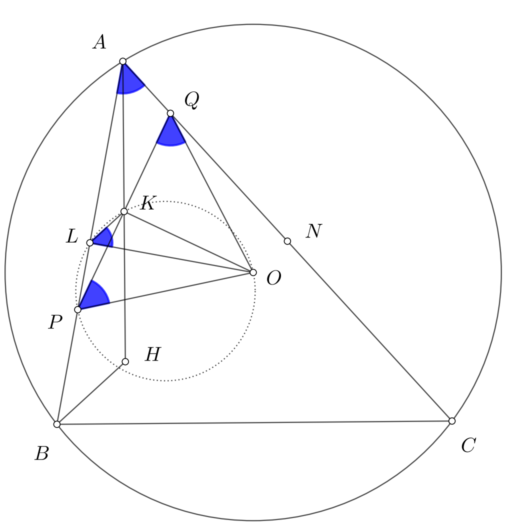
G5. Let ABC be a triangle with circumcircle ω, circumcenter O, and orthocenter H. Let K be the midpoint of AH. The perpendicular to OK at K intersects AB and AC at P and Q, respectively. The lines BK and CK intersect ω again at X and Y, respectively. Prove that the second intersection of the circumcircles of triangles KPY and KQX lies on ω.

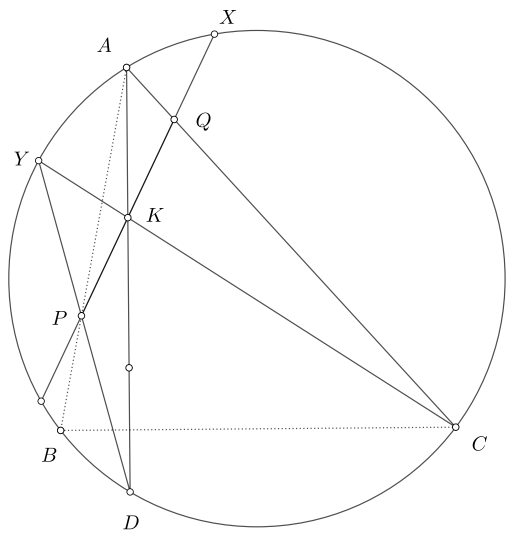
Proposed by ....

Solution.

Claim 1. $PK = KQ$.

Proof of Claim 1. Let L and N be the midpoints of AB and AC, respectively. Since L and K are midpoints of AB and AH, then $LK \parallel BH$ and so $LK \perp AC$. Since also $LO \perp AB$, then $\angle KLO = \angle BAC = \alpha$. Also, $\angle OLP = 90^\circ = \angle OKP$, so the quadrilateral $OKLP$ is cyclic and therefore $\angle KPO = \angle KLO = \alpha$. Similarly, $\angle KQO = \alpha$. Therefore, the triangle OPQ is isosceles, and since $OK \perp PQ$ then K is the midpoint of PQ. $\square$





Claim 2. The intersection of YP and AK lies on ω.

Proof of Claim 2. Let D be the other point of intersection of AK with ω. Since $OK \perp PQ$, then K is the midpoint of the chord ℓ of ω through P, Q. Since AD and YC intersect at K, by the Butterfly theorem the points $P' = YD \cap \ell$ and $Q = AC \cap \ell$ are equidistant from K. Thus $P' = P$ and $YP \cap AK = D \in \omega$. $\square$

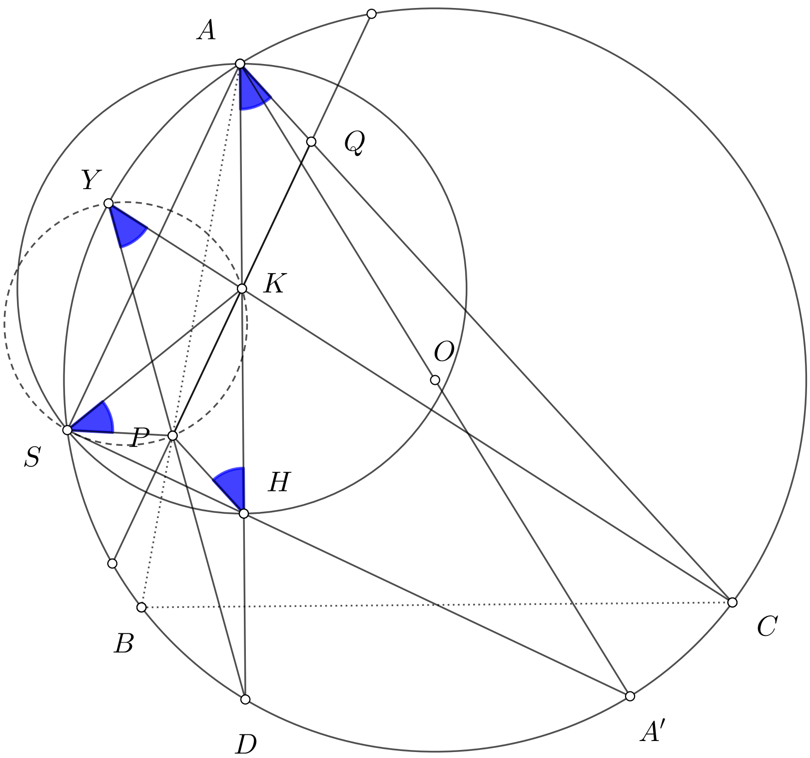
Now let A' be the other point of intersection of AO with ω and let S be the other point of intersection of $A'H$ with ω.

Claim 3. PQ is the perpendicular bisector of HS.

Proof of Claim 3. We have $\angle HSA = \angle A'SA = 90^\circ$ so S lies on the circle ω' with diameter AH centered at K. So $KS = KH$. Since O and K are the circumcenters of ω and ω' respectively, and AS is their common chord, then $OK \perp AS$. But we also have $HS \perp AS$ and $PQ \perp OK$, thus $HS \perp PQ$. Since $KS = KH$, then K belongs on the perpendicular bisector of HS and the result follows. $\square$

From Claim 3 we have $\angle KSP = \angle KHP$. From Claim 1 and the fact that $KP = KQ$ we have that $AQPH$ is a parallelogram and so (using Claim 2 as well)

$$\angle KHP = \angle KAC = \angle DAC = \angle DYC = \angle PYK\,.$$



Since $\angle KSP = \angle KYP$ we get that S belongs on the circumcircle of triangle KPY. Similarly it belongs to the circumcircle of triangle KQX and therefore the result follows.

Note. We can also define S to be the reflection of H on PQ. Now Claim 3 is immediate. Then $AS \parallel PQ$ and so $OK \perp AS$ and $OK \parallel HS$. Since K is the midpoint of AH, then OK is the perpendicular bisector of AS and therefore $S \in \omega$.

G6. Let ABC be a triangle with $AB < AC$ and let D be the other intersection point of the angle bisector of A with the circumcircle of triangle ABC. Let E and F be points on the sides AB and AC respectively, such that $AE = AF$ and let P be the point of intersection of AD and EF. Let M be the midpoint of BC. Prove that AM and the circumcircles of triangles AEF and PMD pass through a common point.

Proposed by ....

Solution 1. Let X the other point of intersection of the circumcircles of the triangles AEF and ABC. We have

$$\angle EXF = \angle EAF = \angle BAC = \angle BXC$$

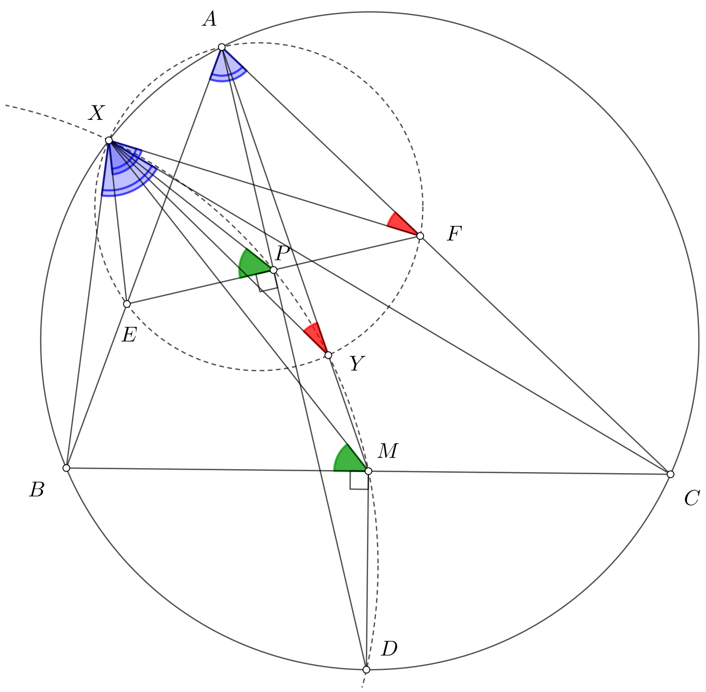
and

$$\angle XFE = \angle XAB = \angle XCB\,,$$

so the triangles BXC and EXF are similar. Since P is the midpoint of the segment EF, and M is the the midpoint of the segment BC, we conclude that and the triangles EXP and BXM are also similar. Therefore, $\angle XPE = \angle XMB$ and so

$$\angle XPD = \angle XPE + 90^\circ = \angle XMB + 90^\circ = \angle XMD\,.$$

Thus the points X, P, M, D are concyclic.



Let Y be the second intersection point of the circumcircle of the triangle AEF and the circle passing through the points X, P, M, D. We will prove that AM passes through Y. Since $\angle AYX = \angle AFX$, it is enough to prove that $\angle XYM = \angle XFC$.

We have

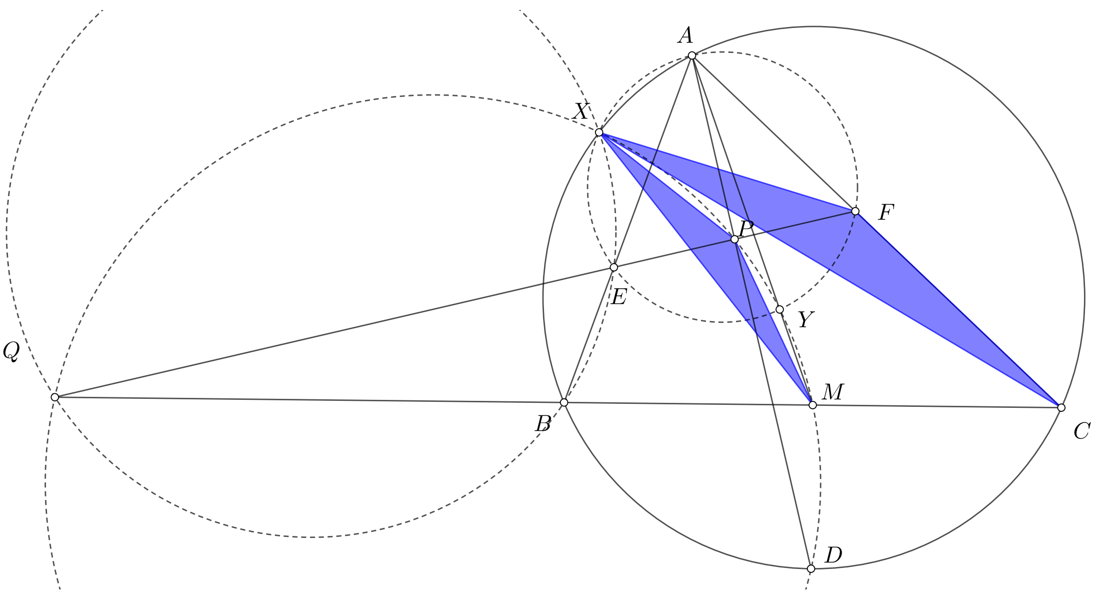
$$\begin{aligned}\angle XYM = \angle XPM = 180^\circ - \angle XDM &= 180^\circ - (\angle BDM - \angle BDX)\\ &= 180^\circ - \frac{1}{2}\angle BDC + \angle BAX = 180^\circ - \frac{1}{2}(180^\circ - \angle BAC) + \angle EFX\\ &= 90^\circ + \angle PAF + \angle EFX = 180^\circ - \angle AFX = \angle XFC\,.\end{aligned}$$

Solution 2. Let Q be the intersection of EF and BC. We have

$$\angle DMQ = \angle DMB = 90^\circ = \angle DPE = \angle DPQ$$

so the quadrrilateral $DMPQ$ is cyclic.

Let X the other point of intersection of the circumcircles of the triangles AEF and ABC. Consider the spiral similarity f_1 which maps BC to EF. Since $A = BE \cap CF$, then the center of f_1 is the second intersection point of the circumcircles of triangles ABC and AEF, i.e. it is the point X. Since M and P are the midpoints of BC and EF, then f_1 maps BM to EP. Since $Q = BM \cap EP$, then the center X of f_1 is the second intersection point of the circumcircles of triangles QBE and QMP. Therefore, we conclude that X, P, M, D, Q are concyclic.



Let Y be the second intersection point of the circumcircles of the quadrilaterals $(AEFX)$ and $(PMDX)$. We will prove that AM passes through Y.

Since spiral similarities come in pairs and f_1 maps MC to PF, there exists another spiral similarity f_2, with the same center X, mapping MP to CF. Therefore, the triangles XMP and XCF are similar and so $\angle XPM = \angle XFC$. We now have

$$\angle AYM = \angle AYX + \angle XYM = \angle AFX + \angle XPM = \angle AFX + \angle FXC = 180^\circ .$$

So $Y \in AM$ as required.

NUMBER THEORY

N1. Let n be a positive integer. What is the smallest sum of digits of $5^n + 6^n + 2022^n$?

Proposed by ....

Solution 1. We will prove that the smallest sum is equal to 8. One case when it is achieved is for $n = 1$. Suppose that for some $n > 1$ it is possible to obtain a smaller sum of 8. Observing the last digit of the number $5^n + 6^n + 2022^n$, we can easily conclude that

$$5^n + 6^n + 2022^n \equiv \begin{cases} 7 \bmod 10, & \text{if } n \equiv 0 \bmod 4 \\ 3 \bmod 10, & \text{if } n \equiv 1 \bmod 4 \\ 5 \bmod 10, & \text{if } n \equiv 2 \bmod 4 \\ 9 \bmod 10, & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

It follows that $n \equiv 1, 2 \bmod 4$. We now consider these two cases.

Case 1: If $n = 4k + 1$, then

$$5^n + 6^n + 2022^n \equiv \begin{cases} 5 \bmod 9, & \text{if } k \equiv 0 \bmod 3 \\ 2 \bmod 9, & \text{if } k \equiv 1 \bmod 3 \\ 8 \bmod 9, & \text{if } k \equiv 2 \bmod 3 \end{cases}$$

From here, due to the last digit being equal to 3, it is only possible for the sum of the digits to be equal to 5. Since $5^n + 6^n + 2022^n \equiv 1 \bmod 4$, the penultimate digit must be equal to 1, and all other digits (except the first) are equal to 0. So the last digits of $5^n + 6^n + 2022^n$ are 0013 and therefore $5^n + 6^n + 2022^n \equiv 13 \bmod 16$. On the other hand, since $n > 1$ and $n \equiv 1 \bmod 4$ we have $5^n + 6^n + 2022^n \equiv 5 \cdot 5^{4k} \equiv 5 \bmod 16$, a contradiction.

Case 2: If $n = 4k + 2$, then

$$5^n + 6^n + 2022^n \equiv \begin{cases} 7 \bmod 9, & \text{if } k \equiv 0 \bmod 3 \\ 1 \bmod 9, & \text{if } k \equiv 1 \bmod 3 \\ 4 \bmod 9, & \text{if } k \equiv 2 \bmod 3 \end{cases}$$

Due to the last digit, the only possibility is $k \equiv 0 \bmod 3$ so $n = 12s+2$ for some $s \in \mathbb{N}_0$. Now the sum of the digits is 7, and the last digit is 5. Let's use the divisibility criterion with 37. (A number when divided by 37 gives the same remainder as the sum of its three-digit blocks that are formed from right to left.) Since $6^n + 5^n + 2022^n$ must be equal to one of $20\cdots05$ or $10\cdots010\cdots05$, according to othe above criterion $6^n+5^n+2022^n \equiv 205, 25, 7, 115, 106, 16 \bmod 37$. I.e. congruent to $20, 25, 7, 4, 32, 16 \bmod 37$.

On the other hand, since it is easy to check that $6^{12} \equiv 1 \bmod 37, 5^{12} \equiv 10 \bmod 37$ and $2022^{12} \equiv 24^{12} \equiv 10 \bmod 37$ we have that

$$5^n + 6^n + 2022^n \equiv 25 \cdot 10^2 + 36 + 24^2 \cdot 10^2 \equiv 36 + 9 \cdot 10^s \equiv \begin{cases} 8 \bmod 37, & \text{if } s \equiv 0 \bmod 3 \\ 15 \bmod 37, & \text{if } s \equiv 1 \bmod 3 \\ 11 \bmod 37, & \text{if } s \equiv 2 \bmod 3 \end{cases}$$

This is a contradiction.

Solution 2. Let $A_n = 5^n + 6^n + 2022^n$. We have $A_n \equiv 1 \bmod 4$. For $n \equiv 1, 2, \ldots, 5 \bmod 5$ we have $6^n \equiv 6, 11, 16, 21, 1 \bmod 25$ and for $n \equiv 1, 2, \ldots, 20 \bmod 20$ we have

$$2022^n \equiv -3, 9, -2, 6, 7, 4, 13, 11, 17, -1, 3, -9, 2, -6, -7, -4, -13, -11, -17, 1 \bmod 25$$

Thus for $n > 1$ and $n \equiv 1, 2, \ldots, 20 \bmod 20$ we have

$$A_n \equiv 3, 20, 14, 27, 8, 10, 24, 27, 38, 0, 9, 2, 18, 15, -6, 2, -2, 5, 4, 2 \bmod 25$$

So for $n > 1$ and $n \equiv 1, 2, \ldots, 20 \bmod 20$ we have

$$A_n \equiv 53, 45, 89, 77, 33, 85, 49, 77, 13, 25, 9, 77, 93, 65, 69, 77, 73, 5, 29, 77 \bmod 100$$

The only possibilities for the sum of the digits of A_n to be less than 8 are $n \equiv 5, 9, 18 \bmod 20$ where the last two digits of A_n are $33, 13, 05$ respectively. (The case $n \equiv 10 \bmod 20$ is rejected since then we must have $A_n = 25$ which is impossible.)

Now for $n > 1$ and $n \equiv 1, 2, \ldots, 6 \bmod 6$ we have $A_n \equiv 5, 7, 8, 4, 2, 1 \bmod 9$. Looking modulo 60, the only possibilities are $n \equiv 5, 9, 18, 25, 29, 38, 45, 49, 58 \bmod 60$ and in those cases the the last two digits of A_n are $33, 13, 05, 33, 13, 05, 33, 13, 05$ respectively and the sum of digits of A_n are $2, 8, 1, 5, 2, 7, 8, 5, 4 \bmod 9$ respectively.

So the only possibilites are $n \equiv 38, 49 \bmod 60$ where the last two digits of A_n are 05 and 13 respectively, and the sum of digits of A_n are 7 and 5 respectively.

The only possibilities are therefore for A_n to be equal to a number of the form $20 \cdots 05$ or $10 \cdots 010 \cdots 05$ or $10 \cdots 013$. In particular, using the alterrnating sum of digits criterion for divisibility by 11 we have that the first number is congruent to $3, 7 \bmod 11$, the second congruent to $3, 5, 7 \bmod 11$ and the third congruent to $1, 3 \bmod 11$.

We now look at $A_n \bmod 11$. It can be easily checked that for $n \equiv 8 \bmod 10$ we have

$$A_n \equiv 4 + 4 + 3 \equiv 0 \bmod 11$$

and for $n \equiv 9 \bmod 10$ we have

$$A_n \equiv 9 + 2 + 5 \equiv 5 \bmod 11$$

So all three cases lead to a contradiction.

N2. Let a, b, n be positive integers such that:

(i) $a^{2021} \mid n$ and $b^{2021} \mid n$

(ii) $2022 \mid a-b$ and $a > b$.

Prove that there is a subset of the divisors of the number n having sum of elements divisible by 2022 but not by 2022^2.

Proposed by ....

Solution 1. Write $a = dr$ and $b = ds$ where $d = \gcd(a, b)$ and $(r, s) = 1$. Then $d^{2021}r^{2021}s^{2021}$ divides n. Furthermore $2 \cdot 3 \cdot 337 \mid d(r-s)$.

Case 1: Assume $337|d$. Since $(r, s) = 1$, we may assume that r is odd. Then

$$\{337r^2, 337r^4, \ldots, 337r^{10}, 337r^{12}\}$$

works. Indeed each of these six divisors of n is congruent to 1 mod 4 so their sum is a multiple of 2 but not of 4. If $3 \mid r$ then the sum is 0 mod 3 while if $3 \nmid r$ then each of these divisors is 1 mod 3 so the sum is again 0 mod 3. Therefore the sum is a multiple of 2022 but not of 2022^2.

Case 2: Assume $337 \nmid d$. Then $337|r-s$ and since $(r, s) = 1$ then $337 \nmid rs$. Consider the 2022^2 divisors of n of the form $r^k s^\ell$ where $k, \ell \in \{0, 1, 2, \ldots, 2021\}$. Since none of them is a multiple of 337, they have at most $2 \cdot 3 \cdot 336$ distinct remainders modulo 2022. Therefore at least $\frac{2022^2}{2 \cdot 3 \cdot 336} > 2022$ of them have the same remainder modulo 2022.

Pick 2023 out of those, say $d_1, d_2, \ldots, d_{2023}$. Let S be their sum. We claim that there is a subset of 2022 of them that will work. Note that the sum of any such subset is a multiple of 2022. It is enough to show that there is such a subset whose sum is not divisible by 337^2. If this is not the case then $S - d_i \equiv 0 \bmod 337^2$ for each $i = 1, 2, \ldots, 2023$. In particular, all d_i are congruent mod 337^2. Say that $d_i \equiv k \bmod 337^2$ for each i. Then $337 \nmid k$ and so the sum of any 2022 of them is congruent to $2022k \not\equiv 0 \bmod 337^2$.

Solution 2. We start with the following claim:

Claim. If k is a positive integer, then $a^k b^{2021-k} \mid n$.

Proof of the Claim. We have that $n^{2021} = n^k \cdot n^{2021-k}$ is divisible by $a^{2021k} \cdot b^{2021(2021-k)}$ and taking the 2021-root we get the desired result. □

Back to the problem, we will prove that the set $T = \{a^k b^{2021-k},\ k \geqslant 0\}$ consisting of 2022 divisors of n, has the desired property. The sum of its elements is equal to

$$S = \sum_{k=0}^{2021} a^k b^{2021-k} \equiv \sum_{k=0}^{2021} a^{2021} \equiv 0 \bmod 2022\,.$$

On the other hand, the last sum is equal to $\frac{a^{2022}-b^{2022}}{a-b}$. We will prove that this is not divisible by 9. Indeed, if $3^t \mid\mid a-b$ then, since $3^1 \mid\mid 2022$, by the Lifting the Exponent Lemma, we have that $3^{t+1} \mid\mid a^{2022} - b^{2022}$. This implies that S is not divisible by 9, therefore, 2022^2 doesn't divide S.

N3. For every natural number x, let $P(x)$ be the product of the digits of the number x. Is there a natural number n such that the numbers $P(n)$ and $P(n^2)$ are non-zero squares of natural numbers, where the number of digits of the number n is equal to

(a) 2021

(b) 2022

Proposed by ....

Solution. The answers are affirmative in both cases.

(a) Take $n=\overline{\underbrace{33\cdots3}_{2019}68}$. Then $P(n)=(4\cdot3^{1010})^2$. Also,

$$\begin{aligned}
n^2&=\left(\frac{10^{2021}-1}{3}+35\right)^2=\frac{\left(10^{2021}+104\right)^2}{9}\\
&=\frac{10^{4042}+208\cdot10^{2021}+10816}{9}\\
&=\frac{10^{4042}-10^{2021}}{9}+209\cdot\frac{10^{2021}-1}{9}+1225\\
&=\overline{\underbrace{1\cdots1}_{2021}\underbrace{0\cdots0}_{2021}}+\overline{\underbrace{2\cdots2}_{2021}00}+\overline{\underbrace{9\cdots9}_{2021}}+1225\\
&=\overline{\underbrace{1\cdots1}_{2019}33\underbrace{2\cdots2}_{2019}00}+10^{2021}+1224\\
&=\overline{\underbrace{1\cdots1}_{2019}34\underbrace{2\cdots2}_{2017}3424}\,.
\end{aligned}$$

Thus $P(n^2)=(3\cdot2^{2012})^2$.

(b) Take $n=\overline{11\underbrace{33\cdots3}_{2020}}$. Then $P(n)=(3^{1010})^2$. Also,

$$\begin{aligned}
n^2&=\frac{\left(34\cdot10^{2020}-1\right)^2}{9}=\frac{1156\cdot10^{4040}-68\cdot10^{2020}+1}{9}\\
&=128\cdot10^{4040}+\frac{4(10^{4040}-10^{2020})}{9}-7\cdot10^{2020}-\frac{10^{2020}-1}{9}\\
&=\overline{128\underbrace{0\cdots0}_{4040}}+\overline{\underbrace{4\cdots4}_{2020}\underbrace{0\cdots0}_{2020}}-7\cdot10^{2020}-\overline{\underbrace{1\cdots1}_{2020}}\\
&=\overline{128\underbrace{4...4}_{2018}36\underbrace{8...8}_{2019}9}\,.
\end{aligned}$$

Thus $P(n^2)=(9\cdot2^{5049})^2$.

Remark. In part (a) we can also take $n=\underbrace{66\cdots66}_{2020}1$ or $n=1\underbrace{33\cdots33}_{2018}28$.

N4. A hare and a tortoise run in the same direction, at constant but different speeds, around the base of a tall square tower. They start together at the same vertex, and the run ends when both return to the initial vertex simultaneously for the first time. Suppose the hare runs with speed 1, and the tortoise with speed less than 1. For what rational numbers x is it true that, if the tortoise runs with speed x, the fraction of the entire run for which the tortoise can see the hare is also x?

Proposed by ....

Solution. Suppose that $x = \frac{p}{q}$ where p, q are positive integers with $p < q$ and $\gcd(p, q) = 1$. Suppose that the hare takes p minutes for a full turn about the tower. Then the tortoise takes q minutes for a full turn. They will meet again at the same vertex pq minutes when the hare will make q full turns and the tortoise will make p full turns. In particular, the hare will overtake the tortoise exactly $k = q - p$ times taking into account the start of the race but not the end of the race as an overtake.

The overtakes should occur at minutes $0, \frac{pq}{k}, \frac{2pq}{k}, \ldots, \frac{(k-1)pq}{k}$. In those minutes the tortoise would have made $0, \frac{p}{k}, \frac{2p}{k}, \ldots, \frac{(k-1)p}{k}$ full turns about the tower. Since $(p, q) = 1$, then $(p, k) = 1$ and therefore at these meeting points the tortoise would have in some order made some full turns about the tower plus another $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$ fraction of a full turn.

Case 1: Suppose k is odd. We claim that the $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$ fractions of a full turn correspond, in some order to $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$ fractions of a side. To see this, given $i = 0, 1, \ldots, k-1$, note that if $\frac{j-1}{4} \leqslant \frac{i}{k} < \frac{j}{4}$ for some $j = 1, 2, 3, 4$ then the meeting point is on the j-th side at a fraction of $4\left(\frac{i}{k} - \frac{j-1}{4}\right) = \frac{4i-k(j-1)}{k}$ of the side. No two such fractions can be equal. Indeed if

$$\frac{4i - k(j-1)}{k} = \frac{4i' - k(j'-1)}{k}$$

then $4(i' - i) = k(j' - j)$ and since (for $i' > i$ say) $j' - j \in \{1, 2, 3\}$, then $2|k$, a contradiction.

Now if the hare meets the tortoise at a fraction of $\frac{i}{k}$ of the side, then the tortoise can see the hare for $\frac{k-i}{k} \cdot \frac{p}{4}$ minutes. I.e. the time it takes the hare to reach the endpoint of the side. Furthermore, if i is large enough, it is possible for the tortoise to also reach the endpoint before the hare reaches the next endpoint and thus see the hare for a little bit more. The tortoise takes $\frac{k-i}{k} \cdot \frac{q}{4}$ minutes to reach the endpoint. The hare takes $\frac{2k-i}{k} \cdot \frac{p}{4}$ minutes in total to reach the next endpoint. So the tortoise can see the hare for another

$$\frac{(2k-i)p - q(k-i)}{4k} = \frac{p}{4} + \frac{(p-q)(k-i)}{4k} = \frac{p+i-k}{4}$$

minutes, provided that this is non-negative. So the total meeting time is

$$\frac{p}{4}\left(\frac{1}{k} + \frac{2}{k} + \cdots + \frac{k}{k}\right) + \frac{1}{4}\left(1 + 2 + \cdots + (p-1)\right) = \frac{p(k+1)}{8} + \frac{(p-1)p}{8} = \frac{pq}{8}$$

minutes. So we need $x = \frac{1}{8}$ which is accepted as $k = 7$ is odd in this case.

Case 2: Suppose $k = 2r$ where r is odd. We claim that the $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$ fractions of a full turn correspond, in some order to $0, 0, \frac{1}{r}, \frac{1}{r}, \ldots, \frac{r-1}{r}, \frac{r-1}{r}$ fractions of a side. The proof is similar to Case 1 with the meeting points being at fractions $\frac{4i-k(j-i)}{k} = \frac{2i-r(j-i)}{r}$ of the sides. Two of these fractions are equal if and only if $4(i' - i) = k(j' - j)$ which (for $i' > i$ say) can occur when $j' = j + 2$ and $i' = i + r$.

If they meet at a fraction of $\frac{i}{r}$ of the side, the tortoise meets that hare for $\frac{r-i}{r} \cdot \frac{p}{4}$ minutes plus possibly another

$$\frac{(2r-i)p - q(r-i)}{4r} = \frac{p}{4} + \frac{(p-q)(r-i)}{4r} = \frac{p+2i-2r}{4}$$

minutes provided this is non-negative. So (noting that p is odd in this case) the tortoise can see the hare for

$$\frac{2p}{4}\left(\frac{1}{r} + \frac{2}{r} + \cdots + \frac{r}{r}\right) + \frac{2}{4}\left(1 + 3 + \cdots + (p-2)\right) = \frac{p(r+1)}{4} + \frac{(p-1)^2}{8}$$

minutes. This is equal to

$$\frac{p(2r+2) + (p-1)^2}{8} = \frac{p(q-p+2) + (p-1)^2}{8} = \frac{pq+1}{8}$$

minutes. So we need

$$\frac{p}{q} = x = \frac{1}{8} + \frac{1}{8pq} \implies 8p^2 = pq + 1 \implies p = 1, q = 7\,.$$

Thus $x = \frac{1}{7}$ which is accepted since $k = 6 \equiv 2 \bmod 4$.

Case 3: Suppose $k = 4s$. Similarly to Cases 1 and 2, the $0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}$ fractions of a full turn correspond, in some order to $0, 0, 0, 0, \frac{1}{s}, \frac{1}{s}, \frac{1}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}, \frac{s-1}{s}, \frac{s-1}{s}, \frac{s-1}{s}$ fractions of a side.

If they meet at a fraction of $\frac{i}{s}$ of the side, the tortoise meets that hare for $\frac{s-i}{s} \cdot \frac{p}{4}$ minutes plus possibly another

$$\frac{(2s-i)p - q(s-i)}{4r} = \frac{p}{4} + \frac{(p-q)(s-i)}{4s} = \frac{p+4i-4r}{4}$$

minutes provided this is non-negative. Noting that p is odd in this case, if $p \equiv 1 \bmod 4$ this is equal to

$$\begin{aligned} p\left(\frac{1}{s} + \frac{2}{s} + \cdots + \frac{s}{s}\right) + (1 + 5 + \cdots + (p-4)) &= \frac{p(s+1)}{2} + \frac{(p-3)(p-1)}{8} \\ = \frac{p(q-p+4) + (p-3)(p-1)}{8} = \frac{pq+3}{8} \end{aligned}$$

minutes, while if $p \equiv 3 \bmod 4$ this is equal to

$$p\left(\frac{1}{s} + \frac{2}{s} + \cdots + \frac{s}{s}\right) + (3 + 7 + \cdots + (p-4)) = \frac{p(s+1)}{2} + \frac{(p-1)(p-3)}{8} = \frac{pq+3}{8}$$

minutes as well. So we need

$$\frac{p}{q} = x = \frac{1}{8} + \frac{3}{8pq} \implies 8p^2 = pq + 3 \implies p|3$$

This gives the solutions $p = 1, q = 5$ and $p = 3, q = 23$ giving $x = \frac{1}{5}$ and $x = \frac{3}{23}$ which are both accepted.

Solution 2. If we run the process in reverse, the dynamics are the same except that the toirtoise can see the hare at some time in the reversed process precisely if the hare could see the tortoise at the same time in the original process. From this observation, the proportion of the race for which the tortoise can see the hare is precisely half the proportion of the race for which the two runners are on the same side of the square. It suffices to show that this proportion is:

(a) $\frac{1}{4}$, when $p-q$ is odd;

(b) $\frac{1}{4}+\frac{1}{4pq}$ when $p-q$ is even but not divisible by 4;

(c) $\frac{1}{4}+\frac{3}{4pq}$ when $4 \mid p-q$.

The proof can then be completed exactly as in Solution 1 to get that $x=\frac{1}{8},\frac{1}{7},\frac{1}{5},\frac{3}{23}$.

To streamline the argument, we assume that the square (always meaning the boundary) has side length pq units, and that in a *time-step*, the tortoise moves p units, and the hare moves q units. Note that a runner can only be at the vertex of the square at the start or end of a step. We say that a vertex of the square is on the side of the square that lies clockwise from the vertex, and we refer to that as the side's associated vertex. So every point on the square is on exactly one 'side'.

Now, we index all points on the square by their distance from the vertex associated to the side containing that point. So each label occurs exactly four times. So, after n steps of the process, we study the indices of T and H's current locations, which must have the form (ap, bq) for $a \in \{0,1,\ldots,q-1\}$ and $b \in \{0,1,\ldots,p-1\}$. Thus the total distances travelled by T and H have the forms

$$ap+mpq, \quad bq+m'pq, \text{ respectively}, \quad m,m' \in \mathbb{N}$$

which means that the number of steps n satisfies

$$n = a+mq = b+m'p\,. \tag{1}$$

T and H are on the same side of the square precisely when $4 \mid m'-m$ and are on the same side or on opposite sides of the square precisely when $2 \mid m'-m$, equivalently when $2 \mid m'+m$.

Now, after pq steps, both runners are again at a vertex of the square. We return to the case distinction introduced earlier.

(a) Here, the two vertices are adjacent. Therefore, for any time $0 \leqslant t < pq$, T and H are on the same side of the square at exactly one of the times $\{t, t+pq, t+2pq, t+3pq\}$. It follows that across the entire run, T and H will be on the same side exactly $\frac{1}{4}$ of the time.

(c) Here, the two vertices are the same. It then suffices to study the proportion of times the runners are on the same side before timestep pq. Note that by the Chinese Remainder Theorem, every $(a,b) \in [0,q-1] \times [0,p-1]$ occurs exactly once as the indexing of the runners' locations (ap,bq) for $n=0,\ldots,pq-1$. But, from (1),

$$a-b = m'p-mq \equiv (m'-m)p \bmod 4.$$

Since p is odd, $4 \mid m'-m$ precisely if $4 \mid a-b$. So it suffices to enumerate

$$K(p,q) := \left|\Big\{(a,b) \in [0,q-1]\times[0,p-1] \,:\, 4 \mid a-b\Big\}\right|.$$

By considering the number of times each congruence class appears for a and for b, we find, for $p \equiv q \equiv 1$:

$$K(p,q) = \frac{p+3}{4}\times\frac{q+3}{4}+3\left(\frac{p-1}{4}\times\frac{q-1}{4}\right),$$

and for $p \equiv 1$, $q \equiv 3$,

$$K(p,q) = \frac{p+3}{4}\times\frac{q+1}{4}+2\left(\frac{p-1}{4}\times\frac{q+1}{4}\right)+\frac{p-1}{4}\times\frac{q-3}{4}\,.$$

In both cases, a calculation shows $\frac{K(p,q)}{pq}=\frac{1}{4}+\frac{3}{4pq}$, with an obvious symmetric argument for $p \equiv 3$, $q \equiv 1$.

(b) We combine aspects of the two previous cases. Here, the two vertices are opposite. Therefore, for any time $0 \leqslant t < pq$, T and H are on the same side or on opposite sides of the square at time t iff they are on the same side or on opposite sides of the square at time $t + pq$. Furthermore, on this event, they are on the same side at precisely one of times $\{t, t + pq\}$.

So we calculate the proportion of times the runners are on the same side or opposite sides before timestep pq. Again from (1), using $p \equiv -q$ modulo 4,

$$a - b \equiv m'p - mq \equiv (m' + m)p \bmod 4\,.$$

Since p is odd, $2 \mid m' + m$ precisely if $2 \mid a - b$. We enumerate

$$\begin{aligned} L(p,q) :&= \left|\left\{(a,b) \in [0, q-1] \times [0, p-1] \,:\, 2 \mid a - b\right\}\right| \\ &= \frac{p+1}{2} \times \frac{q+1}{2} + \frac{p-1}{2} \times \frac{q-1}{2}. \end{aligned}$$

So $\frac{L(p,q)}{2pq} = \frac{1}{4} + \frac{1}{4pq}$ as required.

Cyprus Mathematical Society

36 Stasinou street, Office 102

Strovolos 2003, Nicosia, Cyprus

Tel. +357 – 22378101, Fax +357 – 22379122

Email: info@cms.org.cy, cms@cms.org.cy

Website: www.cms.org.cy