Problems and solutions of the 13th European Mathematical Cup

14th December 2024 - 22nd December 2024

Junior Category

Problem 1. Wiske wrote a 2024-digit positive integer on the blackboard. In each round of the game she erases the last digit of the integer, let this digit be d, and writes down the sum of the remaining number and $2d$ in place of the old number. She repeats the same steps with the newly obtained number. After a certain number of rounds, Wiske found that the new number obtained was the same as the number in the last round and she stopped the game. What is the smallest possible 2024-digit integer that Wiske started with in this game?

(Kai Chen)

Solution. Firstly, we claim that the last number Wiske obtained is 19.

In the last round, let the last digit of the number be d where $0 \leqslant d \leqslant 9$, and the remaining digits form an integer x. The number at the beginning of this round is then $10x + d$, and the new number obtained in this round is $x + 2d$. Since the two numbers are the same, $10x + d = x + 2d$, i.e., $9x = d$.
Because $0 \leqslant d \leqslant 9$ and we cannot have $x = d = 0$ because all newly written numbers are positive, the only solution is $x = 1$ and $d = 9$. The last number is then 19.

2 points.

Secondly, we claim that the numbers in all previous rounds are divisible by 19. From $2(10x + d) = 19x + (x + 2d)$, it follows that $2(10x + d) \equiv x + 2d \ (mod\ 19)$. Since the last number is 19, it can be concluded by reverse induction that the numbers in all rounds of the game are divisible by 19.

3 points.

From the same induction we get that the number Wiske started with being divisible by 19 is a sufficient condition as well.

1 point.

Finally, our goal is to find the smallest 2024-digit number which is divisible by 19 because the sequence of the numbers in all rounds is strictly descending:

$$10x + d > x + 2d, \text{ if } x > 1.$$

Per the Fermat's Little Theorem, we get $10^{18} \equiv 1 \ (mod\ 19)$. We have,

$10^{2023} \equiv 10^7 \ (mod\ 19)$, because $2023 = 112 \times 18 + 7$.
$10^7 = 100^3 \times 10 \equiv 5^3 \times 10 = 1250 \equiv 15 \ (mod\ 19)$.
$10^{2023} + 4 \equiv 0 \ (mod\ 19)$.

3 points.

Thus, $10^{2023} + 4$ is the smallest 2024-digit number which is divisible by 19. It is the smallest possible 2024-digit integer that Wiske started with in the game.

1 point.

Problem 2. Let X be the largest possible value of the expression

$$\min\{bc, 2-a^2\} + \min\{ac, 2-b^2\} + \min\{ab, 2-c^2\},$$

where a, b and c are positive real numbers. Similarly, let Y be the smallest possible value of the expression

$$\max\{a^2, 2-bc\} + \max\{b^2, 2-ac\} + \max\{c^2, 2-ab\},$$

where a, b and c are positive real numbers. Prove that $X = Y$. *(Ognjen Tešić)*

First Solution. Observe that $\min\{bc, 2-a^2\} = 2 + \min\{bc-2, -a^2\} = 2 - \max\{-(bc-2), a^2\} = 2 - \max\{2-bc, a^2\}$, so $X = Y$ is equivalent to

$$\begin{aligned}
&\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = \max\left\{\sum_{cyc}\min\{bc, 2-a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = \max\left\{\sum_{cyc} 2-\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 6 + \max\left\{-\sum_{cyc}\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 6 - \min\left\{\sum_{cyc}\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 3
\end{aligned}$$

3 points.

For $a = b = c = 1$ we get that $\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} \leq 3$, so we need to show that

$$\sum_{cyc}\max\{a^2, 2-bc\} \geq 3$$

1 point.

but since $\max\{x, y\} \geq \frac{x+y}{2}$ we have that

2 points.

$$\sum_{cyc}\max\{a^2, 2-bc\} \geq \sum_{cyc}\frac{a^2+2-bc}{2} = 3 + \frac{1}{2}\left(\sum_{cyc}a^2 - \sum_{cyc}bc\right) = 3 + \frac{1}{2}\left(\sum_{cyc}\frac{b^2+c^2}{2} - \sum_{cyc}bc\right) \geq 3 + \frac{1}{2}\left(\sum_{cyc}bc - \sum_{cyc}bc\right) = 3$$

4 points.

Second Solution. An alternative proof of the last inequality. Suppose, for the sake of contradiction, that

$$\sum_{cyc}\max\{a^2, 2-bc\} < 3$$

but since $\min\{a^2, 2-bc\} \leq \max\{a^2, 2-bc\}$ we have that

$$\sum_{cyc}\min\{a^2, 2-bc\} \leq \sum_{cyc}\max\{a^2, 2-bc\} < 3$$

by adding the two inequalities we get

$$\sum_{cyc}\max\{a^2, 2-bc\} + \max\{a^2, 2-bc\} < 6$$

1 point.

now we use the fact that $\min\{x, y\} + \max\{x, y\} = x + y$

1 point.

to get that

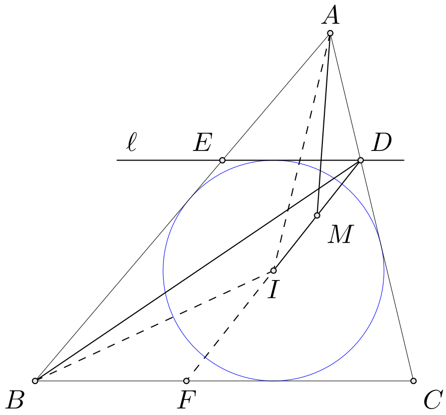
$$\sum_{cyc}a^2 + 2 - bc < 6,$$

contradiction by AG inequality as in the First Solution.

4 points.

Problem 3. Let ABC be a triangle with incenter I and incircle ω. Let ℓ be the tangent to ω parallel to BC and distinct from BC. Let D be the intersection of ℓ and AC, and let M be the midpoint of $\overline{ID}$. Prove that $\angle AMD = \angle DBC$.

(Weihua Wang)



First Solution. Let E be the intersection point of line ℓ with AB, and let F be the intersection point of DI with BC. Since $\ell \parallel BC$ and ℓ is tangent to circle ω, we have

$$\angle CDI = \angle EDI = \angle CFI = 90^\circ - \frac{C}{2},$$

$$\angle ADI = \angle BFI = 90^\circ + \frac{C}{2} = \angle AIB.$$

2 points.

Noting that $\angle DAI = \angle BAI$ and $\angle ABI = \angle FBI$, we obtain $\triangle ADI \sim \triangle AIB \sim \triangle IFB$. Therefore,

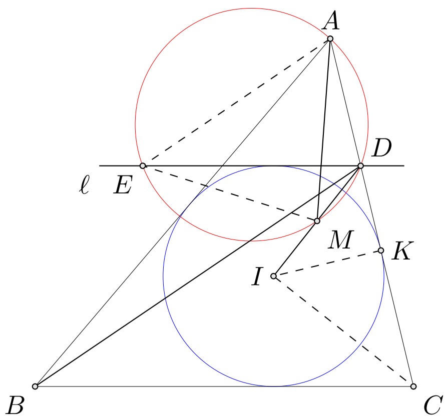
2 points.

$$\frac{AD}{DI} = \frac{IF}{FB} \Rightarrow \frac{AD}{DM} = \frac{2AD}{DI} = \frac{2IF}{BF} = \frac{DF}{BF}.$$

4 points.

Thus, $\triangle AMD \sim \triangle DBF$, so $\angle AMD = \angle DBC$.

2 points.



Second Solution. Let $BC = a$, $CA = b$, $AB = c$, and let the inradius of $\triangle ABC$ be r. Suppose ω is tangent to AC at K, and let line ℓ intersect the circumcircle of $\triangle ADM$ again at E. Let the area of $\triangle ABC$ be S. Then, from the formula

$$S = \frac{1}{2}(a+b+c) \cdot r = \frac{1}{2}ab\sin C,$$

we have

$$\frac{a+b+c}{a} = \frac{b\sin C}{r} \Rightarrow \frac{(b+c-a)/2 + a}{a} = \frac{b}{(2r)/\sin C},$$

2 points.

which implies

$$\frac{AK+BC}{BC}=\frac{AC}{CD}\Rightarrow\frac{AK}{BC}=\frac{AD}{CD}.$$

2 points.

Noting that $\angle AEM=\angle CDI=\angle EDI=\angle EAM=90^\circ-\frac{C}{2}$, we conclude $ME=MA$. Applying Ptolemy's theorem to quadrilateral $ADME$ and noting that $AE=2AM\cdot\sin\frac{C}{2}$, we get

$$AM\cdot DE=EM\cdot AD+AE\cdot DM=AM\cdot AD+2AM\cdot\sin\frac{C}{2}\cdot DM.$$

3 points.

Thus,

$$DE=AD+2DM\cdot\sin\frac{C}{2}=AD+DI\cdot\sin\frac{C}{2}=AD+DK=AK.$$

1 point.

Therefore, $\frac{DE}{BC}=\frac{AD}{CD}$, which implies $\triangle AED\sim\triangle DBC$. Hence, $\angle AMD=\angle AED=\angle DBC$.

2 points.

Problem 4. Let $\mathcal{F}$ be a family of (distinct) subsets of the set $\{1, 2, \ldots, n\}$ such that for all $A, B \in \mathcal{F}$ we have that $A^c \cup B \in \mathcal{F}$, where A^c is the set of all members of $\{1, 2, \ldots, n\}$ that are not in A.

Prove that every $k \in \{1, 2, \ldots, n\}$ appears in at least half of the sets in $\mathcal{F}$.

(Stijn Cambie, Mohammad Javad Moghaddas Mehr)

First Solution. We start out by "cleaning up" our set family. We denote $[n] = \{1, 2, \ldots, n\}$, and refer to it as the *ground set*.

Firstly, if there exists a number $x \in \{1, 2, \ldots n\}$ which appears in every member of the family $\mathcal{F}$, remove it from all members of the family. Proving the claim of the problem for the remaining family clearly suffices, as x is in all sets of the family, and so in at least half of them.

0 points.

Additionally, while there exist two elements x, y such that for every $A \in \mathcal{F}$ we have

$$x \in A \iff y \in A$$

remove one of them from all the sets of the family, and do this until no such pairs remain. Proving the problem claim for the remaining family of sets clearly suffices, as every removed number has a corresponding number that is still in the ground set and appears in exactly as many sets of the family as the removed member originally did.

1 point.

Now, fix any pair of distinct elements $\{x, y\}$ of the ground set. We wish to show that there exist sets $A_{x,y}, A_{y,x} \in \mathcal{F}$ such that $x \in A_{x,y}, x \notin A_{y,x}$ and $y \in A_{y,x}, y \notin A_{x,y}$. As we ensured that x, y do not always appear together, one of them must exist. Assume without loss of generality that it is $A_{x,y}$.

Assuming that any set $A_{y,x} \in \mathcal{F}$ containing y but not x does not exist, this implies that for every $A \in \mathcal{F}$ we have

$$x \in A \implies y \in A.$$

Now, as y is not in every set of $\mathcal{F}$, there exists a set B such that $y \notin B$ and the previous implication implies that $x \notin B$. However, if we now consider the set $A^c_{x,y} \cup B \in \mathcal{F}$, it contains y but does not contain x, contradicting the nonexistence of a suitable $A_{y,x} \in F$.

2 points.

We now aim to show that for every $x \in [n]$, we have that $\{x\}^c \in \mathcal{F}$. Fix one such x, take some set $B \in \mathcal{F}$ such that $x \notin B$ and consider the set

$$B \cup \bigcup_{y \neq x} A^c_{x,y}.$$

This set contains all elements $y \neq x$, so it must be equal to $\{x\}^c$. It is a member of $\mathcal{F}$ by repeated $n-1$-fold application of the condition on members of $\mathcal{F}$.

4 points.

To finish, consider some $x \in [n]$ and some $B \in \mathcal{F}$ not containing x. We then have that $\{x\} \cup B \in \mathcal{F}$, so for every set in $\mathcal{F}$ that does not contain x we can find a unique one that does and we are done.

3 points.

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Second Solution. First we observe that for $A, B \in \mathcal{F}$ by using the rule on $A^c \cup B$ and B we get that

$$(A^c \cup B)^c \cup B = (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) = A \cup B \in \mathcal{F}. \tag{1}$$

2 points.

Now we take an arbitrary $x \in [n]$, let $\mathcal{A} = \{S \in \mathcal{F} : x \in S\}$ and $\mathcal{B} = \{S \in \mathcal{F} : x \notin S\}$. Then the set $T := \bigcup_{S \in \mathcal{B}} S$ is in $\mathcal{F}$ by using (1). Note that $x \notin T$ and that every $S \in \mathcal{B}$ is a subset of T.

3 points.

Now we'll prove that the function $f : \mathcal{B} \to \mathcal{A}$ defined by $f(S) = T^c \cup S$ is an injection.

3 points.

Suppose there exists $B_1, B_2 \in \mathcal{B}$ such that $T^c \cup B_1 = T^c \cup B_2$

$$\begin{aligned}
&\Longrightarrow T \cap (T^c \cup B_1) = T \cap (T^c \cup B_2)\\
&\Longrightarrow (T \cap T^c) \cup (T \cap B_1) = (T \cap T^c) \cup (T \cap B_2)\\
&\Longrightarrow \emptyset \cup B_1 = \emptyset \cup B_2\\
&\Longrightarrow B_1 = B_2
\end{aligned}$$

where the third implication holds since $B_1, B_2 \subseteq T$. So f is injective $\implies |\mathcal{A}| \geqslant |\mathcal{B}|$.

2 points.

Third Solution. Obtain that for no two elements $x, y \in [n]$ holds $x \in A \iff y \in A$ for every $A \in \mathcal{F}$ as in the first solution.

1 point.

For arbitrary x, let T be the largest set not containing x. We claim $T = \{x\}^c$.
Assume the opposite, then every set $A \in \mathcal{F}$ containing x needs to contain T^c because otherwise $|A^c \cup T| > |T|$.

2 points.

Show that for any two sets $A, B \in \mathcal{F}$ their union $A \cup B \in \mathcal{F}$ is also in the family as shown in the Second Solution.

2 points.

If $A \in \mathcal{F}$ contains an element of T^c, then $|A \cup T^c| > |T|$ so A must contain x.

1 point.

These to combined imply that $x, y \in T^c$ belong to the exact same sets in $\mathcal{F}$ which is a contradiction with the claim at the beginning.

1 point.

Since we have $\{x\}^c \in \mathcal{F}$ we can finish the solution as in the First Solution.

3 points.

Problem 1. We call a pair of distinct numbers (a, b) a *binary pair* if $ab + 1$ is a power of two. Given a set S of n positive integers, what is the maximum possible number of binary pairs in S?

(Oleksii Masalitin)

First Solution. The answer is $n-1$, achieved by choosing $2^k - 1$ for $1 \le k \le n$. One can then easily see that $1, 2^k - 1$ makes a binary pair for $n \ge k > 1$. For the bound, make a graph G with vertices $a_1, \ldots a_n$, and connect (a, b) if a, b makes a binary pair.

2 points.

The key is the following:

Claim 1. *G does not contain a cycle.*

Proof. Assume otherwise and suppose:

$$\begin{aligned} x_1x_2 + 1 &= 2^{a_1} \\ x_2x_3 + 1 &= 2^{a_2} \\ &\ldots \\ x_nx_1 + 1 &= 2^{a_n} \end{aligned}$$

for positive integers $a_1, \ldots a_n$ and $n \geqslant 3$. WLOG let a_1 be the greatest (not necessarily unique). Notice that all x_i are odd, this is just parity. Focus on the following three equations:

$$x_nx_1 + 1 = 2^{a_n} \quad (1)$$

$$x_1x_2 + 1 = 2^{a_1} \quad (2)$$

$$x_2x_3 + 1 = 2^{a_2} \quad (3)$$

Subtracting (1) from (2) and (3) from (2) results in:

$$x_1(x_2 - x_n) = 2^{a_n}(2^{a_1 - a_n} - 1) \implies x_1 \mid 2^{a_1 - a_n} - 1 \text{ as } x_1 \text{ is odd}$$

$$x_2(x_1 - x_3) = 2^{a_2}(2^{a_1 - a_2} - 1) \implies x_2 \mid 2^{a_1 - a_2} - 1 \text{ as } x_2 \text{ is odd}$$

3 points.

The bound $x_1x_2 + 1 < (x_nx_1 + 1)(x_2x_3 + 1)$ can be verified by simple expansion. In turn, this gives $a_2 + a_n > a_1$

3 points.

Finally:

$$2^{a_1} - 1 = x_1x_2 \mid (2^{a_1 - a_2} - 1)(2^{a_1 - a_n} - 1) < 2^{a_1} - 2^{a_1 - a_2} - 2^{a_1 - a_n} + 1 \leqslant 2^{a_1} - 1$$

As $(2^{a_1 - a_2} - 1)(2^{a_1 - a_n} - 1)$ is nonnegative, the divisibility can hold only if it is equal to 0. This would give $a_1 = a_n$ or $a_1 = a_2$, contradicting $x_n \neq x_2$ and $x_1 \neq x_3$ respectively.

□

As a graph with n vertices not containing cycles can have at most $n-1$ edges, the proof of the bound is finished.

2 points.

Second Solution. Here we present an approach using Zsigmondy's theorem. Assume that $T = \{b_1, \dots b_k\}$ is the set of all indices with the maximal value of a_i, call it M. Observe that a_j-s in T must not be consecutive, otherwise they would violate the distinct x_i condition. Now we can multiply the equations to get the following:

$$\prod_{i \in T}(2^M - 1) \mid \prod_{i \notin T}(2^{a_i} - 1)$$

2 points.

We get a contradiction with Zsigmondy's theorem as $2^M - 1$ has a primitive prime divisor not dividing $2^n - 1$ for $n < M$, except for the case $M = 6$

4 points.

. Assume $x_i x_{i+1} = 63$ for some i.

- if $x_i = 63$, then $x_{i+2} = 63$ as 2^6 is the greatest possible value, contradicting distinctness.
- if $x_i = 21$, one similarly gets $x_{i+2} = 21$, again a contradiction.
- the rest of the cases are similar and bruteforce

2 points.

Third Solution. Again, the construction and the comment is the same as in the first solution. For the bound, we claim the following:
If $(a, b), (a, c)$ are two binary pairs with $a > b, c$, then $b = c$.

1 point.

Proof. WLOG assume $b \geq c$. As $ac + 1$ and $ab + 1$ are powers of 2, we have:

$$ac + 1 \mid ab + 1$$
$$ac + 1 \mid a(b - c)$$
$$ac + 1 \mid b - c$$

4 points.

But, if $b > c$, then

$$0 < b - c < b < a < ac + 1$$

2 points.

, contradiction. □

Now, all elements of S except the smallest can be the larger element in at most one binary pair, giving the bound $n - 1$

1 point.

Comment: One can show no cycles of odd length with little effort: All powers of 2 involved are at least 4. Now subtract one and multiply all equations together, and finish by mod 4.

Problem 2. Let n be a positive integer. The numbers $1, 2, \ldots, 2n+1$ are arranged in a circle in that order, and some of them are *marked*.

We define, for each k such that $1 \leqslant k \leqslant 2n+1$, the interval I_k to be the closed circular interval starting at k and ending at $k+n$ (taking remainders modulo $2n+1$ if $k+n > 2n+1$). We call an interval I_k *magical* if it contains strictly more than half of all the marked elements.

Prove that the following two statements are equivalent:

1. At least $n+1$ of the intervals $I_1, I_2, \ldots, I_{2n+1}$ are magical.
2. The number of marked numbers is odd.

(Andrei Constantinescu)

First Solution. Let S be the set containing all the marked numbers and $S_i = S \cap I_i$. Note that $S_i \cap S_{i+n} = \emptyset$ or $\{i+n\}$. So for each i we have that

$$|S_{i+n}| = \begin{cases} |S| - |S_i| + 1, & \text{if } i+n \in S \\ |S| - |S_i|, & \text{otherwise.} \end{cases}$$

2 points.

Suppose $|S|$ is odd. For each interval, I_i, that isn't magical we have that $|S_i| < \frac{|S|}{2}$ (since the equality can't hold) hence $|S_{i+n}| \geq |S| - |S_i| > |S| - \frac{|S|}{2} = \frac{|S|}{2}$ so I_{i+n} is magical. So for each non magical interval we can find a unique magical one, therefor we must have at least $n+1$ magical intervals.

4 points.

Now suppose $|S|$ is even. For each magical interval I_i, we have that $|S_i| > \frac{|S|}{2}$ hence $|S_{i+n}| \leq |S| - |S_i| + 1 < |S| - \frac{|S|}{2} + 1 = \frac{|S|}{2} + 1 \implies |S_{i+n}| \leq \frac{|S|}{2}$ so I_{i+n} is not magical. Since for each magical interval we can find a unique non magical one, there must be at least $n+1$ non magical intervals, so less that $n+1$ magical ones.

4 points.

Second Solution. Represent the remainders modulo $2k+1$ in a circle in ascending order. For the rest of the solution, *t-good* means the interval $[t, t+k]$ is magical, and the set being very good means it satisfies property (1). Label the vertices of the graph as $(0, 1, 2, \ldots, 2k)$. If S is t-good, draw an edge between t and $t+k$ (taken modulo $2k+1$). Now we prove both directions:

- Let $|S| = 2l$ be **very good**. The critical observation is the following: there is a node with degree 2. Since S is **very good**, there are at least $k+1$ edges. Thus, the total sum of degrees is at least $2k+2$.

  1 point.

  By the pigeonhole principle, there is a node with degree 2. Let a be the value of this node.

  1 point.

  By definition, the intervals $[a, a+k]$ and $[a-k, a]$ each contain more than half of the elements of S, i.e., at least $l+1$ elements each. These two intervals share exactly one element.

  2 points.

  Thus, the total number of distinct elements of S in these intervals is at least:

  $$2l+2 \quad (\text{if } a \notin S), \quad \text{or} \quad 2l+1 \quad (\text{if } a \in S).$$

  This contradicts $|S| = 2l$. Hence, if S is **very good**, S must have an odd number of elements.

  1 point.

- Now let $|S| = 2l-1$. A similar lemma applies: every node has degree at least 1. Fix a value x. The intervals $[x-k, x]$ and $[x, x+k]$ cover the entire residue class modulo $2k+1$ and share exactly one element. Now we split into two cases:

  - If $x \in S$, the remaining elements of S are in two disjoint intervals $[x-k, x-1]$ and $[x+1, x+k]$. By the pigeonhole principle, one of these intervals contains at least $l-1$ elements. Adding x creates a good interval with one of its endpoints at x, so x has degree at least 1.
  - If $x \notin S$, the elements of S are in two disjoint intervals $[x-k, x-1]$ and $[x+1, x+k]$. By the pigeonhole principle, one of these intervals contains at least l elements, so the conclusion is the same as above.

  In either case, x has degree at least 1.

  3 points.

  Similarly, every node has degree at least 1, so the total sum of degrees is at least $2k+1$. By the handshake lemma, the total sum of degrees is at least $2k+2$. This gives at least $k+1$ edges. Thus, S is indeed x **very good**.

  2 points.

Problem 3. Let ω be a semicircle with diameter $\overline{AB}$ and let M be the midpoint of $\overline{AB}$. Let X, Y be the points in the same half-plane as ω with respect to the line AB such that $AMXY$ is a parallelogram. Let I be the incenter of the triangle MXY. Lines MX, MY intersect ω in points C, D respectively. Let T be the intersection of AC and BD. The line MT intersects XY in E. If P is the intersection of EI and AB, and Q is the projection of E onto the line AB, show that M is the midpoint of $\overline{PQ}$.

(Michal Pecho)

Solution. We will work in the context of a triangle MXY. The solution is in two parts:

Claim 2. *E is the M-excircle touch point with XY*

We present two proofs:

Proof. We first claim that TCD is tangent to both MX and MY. Observe:

$$\angle MCT = \angle MCA = \angle CAM = \angle CAB = \angle CDB = \angle CDT$$

where the angles are directed. This shows that MX is tangent to DCT. Analogous proof gives MY tangent to CDT.

2 points.

Now we claim that the tangent to DCT at T is parallel to XY. The cleanest way is through negative inversion in T fixing the circle with diameter AB. This sends (CDT) into AB and fixes the tangent. The tangency is preserved, so the two lines are parallel, as desired.

1 point.

If the mentioned tangent meets MX and MY at R, S respectively, we have shown that CDT is M-excircle in MRS. Consider the homothety centred at M that maps RS to XY. It also maps T into E, but it also sends the excircle of MRS into the excircle of MXY, hence sending the M-touchpoint in MRS (i.e. T) into E, which is what we wanted to show.

3 points.

□

Proof. Let U, V be points of XY such that U, X, Y, V lie on XY in that order, and $UX = XM, VY = YM$. Also let R, S be the intersections of XY with AC, BD respectively (note that these do not correspond to R, S in the previous proof). Easy angle chase gives:

$$\begin{aligned} \angle YRT = \angle XRC &= \angle MXY - \angle MCA \\ &= \angle BMC - \angle MCA \\ &= \frac{1}{2}\angle YXM \\ &= \angle XUM \end{aligned}$$

, so $MU \parallel TR$. Similarly $MV \parallel TS$. Now the triangles TRS and MUV are homothetic

4 points.

, with E being the homothety center. The idea is that we can now express all the relevant lengths in terms of MXY. Let $a = XY, b = YM, c = MX$. Compute:

$$\begin{aligned} XR &= XC = a - c \\ YS &= YD = a - b \\ RS = XY - XR - SY &= b + c - a \end{aligned}$$

From Thales,

$$\frac{ER}{ES} = \frac{RU}{SV} = \frac{a}{a} = 1$$

and finally $XE = XR + \frac{1}{2}RS = a - c + \frac{b+c-a}{2} = \frac{a+b-c}{2}$, which is precisely the distance from X to M-extouchpoint.

2 points.

□

Having established that, let Z be the midpoint of M-altitude in MXY and N be its foot. It is well-known that Z, I, E are collinear, one can get that for example with homothety mapping the incircle to M-excircle

2 points.

. We are now done, check that $PM = NE$ from congruent MPZ and ZNE and $MQ = NE$ from the rectangle.

2 points.

Problem 4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + yf(x)) = xf(1 + y)$$

for all $x, y \in \mathbb{R}^+$.

Remark. We denote by $\mathbb{R}^+$ the set of all positive real numbers. *(Ioannis Galamatis)*

First Solution. The function $f(x) = x$ for all $x \in \mathbb{R}^+$ ssatisfies the condition, and we will show it is the only such function.

Firstly, note that if $f(x) \neq 1$ and we have that either $x > 1, f(x) < 1$ or $x < 1, f(x) > 1$, plugging in x and

$$y = \frac{1 - x}{f(x) - 1} > 0$$

gives that $x = 1$, a contradiction. Therefore, for all $x < 1$ we have $f(x) \leqslant 1$ and for all $x > 1$ we have $f(x) \geqslant 1$.

1 point.

Now, assume that $f(t) < t$ for some $t \in \mathbb{R}^+$, and take

$$x = f(t), y = \frac{t - f(t)}{f(f(t))}$$

in the starting equation. If we denote $s = \frac{t-f(t)}{f(f(t))}$ this gives

$$f(t) = f(t)f(1 + s) \implies f(1 + s) = 1$$

as the left-hand side and the $f(t)$ term on the right cancel out. If we now plug in

$$x = 1 + \frac{s}{2}, y = \frac{s}{2f\left(1 + \frac{s}{2}\right)}$$

we obtain that

$$1 = f(1 + s) = \left(1 + \frac{s}{2}\right) \cdot f(1 + y) \geqslant 1 + \frac{s}{2}$$

which is a contradiction.

Therefore, $f(x) \geqslant x$ for all $x \in \mathbb{R}^+$.

2 points.

If we apply this inequality to the left-hand side, we obtain that

$$xf(1 + y) \geqslant x + yf(x) \iff \frac{f(x)}{x} \leqslant \frac{f(y + 1) - 1}{y}$$

for all $x, y \in \mathbb{R}^+$. Plugging in $y = 1$, we obtain that $f(x)/x$ is bounded by $f(2) - 1$ for all $x \in \mathbb{R}^+$, and we already know it's always at least 1. Define

$$C = \limsup_{x\to\infty} \frac{f(x)}{x}.$$

We easily obtain that $\frac{f(y')-1}{y'-1} \geqslant C$ for all $y' = y + 1 \in \langle 1, \infty\rangle$, which rearranges to $f(y') \geqslant Cy' + 1 - C$ for all $y' > 1$. Additionally, by definition of C, for all $\varepsilon > 0$ there exists some $T_\varepsilon > 0$ such that $x > T_\varepsilon \implies f(x) \leqslant (C + \varepsilon)x$. Now, take some $x, y > \max\{1, T_\varepsilon\}$ and notice that $x + yf(x) > 1$ and $y + 1 > T_\varepsilon$ implies that we have

$$Cx + C^2xy + (1 - C)(y + 1) \leqslant f(x + yf(x)) = xf(y + 1) \leqslant x(C + \varepsilon)(y + 1) = Cxy + Cx + \varepsilon xy + \varepsilon x$$

which rearranges to

$$(C^2 - C - \varepsilon)y \leqslant \frac{(C - 1)(y + 1)}{x} + \varepsilon$$

for all x, y large enough. By fixing y and taking $x \to \infty$ we see that $C^2 - C \leqslant \varepsilon + \frac{\varepsilon}{y} < 2\varepsilon$ and as ε was arbitrary, we have that $C^2 - C = 0$ so $C = 1$.

5 points.

To finish, note that there now exists, for every $\varepsilon > 0$, a T_ε such that $y + 1 > T_\varepsilon$ implies $f(y + 1) \leqslant (1 + \varepsilon)(y + 1)$ and inserting this y into the inequality we earlier obtained gives that

$$\frac{f(x)}{x} \leqslant \frac{y + \varepsilon y + \varepsilon}{y} = (1 + \varepsilon) + \frac{\varepsilon}{y} < 1 + 2\varepsilon$$

for every $x \in \mathbb{R}^+$ and as ε is arbitrary, we obtain $f(x) \leqslant x$ for all $x \in \mathbb{R}^+$ and we are done.

2 points.

Second Solution. We shall firstly prove the following lemma about the behavior of f.

Lema 1. *The function f is non-decreasing.*

Proof. Assume on the contrary, there are $a > b$ such that $f(a) < f(b)$. Then, $c = \frac{a-b}{f(b)-f(a)}$ is positive. Now, plugging $(x, y) = (a, c)$ yields

$$f(\frac{af(b) - bf(a)}{f(b) - f(a)}) = f(a + \frac{a - b}{f(b) - f(a)}f(a)) = af(1 + c)$$

Plugging $(x, y) = (b, c)$ yields;

$$f(\frac{af(b) - bf(a)}{f(b) - f(a)}) = f(b + \frac{a - b}{f(b) - f(a)}f(b)) = bf(1 + c)$$

Yielding, $a = b$, a contradiction. This completes our proof.

3 points.

□

Now, it is easy to find that f is *surjective*, indeed, $f(x/f(2) + f(x/f(2)) = x$.

1 point.

Thus, f would be *continuous* [1].

2 points.

Hence, $f(x) = \lim_{y\to 0^+} f(x + yf(x)) = x \lim_{y\to 0^+} f(1 + y) = xf(1)$.

3 points.

That is, $f(x) = Cx$, for some $C > 0$. Hence, $C(x + C^2xy) = Cx(1 + y)$, yielding $C = 1$. It is easy to verify that $f(x) = x$ indeed satisfies the statement of the problem.

1 point.

Third Solution. For $x > 1$ we have $f(x) \geq 1$ otherwise plugging in $y = \frac{1-x}{f(x)-1}$ gives us a contradiction.
Applying this to the *RHS* of the original equation we get $f(x + yf(x)) \geq x$.
Putting $x = s - \epsilon, y = \frac{\epsilon}{f(s-\epsilon)}, \epsilon \to 0$ we get $f(s) \geq s$ for all $s \in \mathbb{R}^+$.

3 points.

Applying this to the *LHS* of the original equation, yielding;

$$f(y + 1) \geq 1 + y\frac{f(x)}{x} \qquad \forall x, y \in \mathbb{R}^+$$

1 point.

If there exist $c > 1$ such that $\frac{f(c)}{c} > 1$. Put $K = \frac{f(c)}{c}$.

Claim 3. $f(y + 1) \geq yK^n \qquad \forall n \in \mathbb{N}, \forall y \in \mathbb{R}^+$.

Proof. We shall prove it through induction. The base is clear. Putting $(x, y) = (c, y)$ into original equation and assuming $f(y + 1) \geq yK^n$ for all y and fixed n:

$$cf(y + 1) = f(c + yf(c)) \geq (c + yf(c) - 1)K^n \geq yf(c)K^n$$

gives us $f(y + 1) \geq yK^n \implies f(y + 1) \geq yK^{n+1}$.

4 points.

□

Back to our problem, fixing y and using the claim letting $n \to \infty$ we get that such K can't exist, that is $f(x) \leq x$ for $x > 1$. Since we already proved $f(x) \geq x$ we have $f(x) = x$ for $x > 1$, but returning to $f(y+1) = y+1 \geq 1+y\frac{f(x)}{x}$ gives us $f(x) = x$ for all x. It is easy to check that $f(x) = x$ indeed satisfies the original equation.

2 points.

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[1] For sake of completeness, in the following we shall provide the *outline* of the proof of this claim: Since f is non-deceasing, for an arbitrary positive a, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ exist and $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$. Now, we prove these two limits are equal. Assume for contradiction, $b = \lim_{x\to a^-} f(x) < \lim_{x\to a^+} f(x) = c$. Thus, for all $x < a$ we have $f(x) \leq b$, for all $x > a$, we have $f(x) \geq c$. Hence the image of function is a subset of $(0,\ b) \cup (c,\ +\infty) \cup \{f(a)\}$. This, can not be the whole $\mathbb{R}^+$. The derived contradiction, completes our proof.