PROBLEM SHORTLIST

(with solutions)

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Contents

ALGEBRA and LINEAR ALGEBRA

Problem 1. Let A, B and X be $n \times n$ matrices over the same field, with X being nonsingular. Prove that $AB = AX + X^{-1}B$ if and only if $BA = XA + BX^{-1}$.

Solution. It is enough to prove that $AB = AX + X^{-1}B$ implies $BA = XA + BX^{-1}$. For the sake of this, consider the product

$$(A - X^{-1})(B - X) = AB - AX - X^{-1}B + I$$

where I is the identity $n \times n$ matrix. Thus $(A - X^{-1})(B - X) = I$, which means that

$$(A - X^{-1}) = (B - X)^{-1}.$$

Therefore, we have

$$(A - X^{-1})(B - X) = (B - X)(A - X^{-1}) = I .$$

It follows now that

$$I = (B - X)(A - X^{-1}) = BA - XA - BX^{-1} + I ,$$

and finally

$$BA = XA + BX^{-1}.$$

Problem 2. Prove that given two matrices $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ have a common eigenvalue if and only if there exists a non-zero matrix $C \in M_{m\times n}(\mathbb{C})$ such that $AC = CB$.

Solution. Let $\lambda \in \mathbb{C}$ be a common eigenvalue of A and B. Since $\det(B - \lambda I_n) = 0$, it follows that $\det(B^t - \lambda I_n) = 0$ (where B^t denotes the transpose of B), hence λ is an eigenvalue of B^t. Next, let $X \in \mathbb{C}^m$ and $Y \in \mathbb{C}^n$ be eigenvectors associated to the eigenvalue λ and the matrices A and B^t, respectively, i.e.

$$AX = \lambda X, \quad X \neq 0,$$

$$B^t Y = \lambda Y, \quad Y \neq 0.$$

Setting $C = XY^t \in M_{m\times n}(\mathbb{C})$, it follows that

$$AC = AXY^t = \lambda XY^t = X(\lambda Y)^t = X(B^t Y)^t = XY^t B = CB.$$

Conversely, let $C \in M_{m\times n}(\mathbb{C})$ be a non-zero matrix such that $AC = CB$. It is easily deducible (e.g., by induction) that $A^k C = CB^k$ holds for all non-negative integers *k*, hence $P(A)C = CP(B)$ for any polynomial P over $\mathbb{C}$. Choose P to be the characteristic polynomial of A, and $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{C}$ to be its eigenvalues (repeated according to their algebraic multiplicities). Then

$$P(x) = \det(xI_m - A) = (x - \lambda_1)(x - \lambda_2)...(x - \lambda_m)$$

and $P(A) = 0$, hence $CP(B) = 0$. This leads to

$$0 = C(B - \lambda_1 I_n)(B - \lambda_2 I_n)...(B - \lambda_m I_n).$$

Suppose that none of $\lambda_1, \lambda_2, ..., \lambda_m$ is an eigenvalue of B. Thus, for every i, the matrix $B - \lambda_i I_n$ is invertible, which implies that $C = 0$. This contradiction concludes the proof.

Problem 3. Let $f_1(x) = 3x - 4x^3$ and

$$f_n(x) = f_1(f_{n-1}(x)).$$

Solve the equation $f_n(x) = 0$.

Solution. First, we prove that $|x| > 1 \Rightarrow |f_n(x)| > 1$ holds for every positive integer n. It suffices to demonstrate the validity of this implication for $n = 1$. But, by assuming $|x| > 1$, it readily follows that

$$|f_1(x)| = |x| \cdot |3 - 4x^2| \geq |3 - 4x^2| > 1,$$

which completes the demonstration. We conclude that every solution of the equation $f_n(x) = 0$ lies in the closed interval $[-1,1]$. For an arbitrary such x, set $x = \sin t$ where $t = \arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We clearly have $f_1(\sin t) = \sin 3t$, which gives

$$f_n(x) = \sin 3^n t = \sin(3^n \arcsin x).$$

Thus, $f_n(x) = 0$ if and only if $\sin(3^n \arcsin x) = 0$, i.e. only when $3^n \arcsin x = k\pi$ for some $k \in \mathbb{Z}$. Therefore, the solutions of the equation $f_n(x) = 0$ are given by

$$x = \sin \frac{k\pi}{3^n},$$

where k acquires every integer value from -3^{n-1} up to 3^{n-1}.

Problem 4. For an integer $n > 2$ let $A, B, C, D \in M_n(\mathbb{R})$ be matrices satisfying:

$$AC - BD = I_n,$$
$$AD + BC = O_n.$$

Prove that:

a) $CA - DB = I_n$ and $DA + CB = O_n$,

b) $\det(AC) \geq 0$ and $(-1)^n \det(BD) \geq 0$.

Solution. a) We have

$$AC - BD + i(AD + BC) = I_n \Leftrightarrow (A + iB)(C + iD) = I_n,$$

which implies that the matrices $A + iB$ and $C + iD$ are inverses to one another. Thus,

$$(C + iD)(A + iB) = I_n \Leftrightarrow CA - DB + i(DA + CB) = I_n$$
$$\Leftrightarrow CA - DB = I_n,\ DA + CB = O_n.$$

b) We have

$$\begin{aligned}\det((A + iB)C) &= \det(AC + iBC)\\ &\overset{AD+BC=O_n}{=} \det(AC - iAD)\\ &= \det(A(C - iD)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\det C &\overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)C) = \det((C + iD)A(C - iD))\\ &= \det(A)\,|\det(C + iD)|^2.\end{aligned}$$

Thus,

$$\det(AC) = (\det A)^2 |\det(C + iD)|^2 \geq 0.$$

Similarly

$$\begin{aligned}\det((A + iB)D) &= \det(AD + iBD)\\ &\overset{AD+BC=O_n}{=} \det(-BC + iBD)\\ &= (-1)^n \det(B(C - iD)).\end{aligned}$$

This implies that

$$\begin{aligned}\det D &\overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)D) = (-1)^n \det((C + iD)B(C - iD))\\ &= (-1)^n \det(B)\,|\det(C + iD)|^2.\end{aligned}$$

Thus, $(-1)^n \det(BD) = (\det B)^2\, |\det(\mathrm{C} + iD)|^2 \geq 0$.

Remark. Since $(C+iD)(A+iB)=I_n$, we have that

$$\det(C+iD)\neq 0 \text{ and } |\det(C+iD)|\neq 0 .$$

It follows that $\det A=0 \Leftrightarrow \det C=0$ and $\det B=0 \Leftrightarrow \det D=0$.

The two matrix formulae, given in the statement of the problem, can also be written as

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} C & -D \\ D & C \end{bmatrix}=I_{2n} .$$

ANALYSIS

Problem 1. For an integer $n \ge 1$, denote by

$$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!},$$

the $(2n-1)$-th Taylor polynomial of the sine function at 0, and let

$$I_n = \int_0^\infty \frac{T_n(x)-\sin x}{x^{2n+1}}\,dx .$$

a) Prove that $I_n = -\frac{1}{2n(2n-1)} I_{n-1},\ \ n \ge 1.$

b) Calculate I_n.

Solution. a) Integration by parts twice gives

$$\begin{aligned}
I_n &= \int_0^\infty \frac{T_n(x)-\sin x}{x^{2n+1}}\,dx = -\frac{T_n(x)-\sin x}{2nx^{2n}}\Bigg|_0^\infty + \frac{1}{2n}\int_0^\infty \frac{T_n'(x)-\cos x}{x^{2n}}\,dx \\
&= \frac{1}{2n}\int_0^\infty \frac{T_n'(x)-\cos x}{x^{2n}}\,dx = -\frac{T_n'(x)-\cos x}{2n(2n-1)x^{2n-1}}\Bigg|_0^\infty + \frac{1}{2n(2n-1)}\int_0^\infty \frac{T_n''(x)+\sin x}{x^{2n-1}}\,dx \\
&= -\frac{1}{2n(2n-1)}\int_0^\infty \frac{T_{n-1}(x)-\sin x}{x^{2n-1}}\,dx = -\frac{1}{2n(2n-1)} I_{n-1},
\end{aligned}$$

since $T_n''(x) = -T_{n-1}(x)$. It follows that

$$I_n = -\frac{1}{2n(2n-1)} I_{n-1} \text{ and } I_n = \frac{(-1)^{n-1}}{2n(2n-1)\cdot \ldots \cdot 4\cdot 3} I_1 . \tag{1}$$

b) The integral equals $\frac{(-1)^{n-1}\pi}{2(2n)!}$. To calculate I_1 we integrate by parts twice

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{x-\sin x}{x^3}\,dx = \frac{x-\sin x}{-2x^2}\Bigg|_0^\infty + \frac{1}{2}\int_0^\infty \frac{1-\cos x}{x^2}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{1-\cos x}{-2x}\Bigg|_0^\infty + \frac{1}{2}\int_0^\infty \frac{\sin x}{x}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{\sin x}{x}\,dx \\
&= \frac{\pi}{4}.
\end{aligned} \tag{2}$$

Combining (1) and (2) we deduce that $I_n = \frac{(-1)^{n-1}\pi}{2(2n)!}$.

Problem 2. Prove that for every $x \in (0,1)$ holds the inequality

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy > \sqrt{x^2+(\sin x)^2} .$$

Solution. Clearly

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy \ge \int_0^x \sqrt{1+(\cos y)^2}\,dy .$$

Define a function $F:[0,1] \to \mathbb{R}$ by setting:

$$F(x) = \int_0^x \sqrt{1+(\cos y)^2}\,dy - \sqrt{x^2+(\sin x)^2} .$$

Since $F(0)=0$, it suffices to show that $F'(x) \ge 0$. By the fundamental theorem of Calculus, holds

$$F'(x) = \sqrt{1+(\cos x)^2} - \frac{x+\sin x\cos x}{\sqrt{x^2+(\sin x)^2}} .$$

Thus, it is enough to prove that

$$(1+(\cos x)^2)(x^2+(\sin x)^2) \ge (x+\sin x\cos x)^2 .$$

But this is a straightforward application of the Cauchy-Schwarz inequality.

Problem 3. Find all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f'(x)-2xf(-x)=x, \text{ for all } x\in\mathbb{R}.$$

Solution. By replacing x with $-x$, we obtain the following two equalities:

$$f'(x)-2xf(-x)=x, \text{ for all } x\in\mathbb{R}$$

$$f'(-x)+2xf(x)=-x, \text{ for all } x\in\mathbb{R}$$

After adding these two, we obtain

$$f'(x)+f'(-x)+2x(f(x)-f(-x))=0, \text{ for all } x\in\mathbb{R}. \tag{1}$$

Let $g:\mathbb{R}\to\mathbb{R}$ be the function $g(x)=f(x)-f(-x)$. Based on (1), we have

$$g'(x)+2xg(x)=0, \text{ for all } x\in\mathbb{R}.$$

Multiply the preceding equality by e^{x^2} to deduce that

$$(g(x)e^{x^2})'=0, \text{ for all } x\in\mathbb{R}.$$

Hence

$$g(x)=Ce^{-x^2}, \text{ for all } x\in\mathbb{R},$$

where C is an arbitrary constant. This implies

$$f(x)-f(-x)=Ce^{-x^2}, \text{ for all } x\in\mathbb{R}. \tag{2}$$

Replacing x by $-x$ in (2) gives

$$f(-x)-f(x)=Ce^{-x^2}, \text{ for all } x\in\mathbb{R}. \tag{3}$$

Add (2) and (3) to conclude that $C=0$, thus $f(-x)=f(x)$, for all $x\in\mathbb{R}$. Therefore, f is an even function, and the initial differential equation implies

$$f'(x)-2xf(x)=x, \text{ for all } x\in\mathbb{R}.$$

We multiply the preceding equality by e^{-x^2} and obtain

$$(f(x)e^{-x^2})'=xe^{-x^2}=(-\tfrac{1}{2}e^{-x^2})', \text{ for all } x\in\mathbb{R},$$

which in turn yields that $f(x)=-\frac{1}{2}+ce^{x^2}$, for all $x\in\mathbb{R}$.

Problem 4. Find the limit:

$$\lim_{n\to\infty} n^2 \int_1^{2015} \frac{1-u^{\frac{1}{n(n+1)}}}{u^{\frac{n}{n+1}}} du .$$

Solution 1. One has

$$\int_1^{2015} \frac{1-u^{\frac{1}{n(n+1)}}}{u^{\frac{n}{n+1}}} du = (n+1)\cdot 2015^{\frac{1}{n+1}} - n\cdot 2015^{\frac{1}{n}} - 1$$

$$= (n+1)\left(1+\frac{\ln 2015}{n+1}+\frac{\ln^2 2015}{2(n+1)^2}+o\left(\frac{1}{n^2}\right)\right) - n\left(1+\frac{\ln 2015}{n}+\frac{\ln^2 2015}{2n^2}+o\left(\frac{1}{n^2}\right)\right) - 1$$

$$= -\frac{\ln^2 2015}{2n(n+1)} + o\left(\frac{1}{n^2}\right).$$

Thus,

$$\lim_{n\to\infty} n^2 \int_1^{2015} \frac{1-u^{\frac{1}{n(n+1)}}}{u^{\frac{n}{n+1}}} du = -\frac{\ln^2 2015}{2} .$$

Solution 2. One has

$$\frac{1-u^{\frac{1}{n(n+1)}}}{u^{\frac{n}{n+1}}} = -\frac{\sum\limits_{k=1}^{\infty} \frac{\ln^k u}{(n(n+1))^k}}{u^{\frac{n}{n+1}}} = -\sum_{k=1}^{\infty} \frac{\ln^k u}{u^{\frac{n}{n+1}}(n(n+1))^k} .$$

The last series is uniformly convergent. Thus,

$$\lim_{n\to\infty} n^2 \int_1^{2015} \frac{1-u^{\frac{1}{n(n+1)}}}{u^{\frac{n}{n+1}}} du = \lim_{n\to\infty} n^2 \left(-\sum_{k=1}^{\infty} \int_1^{2015} \frac{\ln^k u}{u^{\frac{n}{n+1}}(n(n+1))^k} du\right)$$

$$= -\int_1^{2015} \frac{\ln u}{u} du = -\frac{\ln^2 2015}{2}.$$

Problem 5. Find all functions $f:\mathbb{R}_0 \to \mathbb{R}_0$ which satisfy the functional equation

$$f(f(x)-2x)=3x,\ \ \forall x\in \mathbb{R}_0,$$

where $\mathbb{R}_0$ denotes the set of nonnegative real numbers.

Solution. In order for the LHS to be defined, the inequality $f(x)\geq 2x$, $\forall x\in \mathbb{R}_0$ must hold. So,

$$f(f(x)-2x)\geq 2(f(x)-2x)=2f(x)-4x \iff f(x)\leq 3{,}5x\,.$$

Therefore $2x\leq f(x)\leq 3{,}5x$, $\forall x\in \mathbb{R}_0$.

Consider a double inequality of type $a_n x\leq f(x)\leq b_n x$. Then

$$a_n(f(x)-2x)\leq f(f(x)-2x)\leq b_n(f(x)-2x)$$

and so

$$(2+\frac{3}{b_n})x\leq f(x)\leq (2+\frac{3}{a_n})x\,.$$

Define the sequences $\{a_n\}_n$ and $\{b_n\}_n$ by

$$a_1=2,\, b_1=3{,}5;\, a_{n+1}=2+\frac{3}{b_n},\, b_{n+1}=2+\frac{3}{a_n}\,.$$

The above calculations imply that for each n, $a_n x\leq f(x)\leq b_n x$, $\forall x\in \mathbb{R}_0$. One can directly verify the inequalities $a_1\leq a_2$ and $b_1\geq b_2$. Using induction we show that the sequence $\{a_n\}$ is non-decreasing, while $\{b_n\}$ is non-increasing. As these sequences are clearly bounded, they must be convergent. Now it is easy to prove that they both converge to 3. Thus $f(x)=3x$, $\forall x\in \mathbb{R}_0$. It is readily checked that this is indeed a solution of the considered functional equation.

Problem 6. Let f be a continuous real function defined on $[0,2]$ such that

$$\int_0^2 tf(t)dt = 2\int_0^2 f(t)dt .$$

Prove that there exists an $a \in (0,2)$ such that

$$\int_0^a tf(t)dt = 0 .$$

Solution. Let $H(x) = \int_0^x tf(t)dt$. For $\varepsilon > 0$, integration by parts gives

$$\int_\varepsilon^2 (2-t)f(t)dt = \int_\varepsilon^2 \tfrac{2-t}{t} H'(t)dt = \tfrac{2-t}{t} H(t)\,|_\varepsilon^2 + \int_\varepsilon^2 \tfrac{2}{t^2} H(t)dt.$$

Taking limits we obtain:

$$\lim_{\varepsilon \to 0} \int_\varepsilon^2 \tfrac{2}{t^2} H(t)dt = \int_0^2 (2-t)f(t)dt = 0 .$$

Consider the function

$$G(x) = \begin{cases} \int_\varepsilon^2 \tfrac{2}{t^2} H(t)dt, & \text{if } x \in (0,2] \\ 0, & \text{if } x = 0. \end{cases}$$

This is a differentiable function, since L'Hospital's rule gives

$$\lim_{x \to 0} \frac{G(x)}{x} = -\lim_{x \to 0} \frac{2H(x)}{x^2} = -f(0).$$

Finally, as $G(2) = G(0)$, Rolle's theorem implies the desired result.

Problem 7. Prove that:

i) there exists only one function $f:\mathbb{R}\to\mathbb{R}$ which satisfies:

$$f^3(x)+3f(x)\sin^2 x=6\sin x+8,\quad \forall x\in\mathbb{R}$$

ii) f is periodic, bounded, and of class $\mathbb{C}^\infty$;

iii) there exists $\varepsilon>0$ such that

$$0<|x|\le\varepsilon \;\Rightarrow\; f(-3x)+3f(-x)+2f(3x)<12.$$

Solution. *i*) Consider the equation

$$g(y)=y^3+3m^2y-6m-8,\quad m\in[-1,1].$$

Observe that for $y\neq 0$, $g'(y)=3y^2+3m^2>0$, $g(0)<0$, $g(3)>0$. Hence, for any $m\in[-1,1]$, this equation admits a unique solution $y(m)\in(0,3)$. Since the equation involving the function f contains $\sin x$, the function we are looking for must depend on $\sin x$, thus it must be of the form $y(\sin x)$.

ii) It follows immediately that f is periodic and bounded, and moreover, by the implicit function theorem, f is differentiable. We calculate

$$f'(x)=\frac{2\cos x(1-f(x)\sin x)}{f^2(x)+\sin^2 x}.$$

Inductively, one obtains that f is of class $\mathbb{C}^\infty$, in particular

$$f''(x)=-\frac{2}{f^2(x)+\sin^2 x}\left(f(x)f'^2(x)+\sin 2x f'(x)+\cos 2x f(x)+\sin x\right).$$

iii) We calculate $f(0)=2,\ f'(0)=\frac{1}{2},\ f''(0)=-\frac{5}{4}$. By denoting

$$h(x)=f(-3x)+3f(-x)+2f(3x),$$

one gets $h(0)=12,\ h'(0)=0,\ h''(0)=-\frac{75}{2}<0,$ hence h has a strict local maximum at the origin.

Problem 8. Given is the functional equation

$$f(x+1)+f(x-1)=2015f(x). \qquad (1)$$

a) Prove that there is no non-constant continuous periodic solution f of the equation (1).

b) Does there exist a periodic function f which satisfies (1). In addition, is it possible to point out a periodic solution with a period $\sqrt{2015}$?

Solution. a) Because f is not a constant, it follows that one can find $0 \neq x_0 \in \mathbb{R}$ for which $f(x_0) \neq 0$. Consider now the restriction of f on the set $x_0+\mathbb{Z}$ and put $a_n = f(x_0+n)$. Thus, we obtain the sequence $\{a_n\}$ defined by the recurrent equation $a_{n+1}+a_{n-1}=2015a_n$. Using standard techniques, one obtains that $a_n = c_1\lambda_1^n + c_2\lambda_2^n$, where c_1 and c_2 are constants with $c_1^2+c_2^2 \neq 0$ (since $f \neq const$) and $\lambda_{1,2}$ are the roots of the characteristic equation

$$\lambda^2 - 2015\lambda + 1 = 0.$$

Note that $\lim\limits_{n\to\infty} a_n = \pm\infty$ or $\lim\limits_{n\to-\infty} a_n = \pm\infty$ because $\lambda_1 = \frac{1}{\lambda_2}$ and $\lambda_1+\lambda_2 = 2015$ (the roots are real numbers). It follows that f is not bounded, a contradiction (f is continuous and periodic, thus surely is bounded).

b) Choose a root of the characteristic equation, say we have chosen λ_1. Denote by $\mathbf{M}$ the subset $\mathbf{M} = \{m+n\sqrt{2015} \mid m,n \in \mathbb{Z}\}$ of $\mathbb{R}$, and define the function f as follows:

$$f(x)=0 \text{ if } x \notin \mathbf{M} \text{ and } f(m+n\sqrt{2015}) = \lambda_1^m \text{ otherwise.}$$

It is easy to prove that f is periodic with period $\sqrt{2015}$ (note that $2015 = 5\cdot 13\cdot 31$, so it is easily seen that $\sqrt{2015}$ is irrational).

Remark. Note that one can point out an example of a periodic solution of (1) with an arbitrary chosen irrational period. In addition, the unbounded periodic functions as above usually have some exotic properties (for example, the closure of the graph of such a function has a nonempty interior and every such function is unbounded at each point).

Problem 9. Let the sequences $\{p_n\}, \{q_n\}$ be defined recursively by

$$p_{-1} = 0,\ p_0 = q_0 = q_{-1} = 1 \quad \text{and}$$

$$p_n = 2p_{n-1} + (2n-1)^2 p_{n-2},\ \ q_n = 2q_{n-1} + (2n-1)^2 q_{n-2}, \text{ for each } n \ge 1.$$

Calculate: $\lim\limits_{n\to\infty} \frac{p_n}{q_n}$.

Solution. By using induction, one easily verifies that

$$q_n = \prod_{k=0}^{n} (2k+1),\ \ p_n = (2n+1)\,p_{n-1} + (-1)^n \prod_{k=0}^{n-1} (2k+1)\ .$$

Therefore,

$$\frac{p_n}{q_n} = \frac{(2n+1)\,p_{n-1} + (-1)^n \prod\limits_{k=0}^{n-1}(2k+1)}{\prod\limits_{k=0}^{n}(2k+1)} = \frac{(2n+1)\,p_{n-1}}{\prod\limits_{k=0}^{n}(2k+1)} + \frac{(-1)^n \prod\limits_{k=0}^{n-1}(2k+1)}{\prod\limits_{k=0}^{n}(2k+1)}$$

$$= \frac{p_{n-1}}{\prod\limits_{k=0}^{n-1}(2k+1)} + \frac{(-1)^n}{2n+1} = \frac{p_{n-1}}{q_{n-1}} + \frac{(-1)^n}{2n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}.$$

So,

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and the last series is convergent by the alternating series test. Since

$$\arctan x = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1},\ \ \forall x \in (-1,1),$$

Abel's theorem assures that

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = \arctan 1 = \frac{\pi}{4}\ .$$

Problem 10. For an integer $n \geq 1$, let

$$I_n = \int_0^\infty \frac{\arctan x}{(1+x^2)^n} dx .$$

a) Prove that $\sum_{n=1}^{\infty} \frac{I_n}{n} = \zeta(2)$, where $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

b) Calculate $\int_0^\infty \arctan x \ln(1+\frac{1}{x^2}) dx$.

Solution. a) First, we prove that the following recurrence relation holds

$$I_n = -\frac{1}{2n} + 2n(I_n - I_{n+1}), \ \ n \geq 1 .$$

We calculate the integral I_n by parts, with

$$f(x) = \frac{\arctan x}{(1+x^2)^n}, \quad f'(x) = \frac{1}{(1+x^2)^{n+1}} - \frac{2nx}{(1+x^2)^{n+1}} \arctan x, \ \ g'(x) = 1, \ \ g(x) = x$$

and obtain

$$\begin{aligned} I_n &= \frac{x \arctan x}{(1+x^2)^n}\bigg|_0^\infty - \int_0^\infty \left(\frac{x}{(1+x^2)^{n+1}} - 2n\frac{x^2}{(1+x^2)^{n+1}} \arctan x\right)dx \\ &= -\int_0^\infty \frac{x}{(1+x^2)^{n+1}} dx + 2n\int_0^\infty \frac{x^2}{(1+x^2)^{n+1}} \arctan x dx \\ &= \frac{1}{2n(1+x^2)^n}\bigg|_0^\infty + 2n\int_0^\infty \left(\frac{\arctan x}{(1+x^2)^n} - \frac{\arctan x}{(1+x^2)^{n+1}}\right)dx \\ &= -\frac{1}{2n} + 2n(I_n - I_{n+1}). \end{aligned}$$

This implies that $\frac{I_n}{n} = -\frac{1}{2n^2} + 2(I_n - I_{n+1})$. Moreover, dominated convergence implies $I_n \to 0$. Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{I_n}{n} &= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\lim_{n\to\infty} \left(I_1 - I_{n+1}\right) \\ &= -\frac{1}{2}\zeta(2) + 2I_1 \\ &= \frac{\pi^2}{6}, \end{aligned}$$

since

$$I_1 = \int_0^\infty \frac{\arctan x}{1+x^2} dx = \frac{\arctan^2 x}{2}\bigg|_0^\infty = \frac{\pi^2}{8} .$$

b) We have,

$$\int_0^\infty \arctan x \ln(1+\frac{1}{x^2})dx = -\int_0^\infty \arctan x \ln(1-\frac{1}{1+x^2})dx$$

$$= \int_0^\infty \arctan x (\sum_{n=1}^\infty \frac{(\frac{1}{1+x^2})^n}{n})dx$$

$$= \sum_{n=1}^\infty \frac{I_n}{n}$$

$$= \zeta(2),$$

where the last equality follows from part a) of the problem. Note that we were allowed to interchange the integral and the sum in the previous calculations in view of Tonelli's theorem, since the terms involved are positive. The problem is solved.

Problem 11. Consider a C^1- function $f:\mathbb{R}\to\mathbb{R}$ such that

$$\left(f'(y)-f'(x)\right)\left(y-x\right)\geq 0,\ \forall x,y\in\mathbb{R} \tag{1}$$

The following statements are equivalent:

a) x_0 is a local extremum point of f ;

b) x_0 is a global minimum point of f ;

c) f is not open at x_0.

(One says that f is open at x if f maps any neighborhood of x to a neighborhood of $f(x)$).

Solution. Since the relation (1) shows that f' is an increasing function, it means that the function f is convex. Indeed, take $x<y$ and denote $z=(1-\alpha)x+\alpha y,$ where $\alpha\in(0,1).$ Then, by Lagrange's Theorem, there exist $t\in(x,z)$ and $s\in(z,y)$ such that

$$f'(t)(x-z)=f(x)-f(z), \text{ and } f'(s)(y-z)=f(y)-f(z).$$

Then

$$\begin{aligned}(1-\alpha)f(x)+\alpha f(y)-f(z)&=(1-\alpha)(f(x)-f(z))+\alpha(f(y)-f(z)\\&=(1-\alpha)f'(t)(x-z)+\alpha f'(s)(y-z)\\&=\alpha(1-\alpha)(x-y)[f'(t)-f'(s)].\end{aligned}$$

Since f' is increasing and $t\in(x,z)$, $s\in(z,y)$, from above it follows that f is convex.

Let us prove first the equivalence $a)\Leftrightarrow b)$. The implication $b)\Rightarrow a)$ is obvious.

Suppose that x_0 is a local extremum point of f. If x_0 is a local minimum point, there is $r>0$ such that $f(x_0)\leq f(x),$ for any $x\in(x_0-r,x_0+r).$ Take an arbitrary number $z\in\mathbb{R}\setminus\{x_0\},$ and find $t\in(0,1)$ sufficiently small such that $(1-t)x_0+tz\in(x_0-r,x_0+r).$ Hence,

$$f(x_0)\leq f((1-t)x_0+tz)\leq(1-t)f(x_0)+tf(z),$$

i.e.

$$f(x_0)\leq f(z).$$

If x_0 is a local maximum, then there is $r>0$ such that $f(x_0)\geq f(x),$ for any $x\in(x_0-r,x_0+r).$ Then, for every $\delta\in(0,r)$ one has

$$f(x_0)=f(\frac{1}{2}(x_0+\delta)+\frac{1}{2}(x_0-\delta))\leq\frac{1}{2}f(x_0+\delta)+\frac{1}{2}f(x_0-\delta)\leq f(x_0),$$

hence f is constant on $(x_0-r,x_0+r).$ It means that x_0 is a local minimum for

f, and by the proof above it is a global minimum point.

Let us prove now that $b) \Leftrightarrow c)$.

If x_0 is a point of (global) minimum of f, it follows that $f((x_0 - 1, x_0 + 1))$ cannot contain points which are smaller than $f(x_0)$, hence $f((x_0 - 1, x_0 + 1))$ is not a neighborhood of $f(x_0)$. This means that f is not open at x_0.

Next we prove that if x_0 is not a global minimum of f, then f is open at x_0. There exists an $x_1 \in \mathbb{R}$ such that $f(x_1) < f(x_0)$. Without loss of generality, assume that $x_0 = 0$ and $x_1 = 1$. Since $f(0) > f(1)$ and f is convex, it follows that for the points in the interval $[0,1]$, the graph of f lies bellow the line passing through $(0, f(0))$ and $(1, f(1))$, and for the points from $[-1,0]$, it lies above this line. Denote $c = f(0) - f(1) > 0$. It means that for every $y \in [0,1]$, the convexity implies that

$$f(y) \le (1-y) f(0) + yf(1) = f(0) - yc \le f(0) + yc \le f(-y).$$

Since f is continuous, it has the Darboux property. Hence, for any point $\lambda \in (f(0) - r, f(0) + r)$, with $r \in (0, c)$, there is a point $\xi \in (-\frac{r}{c}, \frac{r}{c})$ such that $f(\xi) = \lambda$. This means that

$$(f(0) - r, f(0) + r) \subset f((-\tfrac{r}{c}, \tfrac{r}{c})),$$

thus f maps neighborhoods of 0 into neighborhoods of $f(0)$. This means that f is open at 0, which completes the solution.

Problem 12. Let $I \subset \mathbb{R}$ be an open interval which contains 0, and $f : I \to \mathbb{R}$ be a function of class $C^{2016}(I)$ such that

$$f(0)=0,\ f'(0)=1,\ f''(0)=f'''(0)=...=f^{(2015)}(0)=0,\ f^{(2016)}(0)<0.$$

i) Prove that there is $\delta > 0$ such that

$$0 < f(x) < x, \quad \forall x \in (0,\delta). \tag{1.1}$$

ii) With δ as in *i*), define the sequence (a_n) by

$$a_1 = \frac{\delta}{2},\ a_{n+1} = f(a_n),\ \forall n \geq 1. \tag{1.2}$$

Study convergence of the series $\sum_{n=1}^{\infty} a_n^r$, when $r \in \mathbb{R}$ is an arbitrary parameter.

Solution. ***i)*** We claim that there exists $\alpha > 0$ such that $f(x) > 0$ for any $x \in (0,\alpha)$. For this, observe that, since f is of class C^1 and $f'(0)=1>0$, there exists $\alpha > 0$ such that $f'(x) > 0$ on $(0,\alpha)$. Since $f(0)=0$ and f is strictly increasing on $(0,\alpha)$, the claim follows.

Next, we prove that there exists $\beta > 0$ such that $f(x) < x$ for any $x \in (0,\beta)$. Since $f^{(2016)}(0) < 0$ and f is of class C^{2016}, there is $\beta > 0$ such that $f^{(2016)}(t) < 0$, for any $t \in (0,\beta)$. By the Taylor's formula, for any $x \in (0,\beta)$, there is $\theta \in [0,1]$ such that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^{(2015)}(0)}{2015!}x^{2015} + \frac{f^{(2016)}(\theta x)}{2016!}x^{2016}, \tag{1.3}$$

hence

$$g(x) = \frac{f^{(2016)}(\theta x)}{2016!}x^{2016} < 0, \quad \forall x \in (0,\beta).$$

Taking $\delta = \min\{\alpha,\beta\} > 0$, the item *i)* is completely proven.

ii) We will prove first that the sequence (a_n) given by (1.2) converges to 0. Indeed, by relation (1.1) it follows that

$$0 < a_{n+1} < a_n,\ \forall n \geq 1,$$

hence the sequence (a_n) is strictly decreasing and lower bounded by 0. It follows that (a_n) converges to some $\ell \in [0,\frac{\delta}{2})$. Passing to the limit in (1.2), one gets $\ell = f(\ell)$. Taking into account (1.1), we deduce that $\ell = 0$.

In what follows, we calculate

$$\lim_{n\to\infty} n a_n^{2015}.$$

From $a_n \downarrow 0,$ using the Stolz-Cesàro Theorem, we conclude that

$$\lim_{n\to\infty} n a_n^{2015} = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{a_{n+1}^{2015}}-\frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{f(a_n)^{2015}}-\frac{1}{a_n^{2015}}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{f(x)^{2015}}-\frac{1}{x^{2015}}} = \lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}}.$$

Observe that, by (1.3)

$$\frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}} = \frac{(x^2+\frac{f^{(2016)}(\theta x)}{2016!}x^{2017})^{2015}}{-\frac{f^{(2016)}(\theta x)}{2016!}x^{2016}(x^{2014}+x^{2013}f(x)+\ldots+f(x)^{2014})}.$$

Since f is of class C^{2016}, $\lim_{x\to 0} f^{(2016)}(\theta x) = f^{(2016)}(0)$ and

$$\lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}} = -\frac{2016!}{2015 f^{(2016)}(0)} > 0.$$

It means, by the comparison criterion, that the series $\sum_{n=1}^{\infty} a_n^r$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{r}{2015}}}$ converge and/or diverge simultaneously, hence the series $\sum_{n=1}^{\infty} a_n^r$ converges for $r > 2015,$ and diverges for $r \le 2015.$

COMBINATORICS

Problem 1. In $\mathbb{R}^{2015}$, a ball B_0 centered at $P_0(1,1,\dots,1)$ touches all the axis (of $\mathbb{R}^{2015}$). For how many points P from $\mathbb{R}^{2015}$ does there exist a ball centered at P which touches all the axis, and moreover touches the ball B_0 ?

Solution. (We consider the general case $\mathbb{R}^n$, $n \geq 3$) A ball B (in $\mathbb{R}^n$) centered at $P(c_1, c_2, \dots, c_n)$ and of radius r, touches all the axis of $\mathbb{R}^n$ iff $|c_1| = |c_2| = \dots = |c_n| = t > 0$ and $r = t\sqrt{n-1}$. Namely, the equation

$$c_1^2 + c_2^2 + \dots + (x_p - c_p)^2 + \dots + c_n^2 = r^2$$

needs to have a unique solution x_p for every $p = 1,2,\dots,n$. Consequently, P is a point satisfying the conditions of the problem iff $P = t(c_1, c_2, \dots, c_n)$, $c_p = \pm 1$, $t > 0$ and $|PP_0| = (1+t)\sqrt{n-1}$. If O is the origin then $|OP| = t\sqrt{n}$ and $|OP_0| = \sqrt{n}$.

Case 1. all $c_p = +1$. One has $(1+t)\sqrt{n-1} = |PP_0| = \big||OP| - |OP_0|\big| = |t-1|\sqrt{n}$. This equation in t has two positive solutions.

Case 2. For some k, $1 \leq k \leq n-1$, precisely k among the c_p's are equal to -1. Then $\cos \sphericalangle POP_0 = \frac{n-2k}{n}$ and from the cosine theorem we obtain

$$\begin{aligned}(n-1)(1+t)^2 = |PP_0|^2 &= |OP|^2 + |OP_0|^2 - 2|OP|\cdot|OP_0|\cos \sphericalangle POP_0 \\ &= nt^2 + n - 2(n-2k)t,\end{aligned}$$

i.e.

$$t^2 - 2(2n-2k+1)t + 1 = 0 .$$

This equation in t has two positive solutions for $1 \leq k \leq n-2$, and one positive solution (namely $t=1$) for $k = n-1$.

Case 3. all $c_p = -1$. One has $(1+t)\sqrt{n-1} = |PP_0| = \big||OP| + |OP_0|\big| = (t+1)\sqrt{n}$. This equation clearly has no solution in terms of t.

Consequently, the desired number of points P is

$$2\sum_{k=0}^{n-2} \frac{n!}{k!(n-k)!} + n = 2^{n+1} - n - 2 .$$

Problem 2. Let

$$S(N) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \dots \sum_{k_N=1}^{k_{N-1}} 1.$$

Find the limit $\lim_{N\to\infty} \sqrt[N]{S(N)}$.

Solution. Note that $S(N)$ is the number of N-tuples $(k_1, k_2, \dots, k_N)$ of integers satisfying

$$1 \le k_N \le k_{N-1} \le \dots \le k_1 \le N.$$

We define a map F from the set of such N-tuples to the set of N-tuples $(p_1, p_2, \dots, p_N)$ of integers satisfying

$$0 < p_N < p_{N-1} < \dots < p_1 < 2N$$

by "setting" $F(k_m) = k_m + N - m$ (with a slight abuse of notation). It is easily seen that F is a bijection, thus the two sets are of the same cardinality. Therefore, $S(N) = \frac{(2N-1)!}{N!(N-1)!}$. On the other hand, observe that

$$\lim_{N\to\infty} \frac{S(N+1)}{S(N)} = \lim_{N\to\infty} \frac{2N(2N+1)}{(N+1)N} = 4.$$

This yields $\lim_{N\to\infty} \sqrt[N]{S(N)} = 4$.

Problem 3. Consider the following set of sequences of complex numbers

$$\Im = \{(a_n)_{n>0} : a_0 = 0, a_1 = 1, a_{n+1} \in \{(1+i)a_n - ia_{n-1}, (1-i)a_n + ia_{n-1}\} \text{ for all } n \geq 1\}.$$

a) Show that every complex number $z = x + iy$, with $x, y \in \mathbb{Z}$, is an element of some sequence in $\Im$.

b) Find all the positive integers N such that $a_N = 0$ for some sequence $(a_n)_{n\geq 0} \in \Im$.

c) For a fixed positive integer N, find the cardinality of the set

$$\{(a_0, a_1, ..., a_N) \in \mathbb{C}^{n+1} : (a_n)_{n\geq 0} \in \Im \text{ and } a_N = 0\}.$$

Solution. For a sequence $(a_n)_{n\geq 0} \in \Im$ and arbitrary $n \geq 0$, denote by A_n the point in the Gaussian plane corresponding to the complex number a_n. By definition, we have that $a_{n+1} - a_n = \pm i(a_n - a_{n-1})$, hence $|a_{n+1} - a_n| = |a_n - a_{n-1}|$ for all $n \geq 1$, which leads to $|A_n A_{n+1}| = |a_{n+1} - a_n| = |a_1 - a_0| = 1$, for all $n \geq 0$. Moreover, the segment $A_n A_{n+1}$ is perpendicular to the segment $A_{n-1}A_n$. From here, we conclude that any sequence $(a_n)_{n\geq 0} \in \Im$ is represented in the plane by a path $A_0 A_1 ... A_n ...$ going through the points of $\mathbb{Z}[i]$, starting at $A_0(0,0)$, then passing through $A_1(1,0)$, and so on, always making steps of length 1 in a direction perpendicular to the previous one.

Alternatively, any sequence $(a_n)_{n\geq 0} \in \Im$ can be encoded as a sequence $(b_n)_{n\geq 1}$ (of steps) of the type $1, \pm i, \pm 1, \pm i, \pm 1, ...$ (where $b_n = a_n - a_{n-1}, n \geq 1$), in which case a_n is recovered by adding the first n steps. By denoting with p_n, q_n, r_n, s_n the number of occurrences of the values $1, -1, i, -i$, respectively, in the finite sequence $b_2, b_3, ..., b_n (n \geq 2)$, we have

$$a_n = (1 + p_n - q_n) + i(r_n - s_n)$$

$$p_n + q_n = [\tfrac{n-1}{2}]$$

$$r_n + s_n = [\tfrac{n}{2}]$$

where $[x]$ denotes the integer part of any real number x.

Assuming $a_n = x + iy$, we can express p_n, q_n, r_n, s_n from the above relations as follows:

$$p_n = \frac{x-1+[\frac{n-1}{2}]}{2}, \qquad q_n = \frac{-(x-1)+[\frac{n-1}{2}]}{2}$$

$$r_n = \frac{y+[\frac{n}{2}]}{2}, \qquad s_n = \frac{-y+[\frac{n}{2}]}{2}.$$

These four numbers are non-negative if and only if $[\frac{n-1}{2}] \geq |x-1|$ and $[\frac{n}{2}] \geq |x|$, while they are integers if and only if $x, y \in \mathbb{Z}$ and the following parity table is satisfied:

n	x	y
$4k$	even	even
$4k+1$	odd	even
$4k+2$	odd	odd
$4k+3$	even	odd

Hence, whenever $x, y \in \mathbb{Z}$, and n is sufficiently large (so that $[\frac{n-1}{2}] \geq |x-1|$ and $[\frac{n}{2}] \geq |x|$), there surely exists a sequence $(a_n)_{n\geq 0} \in \Im$ such that $a_n = x + iy$ (using the expressions for p_n, q_n, r_n, s_n from above). This completes the proof of the statement a).

Moreover, if a sequence $(a_n)_{n\geq 0} \in \Im$ is to satisfy $a_N = 0$ for some $N \geq 1$, then that happens only for $N = 4k, k \geq 1$, with $p_N = k-1, q_N = r_N = s_N = k$. On the other hand, this can clearly be used to construct such a sequence $(a_n)_{n\geq 0}$ up to $n = N$. Hence, the set of positive integers N that satisfy the requirement of b) is precisely $\{4k : k \in \mathbb{N}\}$.

Furthermore, the arrangement of the values 1 and -1 in the sequence $b_2, b_3, ..., b_N$ can be chosen in $\binom{2k-1}{k}$ ways, while the arrangement of the values i and $-i$ can be chosen in $\binom{2k}{k}$ ways. Therefore, the answer to c) is $\binom{2k-1}{k}\binom{2k}{k}$.

Problem 4. a) Let $\alpha:\mathbb{N}\to\mathbb{N}$ be a permutation of the set of natural numbers $\mathbb{N}=\{1,2,3,\dots\}$. Prove that the limit $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{(x-\alpha(k))^k}{k!}$ does not exist.

b) Prove that the set $\mathbb{Q}$ of rational numbers can be written as $\mathbb{Q}=\{q_1,q_2,\dots\}$ so that the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(x-q_k)^k}{k!}$$

exists for every real *x*.

Solution. We shall use the double inequality $(\frac{n}{3})^n<n!<(\frac{n}{2})^n$, which is valid for every sufficiently large *n* (this can be shown by straightforward induction).

a) Since, the inequality $\sum\limits_{k=1}^{n}\alpha(k)\geq\sum\limits_{k=1}^{n}k$ holds for every natural *n*, for infinitely many such *n* 's one has $\alpha(n)\geq n$. Hence, for an arbitrary (fixed) real *x* we have that

$$\frac{|x-\alpha(n)|^n}{n!}>(\frac{|2x-2\alpha(n)|}{n})^n>(\frac{3}{2})^n(\frac{\alpha(n)}{n})^n$$

holds for every n satisfying $2x<\frac{1}{2}n\leq\frac{1}{2}\alpha(n)$. Therefore, the necessary condition $\lim\limits_{n\to\infty}\frac{(x-\alpha(n))^n}{n!}=0$ is surely not fulfilled, which proves that the series $\sum\limits_{n=1}^{\infty}\frac{(x-\alpha(n))^n}{n!}$ converges for no real x.

b) We prove that the set $\mathbb{Q}$ of rational numbers can be enlisted as $\mathbb{Q}=\{q_1,q_2,\dots,q_n,\dots\}$ such that $|q_n|<\sqrt{n}$. To demonstrate this, assume $\mathbb{Q}$ is already written in some way as $\mathbb{Q}=\{r_1,r_2,\dots,r_n,\dots\}$. To begin with, choose $q_1=r_{i_1}$, where $i_1=\min\{i\mid |r_i|<1\}$. Consequently, take $q_n=r_{i_n}$, where $i_n=\min\{i\mid |r_i|<\sqrt{n}\ \textit{and}\ i\notin\{i_1,i_2,\dots i_{n-1}\}\}$, and continue doing so. Clearly, this procedure enlists all the rational numbers in a bijective way.

We are only left to observe that for every real *x*, the series $\sum\limits_{n=1}^{\infty}\frac{(x-q_n)^n}{n!}$ converges by Cauchy's criterion:

$$\sqrt[n]{\frac{|x-q_n|^n}{n!}}=\frac{|x-q_n|}{\sqrt[n]{n!}}<3(\frac{|x|}{n}+\frac{|q_n|}{n})\to 0\,.$$