# Balkan MO Shortlist 2012

**–** Algebra

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**A1** Prove that

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx),$$

for all positive real numbers $x, y$ and $z$.

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**A2** Let $a, b, c \geq 0$ and $a+b+c=\sqrt{2}$. Show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \geq 2+\frac{1}{\sqrt{3}}$$

In general if $a_1, a_2, \cdots, a_n \geq 0$ and $\sum_{i=1}^{n} a_i = \sqrt{2}$ we have

$$\sum_{i=1}^{n} \frac{1}{\sqrt{1+a_i^2}} \geq (n-1)+\frac{1}{\sqrt{3}}$$

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**A3** Determine the maximum possible number of distinct real roots of a polynomial $P(x)$ of degree $2012$ with real coefficients satisfying the condition

$$P(a)^3+P(b)^3+P(c)^3 \geq 3P(a)P(b)P(c)$$

for all real numbers $a, b, c \in \mathbb{R}$ with $a+b+c=0$

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**A4** Let $ABCD$ be a square of the plane $P$. Define the minimum and the maximum the value of the function $f : P \to R$ is given by $f(P) = \frac{PA+PB}{PC+PD}$

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**A5** Let $f, g : \mathbb{Z} \to [0, \infty)$ be two functions such that $f(n) = g(n) = 0$ with the exception of finitely many integers $n$. Define $h : \mathbb{Z} \to [0, \infty)$ by

$$h(n) = \max\{f(n-k)g(k) : k \in \mathbb{Z}\}.$$

Let $p$ and $q$ be two positive reals such that $1/p + 1/q = 1$. Prove that

$$\sum_{n\in\mathbb{Z}} h(n) \geq \left(\sum_{n\in\mathbb{Z}} f(n)^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} g(n)^q\right)^{1/q}.$$

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**A6** Let $k$ be a positive integer. Find the maximum value of

$$a^{3k-1}b+b^{3k-1}c+c^{3k-1}a+k^2a^kb^kc^k,$$

where $a$, $b$, $c$ are non-negative reals such that $a+b+c=3k$.

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– Combinatorics

**C1** Let $n$ be a positive integer. Let $P_n = \{2^n, 2^{n-1} \cdot 3, 2^{n-2} \cdot 3^2, \dots, 3^n\}$. For each subset $X$ of $P_n$, we write $S_X$ for the sum of all elements of $X$, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that $y$ is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$.
Prove that there is a subset $Y$ of $P_n$ such that $0 \leq y - S_Y < 2^n$

**G1** Let $A$, $B$ and $C$ be points lying on a circle $\Gamma$ with centre $O$. Assume that $\angle ABC > 90$. Let $D$ be the point of intersection of the line $AB$ with the line perpendicular to $AC$ at $C$. Let $l$ be the line through $D$ which is perpendicular to $AO$. Let $E$ be the point of intersection of $l$ with the line $AC$, and let $F$ be the point of intersection of $\Gamma$ with $l$ that lies between $D$ and $E$.
Prove that the circumcircles of triangles $BFE$ and $CFD$ are tangent at $F$.

**G2** Let $ABC$ be a triangle, and let $\ell$ be the line passing through the circumcenter of $ABC$ and parallel to the bisector of the angle $\angle A$. Prove that the line $\ell$ passes through the orthocenter of $ABC$ if and only if $AB = AC$ or $\angle BAC = 120^o$

**G3** Let $ABC$ be a triangle with circumcircle $c$ and circumcenter $O$, and let $D$ be a point on the side $BC$ different from the vertices and the midpoint of $BC$. Let $K$ be the point where the circumcircle $c_1$ of the triangle $BOD$ intersects $c$ for the second time and let $Z$ be the point where $c_1$ meets the line $AB$. Let $M$ be the point where the circumcircle $c_2$ of the triangle $COD$ intersects $c$ for the second time and let $E$ be the point where $c_2$ meets the line $AC$. Finally let $N$ be the point where the circumcircle $c_3$ of the triangle $AEZ$ meets $c$ again. Prove that the triangles $ABC$ and $NKM$ are congruent.

**G4** Let $M$ be the point of intersection of the diagonals of a cyclic quadrilateral $ABCD$. Let $I_1$ and $I_2$ are the incenters of triangles $AMD$ and $BMC$, respectively, and let $L$ be the point of intersection of the lines $DI_1$ and $CI_2$. The foot of the perpendicular from the midpoint $T$ of $I_1I_2$ to $CL$ is $N$, and $F$ is the midpoint of $TN$. Let $G$ and $J$ be the points of intersection of the line $LF$ with $I_1N$ and $I_1I_2$, respectively. Let $O_1$ be the circumcenter of triangle $LI_1J$, and let $\Gamma_1$ and $\Gamma_2$ be the circles with diameters $O_1L$ and $O_1J$, respectively. Let $V$ and $S$ be the second points of intersection of $I_1O_1$ with $\Gamma_1$ and $\Gamma_2$, respectively. If $K$ is point where the circles $\Gamma_1$ and $\Gamma_2$ meet again, prove that $K$ is the circumcenter of the triangle $SVG$.

**G5** G5 The incircle of a triangle $ABC$ touches its sides $BC$,$CA$,$AB$ at the points $A_1$,$B_1$,$C_1$.Let the projections of the orthocenter $H_1$ of the triangle $A_1B_1C_1$ to the lines $AA_1$ and $BC$ be $P$ and $Q$,respectively. Show that $PQ$ bisects the line segment $B_1C_1$

**G6** Let $P$ and $Q$ be points inside a triangle $ABC$ such that $\angle PAC = \angle QAB$ and $\angle PBC = \angle QBA$. Let $D$ and $E$ be the feet of the perpendiculars from $P$ to the lines $BC$ and $AC$, and $F$ be the foot of perpendicular from $Q$ to the line $AB$. Let $M$ be intersection of the lines $DE$ and $AB$. Prove that $MP \perp CF$

**G7** $ABCD$ is a cyclic quadrilateral. The lines $AD$ and $BC$ meet at X, and the lines $AB$ and $CD$ meet at $Y$ . The line joining the midpoints $M$ and $N$ of the diagonals $AC$ and $BD$, respectively, meets the internal bisector of angle $AXB$ at $P$ and the external bisector of angle $BYC$ at $Q$. Prove that $PXQY$ is a rectangle

– Number Theory

**N1** A sequence $(a_n)_{n=1}^{\infty}$ of positive integers satisfies the condition $a_{n+1} = a_n + \tau(n)$ for all positive integers $n$ where $\tau(n)$ is the number of positive integer divisors of $n$. Determine whether two consecutive terms of this sequence can be perfect squares.

**N2** Let the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ satisfy $a_0 = b_0 = 1, a_n = 9a_{n-1} - 2b_{n-1}$ and $b_n = 2a_{n-1} + 4b_{n-1}$ for all positive integers $n$. Let $c_n = a_n + b_n$ for all positive integers $n$.
Prove that there do not exist positive integers $k, r, m$ such that $c_r^2 = c_k c_m$.

**N3** Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:
(i) $f(n!) = f(n)!$ for every positive integer $n$,
(ii) $m - n$ divides $f(m) - f(n)$ whenever $m$ and $n$ are different positive integers.