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Problems

Problem 1: Let A be an $n \times n$ matrix with strictly positive elements and two vectors $u, v \in \mathbb{R}^n$, also with strictly positive elements, such that

$$Au = v \text{ and } Av = u.$$

Prove that $u = v$.

Problem 2: Calculate

$$\lim_{n\to\infty} n \int_0^{\infty} e^{-x} \sqrt[n]{e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}}\, dx.$$

Problem 3: Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^*A^2 = AA^*$. Prove that $A^2 = A$. (Here we denote by A^* the conjugate transpose of A.)

Problem 4: Let $(a_n)_{n\geq 1}$ be a monotone decreasing sequence of real numbers that converges to 0. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent if and only if the sequence $(a_n \ln n)_{n\geq 1}$ is bounded and $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \ln n$ is convergent.

Solutions

Problem 1

Solution 1 (due to Emmanouil Petrakis): A^2 has positive elements (as a simple calculation shows), so let $A^2 = (a_{ij})_{1\leq i,j\leq n}$ with $a_{ij} > 0$ for all i, j.

Note that $A^2u = A(Au) = Av = u$ and $A^2v = v$. Hence $A^2(u - tv) = u - tv,\ (1)$ for all $t \in \mathbb{R}$. We choose $t = \min\left\{\frac{u_i}{v_i},\ i \in \{1, 2, ..., n\}\right\}$, denoting by u_i, v_i the respective elements of u, v.

Hence, $u_i - tv_i \geq 0$ for all $i \in \{1, 2, ..., n\}$, and moreover we can find a $j \in \{1, 2, ...n\}$ such that $u_j - tv_j = 0$. At relation (1), by looking the $j-$th row, we obtain:

$$\sum_{k=1}^{n} a_{jk}(u_k - tv_k) = u_j - tv_j = 0.$$

However, in this relation the left hand side is a sum of non-negative numbers, hence

$$a_{jk}(u_k - tv_k) = 0 \Rightarrow u_k = tv_k,$$

for all $k \in \{1, 2, ..., n\}$.

We conclude that $u = tv$. Now,

$$Au = v \Rightarrow A(tv) = v \Rightarrow t(Av) = v \Rightarrow tu = v \Rightarrow t^2v = v,$$

and so we conclude that $t^2 = 1$, that is $t \in \{-1, 1\}$. However, if $t = -1$ we obtain a clear contradiction since u, v have positive elements, hence $t = 1$, as desired.

Solution 2 (due to Orestis Lignos. In fact, the two solutions are more or less isomorphic to each other): Let us denote $A = (a_{ij})_{1\leq i,j\leq n}$, $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Note that all variables are positive from the problem statement. We know that

$$a_{i1}u_1 + a_{i2}u_2 + \ldots + a_{in}u_n = v_i$$

and

$$a_{i1}v_1 + a_{i2}v_2 + \ldots + a_{in}v_n = u_i$$

for all $1 \leq i \leq n$. Let us denote $k = \min_{1\leq i\leq n}\left(\frac{u_i}{v_i}\right)$, and WLOG let $k = \frac{u_1}{v_1}$. Note that

$$\sum_{1\leq j\leq n} a_{j1}(u_iu_j - v_iv_j) = u_iv_i - u_iv_i = 0,\ (1)$$

for all $1 \leq i \leq n$. Moreover,

$$\sum_{1\leq j\leq n} a_{ij}(u_jv_1 - u_1v_j) = v_iv_1 - u_iu_1$$

and so

$$\sum_{2\leq j\leq n} a_{ij}(u_jv_1 - u_1v_j) = v_iv_1 - u_iu_1, \tag{2}$$

for all $1 \leq i \leq n$. Now, notice that $u_jv_1 - u_1v_j \geq 0$ for all $1 \leq j \leq n$, and so by relation (2) we obtain that $v_iv_1 \geq u_iu_1$ for all $1 \leq i \leq n$. Therefore, putting $i = 1$ in relation (1) we obtain that the left hand side is ≤ 0, and so equality must hold, i.e. $v_iv_1 = u_iu_1$ for all $1 \leq i \leq n$.

Putting $i = 1$ this readily implies that $k = 1$, that is $u_1 = v_1$, and subsequently $u_i = v_i$ for all $1 \leq i \leq n$, as desired.

Problem 2

Solution (due to Panagiotis-Nikolaos Glyptis): From Taylor's theorem (we use the Lagrange remainder form), there exists a $c \in (0, x)$ such that:

$$e^x - \sum_{k=1}^{n} \frac{1}{k!} = \frac{e^c}{(n+1)!} x^{n+1}$$

Now let

$$f_n(x) = \frac{n}{((n+1)!)^{\frac{1}{n}}} e^{\frac{c}{n} - x} x^{1+\frac{1}{n}}$$

and since we easily infer (for example, using Stirling's formula or the Cesaro-Stolz theorem) that

$$\lim_{n \to \infty} \frac{n}{((n+1)!)^{\frac{1}{n}}} = e,$$

we obtain that

$$f_n(x) \to e^{1-x} x$$

pointwise, and moreover

$$e^{\frac{c}{n} - x} < e^{\frac{x}{2} - x} \text{ for all } n \geq 2$$

and

$$64 e^{\frac{x}{4}} > x^{1+\frac{1}{n}} \text{ for all } n \geq 2,$$

and since the sequence $\frac{n}{((n+1)!)^{\frac{1}{n}}}$ converges, it is bounded and so there exists a $M > 0$ such that $\frac{n}{((n+1)!)^{\frac{1}{n}}} < M$ for all $n \geq 1$ All in all, $f_n(x)$ is bounded by a function with a finite integral in the integration interval, and so by the dominated convergence theorem we finally may write

$$\lim_{n \to \infty} \int_0^\infty f_n(x) dx = \int_0^\infty e^{1-x} x dx = e,$$

and so the given limit equals e.

Problem 3

Solution 1 (due to Anastasios Pastos): We know that $A^*A^2 = AA^*,\ (1)$. Our first claim is the following:

Claim 1: $\ker A = \ker A^*,\ (2)$.
Proof: Let $u \in \ker A$. From relation (1) we obtain that

$$AA^*u = A^*A^2u = 0 \Rightarrow \langle AA^*u, u\rangle = 0 \Rightarrow \langle A^*u, A^*u\rangle = 0 \Rightarrow A^*u = 0 \Rightarrow u \in \ker A^*$$

Hence $\ker A \subseteq \ker A^*$. However,

$$\operatorname{rank}(A^*) = \operatorname{rank}\left(\overline{A}^T\right) = \operatorname{rank}\overline{A} = \operatorname{rank} A,$$

hence we obtain $\dim \operatorname{Im} A^* = \dim \operatorname{Im} A \Rightarrow \dim\ker A^* = \dim\ker A$, by virtue of the rank-nullity theorem. To sum up, we have that $\ker A \subseteq \ker A^*$ and their dimensions are equal, hence the claim follows ■

Moving on, we may write

$$(A^*)^2A = (A^*A^2)^* = (AA^*)^* = AA^* = A^*A^2,$$

that is $A^*(A^*A - A^2) = O_n$, and so by the above Claim (relation (2)) we obtain $A(A^*A - A^2) = O_n$, that is $AA^*A = A^3,\ (3)$.

Multiplying relation (1) with A from the right, we obtain $A^*A^3 = AA^*A \Rightarrow A^*A^3 = A^3,\ (4)$. Now, we move on to our next Claim:

Claim 2: $\ker A \perp \operatorname{Im} A$.
Proof: It is well-known that $\ker A^* \perp \operatorname{Im} A$ (indeed, if $w \in \ker A^*$ and $Av \in \operatorname{Im} A$, then $\langle w, Av\rangle = \langle A^*w, v\rangle = \langle 0, v\rangle = 0$), hence using relation (2) we conclude that $\ker A \perp \operatorname{Im} A$ ■

Using Claim 2, we infer that $\ker A + \operatorname{Im} A$ is a direct sum of two subspaces. However,

$$\dim(\ker A \oplus \operatorname{Im} A) = \dim\ker A + \dim\operatorname{Im} A = n,$$

which implies that $\ker A \oplus \operatorname{Im} A = \mathbb{C}^n$.

Our next step is to chose orthonormal bases $\hat{u} = (u_1, u_2, \ldots, u_k)$ and $\hat{v} = (v_1, v_2, \ldots, v_m)$, corresponding to the subspaces $\operatorname{Im} A$ and $\ker A$, respectively. Note that if $A = O_n$ then $A^2 = A$ trivially holds. Else, $\operatorname{Im} A \neq \{0\}$. From the above results we know that $\hat{w} = (u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m)$ is an orthonormal base of $\mathbb{C}^n$. We perform a change of basis from the ordinary base of $\mathbb{C}^n$ to base $\hat{w}$.

Thus, we write

$$A = U^{-1}\left(\begin{array}{c|c} B & O_{k\times(n-k)} \\ \hline O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{array}\right)U,$$

with B being a $k \times k$ invertible matrix (this follows as $\operatorname{rank} B = \dim \operatorname{Im} A = k$. Moreover, we know that U is a unitary matrix, since its columns are elements of $\hat{w}$, which themselves constitute an orthonormal basis of $\mathbb{C}^n$.

Hence, we obtain $A = U^* \left(\begin{array}{c|c} B & O \\ \hline O & O \end{array} \right) U$, and $A^* = U^* \left(\begin{array}{c|c} B^* & O \\ \hline O & O \end{array} \right) U$. Returning back, relation (4) easily turns into $B^* B^3 = B^3 \Rightarrow B^* B^3 B^{-3} = B^3 B^{-3}$, that is $B^* = I_k$, which means $B = I_k$. Thus, to sum up, we obtain $A = U^* \left(\begin{array}{c|c} I_k & O \\ \hline O & O \end{array} \right) U$, and now it is trivial to conclude that $A^2 = A$, as desired.

Solution 2 (due to Orestis Lignos): Let $B = AA^*$ and $C = A^*A$. Then, $A^*A^2 = AA^*$ is rewritten as $B = CA$, and taking * in the previous relation we obtain $(A^*)^2 A = AA^*$, that is $B = A^*C$.

Claim 1: $(CB)^2 = B^3$.
Proof: Note that

$$(CB)^2 = CB \cdot CB = A^*A \cdot AA^* \cdot A^*A \cdot AA^* = (A^*A^2) \cdot ((A^*)^2 A^2) \cdot A^* =$$

$$= (AA^*)((A^*)^2 A \cdot A) \cdot A^* = (AA^*)(A \cdot A^*A) \cdot A^* = B^3$$

■

Claim 2: Matrices B and C commute.
Proof: We begin with a preliminary result: we prove that AA^*A is Hermitian. Indeed,

$$AA^*A = ((A^*)^2 A)) \cdot A = (A^*)^2 A^2 = A^* \cdot (A^*A^2) = A^*AA^*,$$

hence $AA^*A = A^*AA^*$. Now, note that

$$BC = A(A^*)^2 A = (AA^* \cdot A^*)A = (((A^*)^2 A) \cdot A^*)A = A^* \cdot (A^*AA^*) \cdot A =$$

$$= A^* \cdot (AA^*A) \cdot A = (A^*A) \cdot (A^*A^2) = (A^*A) \cdot (AA^*) = CB,$$

as desired (note that we used the first result in the end of the first line) ■

Since B and C are Hermitian and commute, they are simultaneously orthogonally diagonalizable. So, we may write

$$B = P\Delta_B P^* \text{ and } C = P\Delta_C P^*,$$

with P being a unitary matrix. Since $B = AA^*$ and $C = A^*A$, a well-known lemma (XY and YX have the same characteristic polynomial) implies that B and C, i.e. Δ_B and Δ_C, have the same characteristic polynomial, that is the same eigenvalues.

Assume that $\Delta_B = \operatorname{diag}\{b_1, b_2, \ldots, b_n\}$ and $\Delta_C = \operatorname{diag}\{c_1, c_2, \ldots, c_n\}$. Then, $\{b_i\}_{1 \le i \le n} = \{c_i\}_{1 \le i \le n}$, and $b_i, c_i \in \mathbb{R}$ (B and C are Hermitian matrices, so they have real eigenvalues). Using Claim 1,

$$(CB)^2 = B^3 \Rightarrow (\Delta_C \Delta_B)^2 = \Delta_B^3 \Rightarrow (c_i b_i)^2 = b_i^3, \text{ for all } i \in \{1, 2, \ldots, n\} \Rightarrow$$

$$\Rightarrow b_i = 0 \text{ or } b_i = c_i^2, \text{ for all } i \in \{1, 2, \ldots, n\}, \ (*).$$

We contend:

Claim 3: $b_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, n\}$.
Proof: Assume otherwise. Assume that $b_1 \neq 0$ and $b_1 \neq 1$. Then, using $(*)$, we obtain $c_1 = b_1^{1/2}$. Since $\{b_i\}_{1 \leq i \leq n} = \{c_i\}_{1 \leq i \leq n}$, there is an index j such that $c_1 = b_j$, hence $b_j = b_1^{1/2} \neq b_1$. Therefore, using $(*)$ again, we obtain that $c_j = b_1^{1/4} \notin \{b_1, b_1^{1/2}\}$.

Continuing, we produce infinitely many mutually distinct eigenvalues $b_1^{1/2^i}$ $(i \geq 0)$, a contradiction ■

Now, returing back to $(*)$, we infer that if $b_i \neq 0$, then $1 = b_i = c_i^2$, hence $c_i = 1$.

This implies that the multiplicity of the eigenvalue 1 in Δ_C is at least as large as the multiplicitity of the eigenvalue 1 in Δ_B. Since these two multiplicities must be equal (XY and YX have the same algebraic multiplicitity in non-zero eigenvalues), we must have equality everywhere: these two matrices must have the zeros and the ones in the exact same places.

To sum up $\Delta_B = \Delta_C$, hence $AA^* = A^*A$, and so A, A^* commute. Therefore, A is orthogonally diagonizable itself, that is $A = QDQ^*$ for some unitary matrix Q. Thus, $A^*A^2 = AA^*$ implies (assume that $\Delta = \text{diag}\{\lambda_i\}_{1 \leq i \leq n}$)

$$\overline{\lambda_i}\lambda_i^2 = \lambda\overline{\lambda_i},$$

and now this last relation easily implies $\lambda_i \in \{0, 1\}$, hence

$$A^2 = QD^2Q^* = QDQ^* = A,$$

as desired.

Problem 4

Solution (due to Orestis Lignos): In the following solution, we merely write $n \geq 1$ in all series, meaning that n ranges from 1 to $+\infty$. Moreover, we write $\log(\cdot)$ instead of $\ln(\cdot)$. We split the solution into two parts.

Part 1: *If the series* $\sum_{n\geq 1} \frac{a_n}{n}$ *is convergent, then* $(a_n \log n)$ *is bounded and* $\sum_{n\geq 1}(a_n - a_{n+1}) \log n$ *is convergent.* Note that

$$(a_1-a_2)\log 1+\ldots+(a_n-a_{n+1})\log n = a_1\log 1+a_2\log\frac{2}{1}+\ldots+a_n\log\frac{n}{n-1}-a_{n+1}\log n,$$

and since the sequence $((a_n - a_{n+1})\log n)$ is positive, we need to prove that

$$s_n := a_1 \log 1 + a_2 \log\frac{2}{1} + \ldots + a_n \log\frac{n}{n-1} - a_{n+1}\log n$$

is bounded. This follows using the inequality $\log x \leq x - 1$. Indeed,

$$s_n \leq a_1 \cdot 0 + a_2 \cdot \frac{1}{1} + \ldots + a_n \cdot \frac{1}{n-1} - a_{n+1}\log n < \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n},$$

and this last sum is obviously bounded, as the series $\sum_{n\geq 1} \frac{a_n}{n}$ is convergent.

To prove that $(a_n \log n)$ is bounced, note that

$$\log n < \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$$

for all $n \geq 1$. Hence, it suffices to show that $\frac{a_n}{1} + \frac{a_n}{2} + \ldots + \frac{a_n}{n}$ is bounded. However, this is trivially true, as (a_n) is decreasing, and $\frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n}$ is bounded, as we explained above.

Part 2: *If* $(a_n \log n)$ *is bounded and* $\sum_{n\geq 1}(a_n - a_{n+1})\log n$ *is convergent, then the series* $\sum_{n\geq 1} \frac{a_n}{n}$ *is bounded.* The proof of this direction closely follows the proof in Part 1. Let $t = \frac{n}{n-1}$. Then, using the inequality $\log t \geq \frac{t-1}{t}$, we obtain

$$\log n - \log(n-1) = \log\frac{n}{n-1} = \log t \geq \frac{t-1}{t} = \frac{1}{n}.$$

Therefore,

$$\sum_{1\leq n\leq k} \frac{a_n}{n} \leq a_1 + \sum_{2\leq n\leq k} a_n(\log n - \log(n-1)) = a_1 + \sum_{1\leq n\leq k-1} (a_n - a_{n+1})\log n + a_k \log k.$$

To conclude, we observe that $a_k \log k$ and $\sum_{1 \leq n \leq k-1} (a_n - a_{n+1}) \log n$ are bounded when $k \to +\infty$, hence the partial sums of the series $\sum_{n \geq 1} \frac{a_n}{n}$ are bounded, and since the series has positive terms we conclude it must be convergent, as desired.