22$^{\text{nd}}$ Junior Balkan Mathematical Olympiad

June 19-24 2018, Rhodes, Greece

Shortlisted problems

with Solutions

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Note of Confidentiality

The shortlisted problems should be kept strictly confidential until JBMO 2019

Contributing countries

The Organising Committee and the Problem Selection Committee of the JBMO 2018 wish to thank the following countries for contributing problem proposals.

Albania
Bulgaria
Cyprus
The Former Yugoslav Republic of Macedonia
Montenegro
Romania
Serbia
Turkey

Problem Selection Committee

Chairman: Dimitrios Kodokostas
Members: Silouanos Brazitikos
Alexandros Sygkelakis

Contents

A 1. Let x, y and z be positive numbers. Prove that

$$\frac{x}{\sqrt{\sqrt[4]{y}+\sqrt[4]{z}}}+\frac{y}{\sqrt{\sqrt[4]{z}+\sqrt[4]{x}}}+\frac{z}{\sqrt{\sqrt[4]{x}+\sqrt[4]{y}}}\geq\frac{\sqrt[4]{(\sqrt{x}+\sqrt{y}+\sqrt{z})^7}}{\sqrt{2\sqrt{27}}}.$$

Solution. Replacing $x=a^2, y=b^2, z=c^2$, where a, b, c are positive numbers, our inequality is equivalent to

$$\frac{a^2}{\sqrt{\sqrt{b}+\sqrt{c}}}+\frac{b^2}{\sqrt{\sqrt{c}+\sqrt{a}}}+\frac{c^2}{\sqrt{\sqrt{a}+\sqrt{b}}}\geq\frac{\sqrt[4]{(a+b+c)^7}}{\sqrt{2\sqrt{27}}}.$$

Using the Cauchy-Schwarz inequality for the left hand side we get

$$\frac{a^2}{\sqrt{\sqrt{b}+\sqrt{c}}}+\frac{b^2}{\sqrt{\sqrt{c}+\sqrt{a}}}+\frac{c^2}{\sqrt{\sqrt{a}+\sqrt{b}}}\geq\frac{(a+b+c)^2}{\sqrt{\sqrt{b}+\sqrt{c}}+\sqrt{\sqrt{c}+\sqrt{a}}+\sqrt{\sqrt{a}+\sqrt{b}}}. \quad (1)$$

Using Cauchy-Schwarz inequality for three positive numbers α, β, γ, we have

$$\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}\leq\sqrt{3(\alpha+\beta+\gamma)}.$$

Using this result twice, we have

$$\begin{aligned}\sqrt{\sqrt{b}+\sqrt{c}}+\sqrt{\sqrt{c}+\sqrt{a}}+\sqrt{\sqrt{a}+\sqrt{b}} &\leq \sqrt{6(\sqrt{a}+\sqrt{b}+\sqrt{c})}\\ &\leq\sqrt{6\sqrt{3(a+b+c)}}. \quad (2)\end{aligned}$$

Combining (1) and (2) we get the desired result.

□

Alternative solution by PSC. We will use Hölder's inequality in the form

$$\begin{aligned}&(a_{11}+a_{12}+a_{13})(a_{21}+a_{22}+a_{23})(a_{31}+a_{32}+a_{33})(a_{41}+a_{42}+a_{43})\\ &\geq\left((a_{11}a_{21}a_{31}a_{41})^{1/4}+(a_{12}a_{22}a_{32}a_{42})^{1/4}+(a_{13}a_{23}a_{33}a_{43})^{1/4}\right)^4.\end{aligned}$$

where a_{ij} are positive numbers. Using this appropriately we get

$$\begin{aligned}(1+1+1)((\sqrt{b}+\sqrt{c})+(\sqrt{c}+\sqrt{a})+(\sqrt{a}+\sqrt{b}))\left(\frac{a^2}{\sqrt{\sqrt{b}+\sqrt{c}}}+\frac{b^2}{\sqrt{\sqrt{c}+\sqrt{a}}}+\frac{c^2}{\sqrt{\sqrt{a}+\sqrt{b}}}\right)^2\\ \geq(a+b+c)^4.\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$(\sqrt{b}+\sqrt{c})+(\sqrt{c}+\sqrt{a})+(\sqrt{a}+\sqrt{b})=2(\sqrt{a}+\sqrt{b}+\sqrt{c})\leq2\sqrt{3(a+b+c)}.$$

Combining these two inequalities we get the desired result.

A 2. Find the maximum positive integer k such that for any positive integers m, n such that $m^3+n^3 > (m+n)^2$, we have

$$m^3 + n^3 \geq (m+n)^2 + k.$$

Solution. We see that for $m = 3$ and $n = 2$ we have $m^3 + n^3 > (m+n)^2$, thus

$$3^3 + 2^3 \geq (3+2)^2 + k \Rightarrow k \leq 10.$$

We will show that $k = 10$ is the desired maximum. In other words, we have to prove that

$$m^3 + n^3 \geq (m+n)^2 + 10.$$

The last inequality is equivalent to

$$(m+n)(m^2 + n^2 - mn - m - n) \geq 10.$$

If $m + n = 2$ or $m + n = 3$, then $(m, n) = (1, 1), (1, 2), (2, 1)$ and we can check that none of them satisfies the condition $m^3 + n^3 > (m+n)^2$.

If $m + n = 4$, then $(m, n) = (1, 3), (2, 2), (3, 1)$. The pair $(m, n) = (2, 2)$ doesn't satisfy the condition. The pairs $(m, n) = (1, 3), (3, 1)$ satisfy the condition and we can readily check that $m^3 + n^3 \geq (m+n)^2 + 10$.

If $m + n \geq 5$ then we will show that

$$m^2 + n^2 - mn - m - n \geq 2$$

which is equivalent to

$$(m-n)^2 + (m-1)^2 + (n-1)^2 \geq 6.$$

If at least one of the numbers m, n is greater or equal to 4 then $(m-1)^2 \geq 9$ or $(n-1)^2 \geq 9$ hence the desired result holds. As a result, it remains to check what happens if $m \leq 3$ and $n \leq 3$. Using the condition $m + n \geq 5$ we have that all such pairs are $(m, n) = (2, 3), (3, 2), (3, 3)$.

All of them satisfy the condition and also the inequality $m^2 + n^2 - mn - m - n \geq 2$, thus we have the desired result.

□

Alternative solution by PSC. The problem equivalently asks for to find the minimum value of

$$A = (m+n)(m^2 + n^2 - mn - m - n),$$

given that $(m+n)(m^2 + n^2 - mn - m - n) > 0$. If $m = n$, we get that $m > 2$ and

$$A = 2m(m^2 - 2m) \geq 6(3^2 - 6) = 18.$$

Suppose without loss of generality that $m > n$. If $n = 1$, then $m(m+1)(m-2) > 0$, therefore $m > 2$ and

$$A \geq 3 \cdot (3+1) \cdot (3-2) = 12.$$

If $n \geq 2$, then since $m \geq n + 1$ we have

$$A = (m+n)\left(m(m-n-1) + n^2 - n\right) \geq (2n+1)(n^2 - n) \geq 5(2^2 - 2) = 10.$$

In all cases $A \geq 10$ and the equality holds if $m = n + 1$ and $n = 2$, therefore if $m = 3$ and $n = 2$. It follows that the maximum k is $k = 10$.

A 3. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{ab(b+1)(c+1)} + \frac{1}{bc(c+1)(a+1)} + \frac{1}{ca(a+1)(b+1)} \geq \frac{3}{(1+abc)^2}.$$

Solution. The required inequality is equivalent to

$$\frac{c(a+1)+a(b+1)+b(c+1)}{abc(a+1)(b+1)(c+1)} \geq \frac{3}{(1+abc)^2},$$

or equivalently to,

$$(1+abc)^2(ab+bc+ca+a+b+c) \geq 3abc(ab+bc+ca+a+b+c+abc+1).$$

Let $m = a+b+c$, $n = ab+bc+ca$ and $x^3 = abc$, then the above can be rewritten as

$$(m+n)(1+x^3)^2 \geq 3x^3(x^3+m+n+1),$$

or

$$(m+n)(x^6-x^3+1) \geq 3x^3(x^3+1).$$

By the AM-GM inequality we have $m \geq 3x$ and $n \geq 3x^2$, hence $m+n \geq 3x(x+1)$. It is sufficient to prove that

$$\begin{aligned} x(x+1)(x^6-x^3+1) \geq x^3(x+1)(x^2-x+1) &\iff \\ 3(x^6-x^3+1) \geq x^2(x^2-x+1) &\iff \\ (x^2-1)^2 \geq 0, & \end{aligned}$$

which is true.

□

Alternative solution by PSC. We present here an approach without fully expanding.
Let $abc = k^3$ and set $a = k\frac{x}{y}$, $b = k\frac{y}{z}$, $c = k\frac{z}{x}$, where $k, x, y, z > 0$. Then, the inequality can be rewritten as

$$\frac{z^2}{(ky+z)(kz+x)} + \frac{x^2}{(kz+x)(kx+y)} + \frac{y^2}{(kx+y)(ky+z)} \geq \frac{3k^2}{(1+k^3)^2}.$$

Using the Cauchy-Schwarz inequality we have that

$$\sum_{cyclic} \frac{z^2}{(ky+z)(kz+x)} \geq \frac{(x+y+z)^2}{(ky+z)(kz+x)+(kz+x)(kx+y)+(kx+y)(ky+z)},$$

therefore it suffices to prove that

$$\frac{(x+y+z)^2}{(ky+z)(kz+x)+(kz+x)(kx+y)+(kx+y)(ky+z)} \geq \frac{3k^2}{(1+k^3)^2}$$

or

$$\left((1+k^3)^2 - 3k^3\right)(x^2+y^2+z^2) \geq \left(3k^2(k^2+k+1) - 2(1+k^3)^2\right)(xy+yz+zx).$$

Since $x^2+y^2+z^2 \geq xy+yz+zx$ and $(1+k^3)^2 - 3k^3 > 0$, it is enough to prove that

$$(1+k^3)^2 - 3k^3 \geq 3k^2(k^2+k+1) - 2(1+k^3)^2,$$

or

$$(k-1)^2(k^2+1)(k+1)^2 \geq 0,$$

which is true.

A 4. Let $k > 1$, $n > 2018$ be positive integers, and let n be odd. The nonzero rational numbers x_1, $x_2, \ldots, x_n$ are not all equal and satisfy

$$x_1 + \frac{k}{x_2} = x_2 + \frac{k}{x_3} = x_3 + \frac{k}{x_4} = \cdots = x_{n-1} + \frac{k}{x_n} = x_n + \frac{k}{x_1}.$$

Find:

a) the product $x_1 x_2 \ldots x_n$ as a function of k and n

b) the least value of k, such that there exist n, x_1, $x_2, \ldots, x_n$ satisfying the given conditions.

Solution. a) If $x_i = x_{i+1}$ for some i (assuming $x_{n+1} = x_1$), then by the given identity all x_i will be equal, a contradiction. Thus $x_1 \neq x_2$ and

$$x_1 - x_2 = k\frac{x_2 - x_3}{x_2 x_3}.$$

Analogously

$$x_1 - x_2 = k\frac{x_2 - x_3}{x_2 x_3} = k^2 \frac{x_3 - x_4}{(x_2 x_3)(x_3 x_4)} = \cdots = k^n \frac{x_1 - x_2}{(x_2 x_3)(x_3 x_4) \ldots (x_1 x_2)}.$$

Since $x_1 \neq x_2$ we get

$$x_1 x_2 \ldots x_n = \pm\sqrt{k^n} = \pm k^{\frac{n-1}{2}} \sqrt{k}.$$

If one among these two values, positive or negative, is obtained, then the other one will be also obtained by changing the sign of all x_i since n is odd.

b) From the above result, as n is odd, we conclude that k is a perfect square, so $k \geq 4$. For $k = 4$ let $n = 2019$ and $x_{3j} = 4$, $x_{3j-1} = 1$, $x_{3j-2} = -2$ for $j = 1, 2, \ldots, 673$. So the required least value is $k = 4$.

□

Comment by PSC. There are many ways to construct the example when $k = 4$ and $n = 2019$. Since $3 \mid 2019$, the idea is to find three numbers x_1, x_2, x_3 satisfying the given equations, not all equal, and repeat them as values for the rest of the x_i's. So, we want to find x_1, x_2, x_3 such that

$$x_1 + \frac{4}{x_2} = x_2 + \frac{4}{x_3} = x_3 + \frac{4}{x_1}.$$

As above, $x_1 x_2 x_3 = \pm 8$. Suppose without loss of generality that $x_1 x_2 x_3 = -8$. Then, solving the above system we see that if $x_1 \neq 2$, then

$$x_2 = -\frac{4}{x_1 - 2} \quad \text{and} \quad x_3 = 2 - \frac{4}{x_1},$$

leading to infinitely many solutions. The example in the official solution is obtained by choosing $x_1 = -2$.

Comment by PSC. An alternative formulation of the problem's statement could be the following: Let $k > 1$ be a positive integer. Suppose that there exists an odd positive integer $n > 2018$ and nonzero rational numbers $x_1, x_2, \ldots, x_n$, not all of them equal, that satisfy

$$x_1 + \frac{k}{x_2} = x_2 + \frac{k}{x_3} = x_3 + \frac{k}{x_4} = \cdots = x_{n-1} + \frac{k}{x_n} = x_n + \frac{k}{x_1}.$$

Find the minimum value of k.

A 5. Let a, b, c, d and x, y, z, t be real numbers such that

$$0 \leq a, b, c, d \leq 1, \quad x, y, z, t \geq 1 \quad \text{and} \quad a + b + c + d + x + y + z + t = 8.$$

Prove that

$$a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 + t^2 \leq 28.$$

When does the equality hold?

Solution. We observe that if $u \leq v$ then by replacing (u, v) with $(u - \varepsilon,\ v + \varepsilon)$, where $\varepsilon > 0$, the sum of squares increases. Indeed,

$$(u - \varepsilon)^2 + (v + \varepsilon)^2 - u^2 - v^2 = 2\varepsilon(v - u) + 2\varepsilon^2 > 0.$$

Then, denoting

$$E(a, b, c, d, x, y, z, t) = a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 + t^2,$$

and assuming without loss of generality that $a \leq b \leq c \leq d$ and $x \leq y \leq z \leq t$, we have

$$\begin{aligned} E(a, b, c, d, x, y, z, t) &\leq E(0, 0, 0, 0, a + x, b + y, c + z, d + t) \\ &\leq E(0, 0, 0, 0, 1, b + y, c + z, a + d + x + t - 1) \\ &\leq E(0, 0, 0, 0, 1, 1, c + z, a + b + d + x + y + t - 2) \\ &\leq E(0, 0, 0, 0, 1, 1, 1, 5) = 28. \end{aligned}$$

Note that if $(a, b, c, d, x, y, z, t) \neq (0, 0, 0, 0, 1, 1, 1, 5)$, at least one of the above inequalities, obtained by the ϵ replacement mentioned above, should be a strict inequality. Thus, the maximum value of E is 28, and it is obtained only for $(a, b, c, d, x, y, z, t) = (0, 0, 0, 0, 1, 1, 1, 5)$ and permutations of a, b, c, d and of x, y, z, t.

□

Alternative solution by PSC. Since $0 \leq a, b, c, d \leq 1$ we have that $a^2 \leq a$, $b^2 \leq b$, $c^2 \leq c$ and $d^2 \leq d$. It follows that

$$a^2 + b^2 + c^2 + d^2 \leq a + b + c + d. \tag{1}$$

Moreover, using the fact that $y + z + t \geq 3$, we get that $x \leq 5$. This means that

$$(x - 1)(x - 5) \leq 0 \iff x^2 \leq 6x - 5.$$

Similarly we prove that $y^2 \leq 6x - 5$, $z^2 \leq 6z - 5$ and $t^2 \leq 6t - 5$. Adding them we get

$$x^2 + y^2 + z^2 + t^2 \leq 6(x + y + z + t) - 20. \tag{2}$$

Adding (1) and (2) we have that

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 + t^2 &\leq a + b + c + d + 6(x + y + z + t) - 20 \\ &\leq 6(a + b + c + d + x + y + z + t) - 20 = 28. \end{aligned}$$

We can readily check that the equality holds if and only if $(a, b, c, d, x, y, z, t) = (0, 0, 0, 1, 1, 1, 5)$ and permutations of a, b, c, d and of x, y, z, t.

A 6. Let a, b, c be positive numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{a}{\sqrt{a^3+5}} + \frac{b}{\sqrt{b^3+5}} + \frac{c}{\sqrt{c^3+5}} \leq \frac{\sqrt{6}}{2}.$$

Solution. From AM-GM inequality we have

$$a^3 + a^3 + 1 \geq 3a^2 \Rightarrow 2(a^3+5) \geq 3(a^2+3).$$

Using the condition $ab + bc + ca = 3$, we get

$$(a^3+5) \geq 3(a^2+ab+bc+ca) = 3(c+a)(a+b),$$

therefore

$$\frac{a}{\sqrt{a^3+5}} \leq \sqrt{\frac{2a^2}{3(c+a)(a+b)}}. \tag{1}$$

Using again the AM-GM inequality we get

$$\sqrt{\frac{2a^2}{3(c+a)(a+b)}} \leq \sqrt{\frac{2}{3}}\left(\frac{\frac{a}{c+a}+\frac{a}{a+b}}{2}\right) = \frac{\sqrt{6}}{6}\left(\frac{a}{c+a}+\frac{a}{a+b}\right). \tag{2}$$

From (1) and (2) we obtain

$$\frac{a}{\sqrt{a^3+5}} \leq \frac{\sqrt{6}}{6}\left(\frac{a}{c+a}+\frac{a}{a+b}\right).$$

Similar inequalities hold by cyclic permutations of the a, b, c's. Adding all these we get

$$\sum_{cyclic} \frac{a}{\sqrt{a^3+5}} \leq \sum_{cyc} \frac{\sqrt{6}}{6}\left(\frac{a}{c+a}+\frac{a}{a+b}\right) = \frac{\sqrt{6}}{6}\cdot 3 = \frac{\sqrt{6}}{2},$$

which is the desired result.

□

A 7. Let A be a set of positive integers with the following properties:

(a) If n is an element of A then $n \leqslant 2018$.

(b) If S is a subset of A with $|S| = 3$ then there are two elements n, m of S with $|n-m| \geqslant \sqrt{n}+\sqrt{m}$.

What is the maximum number of elements that A can have?

Solution. Assuming $n > m$ we have

$$\begin{aligned}|n-m| \geqslant \sqrt{n}+\sqrt{m} &\Leftrightarrow \left(\sqrt{n}-\sqrt{m}\right)\left(\sqrt{n}+\sqrt{m}\right) \geqslant \sqrt{n}+\sqrt{m} \\ &\Leftrightarrow \sqrt{n} \geqslant \sqrt{m}+1.\end{aligned}$$

Let $A_k = \{k^2, k^2+1, \ldots, (k+1)^2-1\}$. Note that each A_k can contain at most two elements of since if $n, m \in$ with $n > m$ then

$$\sqrt{n}-\sqrt{m} \leqslant \sqrt{(k+1)^2-1}-\sqrt{k^2} < (k+1)-k = 1.$$

In particular, since $\subseteq A_1 \cup \cdots \cup A_{44}$, we have $|S| \leqslant 2 \cdot 44 = 88$.

On the other hand we claim that $A = \{m^2 : 1 \leqslant m \leqslant 44\} \cup \{m^2+m : 1 \leqslant m \leqslant 44\}$ satisfies the properties and has $|A| = 88$. We check property (b) as everything else is trivial.
So let r, s, t be three elements of A and assume $r < s < t$. There are two cases for r.

(i) If we have that $r = m^2$, then $t \geqslant (m+1)^2$ and so $\sqrt{t}-\sqrt{r} \geq 1$ verifying (b).

(ii) If we have that $r = m^2+m$, then $t \geqslant (m+1)^2+(m+1)$ and

$$\begin{aligned}\sqrt{t} \geqslant \sqrt{r}+1 &\Leftrightarrow \sqrt{(m+1)^2+(m+1)} \geqslant \sqrt{m^2+m}+1 \\ &\Leftrightarrow m^2+3m+2 \geqslant m^2+m+1+2\sqrt{m^2+m} \\ &\Leftrightarrow 2m+1 \geqslant 2\sqrt{m^2+m} \\ &\Leftrightarrow 4m^2+4m+1 \geqslant 4m^2+4m.\end{aligned}$$

So property (b) holds in this case as well.

COMBINATORICS

C 1. A set S is called *neighbouring* if it has the following two properties:

a) S has exactly four elements

b) for every element x of S, at least one of the numbers $x-1$ or $x+1$ belongs to S.

Find the number of all neighbouring subsets of the set $\{1,2,\ldots,n\}$.

Solution. Let us denote with a and b the smallest and the largest element of a neighbouring set S, respectively. Since $a-1 \notin S$, we have that $a+1 \in S$. Similarly, we conclude that $b-1 \in S$. So, every neighbouring set has the following form $\{a, a+1, b-1, b\}$ for $b-a \geq 3$. The number of the neighbouring subsets for which $b-a=3$ is $n-3$. The number of the neighbouring subsets for which $b-a=4$ is $n-4$ and so on. It follows that the number of the neighbouring subsets of the set $\{1,2,\ldots,n\}$ is:

$$(n-3)+(n-4)+\cdots+3+2+1=\frac{(n-3)(n-2)}{2}.$$

□

C 2. A set T of n three-digit numbers has the following five properties:

(1) No number contains the digit 0.

(2) The sum of the digits of each number is 9.

(3) The units digits of any two numbers are different.

(4) The tens digits of any two numbers are different.

(5) The hundreds digits of any two numbers are different.

Find the largest possible value of n.

Solution. Let S denote the set of three-digit numbers that have digit sum equal to 9 and no digit equal to 0. We will first find the cardinality of S. We start from the number 111 and each element of S can be obtained from 111 by a string of 6 A's (which means that we add 1 to the current digit) and 2 G's (which means go to the next digit). Then for example 324 can be obtained from 111 by the string $AAGAGAAA$. There are in total

$$\frac{8!}{6! \cdot 2!} = 28$$

such words, so S contains 28 numbers. Now, from the conditions (3), (4), (5), if $\overline{abc}$ is in T then each of the other numbers of the form $\overline{**c}$ cannot be in T, neither $\overline{*b*}$ can be, nor $\overline{a**}$. Since there are $a+b-2$ numbers of the first category, $a+c-2$ from the second and $b+c-2$ from the third one. In these three categories there are

$$(a+b-2)+(b+c-2)+(c+a-2) = 2(a+b+c)-6 = 2 \cdot 9 - 6 = 12$$

distinct numbers that cannot be in T if $\overline{abc}$ is in T. So, if T has n numbers, then $12n$ are the forbidden ones that are in S, but each number from S can be a forbidden number no more than three times, once for each of its digits, so

$$n + \frac{12n}{3} \leq 28 \iff n \leq \frac{28}{5},$$

and since n is an integer, we get $n \leq 5$. A possible example for $n = 5$ is

$$T = \{144, 252, 315, 423, 531\}.$$

□

Comment by PSC. It is classical to compute the cardinality of S and this can be done in many ways. In general, the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

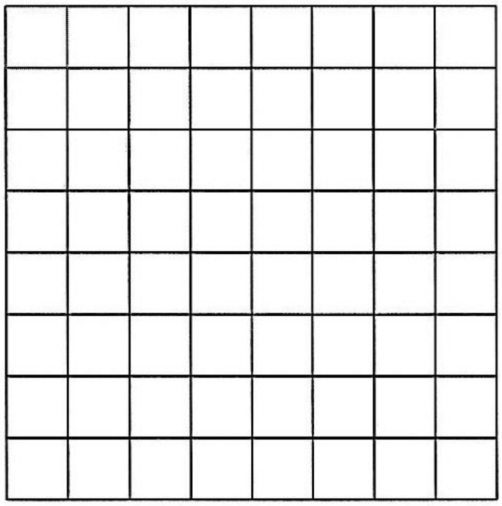
in positive integers, where the order of x_i matters, is well known that equals to $\binom{n-1}{k-1}$. In our case, we want to count the number of positive solutions to $a+b+c=9$. By the above, this equals to $\binom{9-1}{3-1} = 28$. Using the general result above, we can also find that there are $a+b-2$ numbers of the form $\overline{**c}$.

C 3. The cells of a 8×8 table are initially white. Alice and Bob play a game. First Alice paints n of the fields in red. Then Bob chooses 4 rows and 4 columns from the table and paints all fields in them in black. Alice wins if there is at least one red field left. Find the least value of n such that Alice can win the game no matter how Bob plays.

Solution. We will show that the least value of n is $n = 13$.
If $n \leq 12$, Bob wins by painting black the 4 rows containing the highest numbers of red cells. Indeed, if at least 5 red cells remain, then one of the rows not blackened contains at least 2 red cells. Thus, each one of the rows blackened contained at least 2 red cells, and then all blackened cells were at least 8. However, in this case, at most 4 would be not blackened, a contradiction. It follows that at most 4 red cells remain which can be easily blackened by Bob choosing the 4 columns that they are in.
Now let $n = 13$. Enumerate the rows and the columns from 1 to 8 and each field will be referred to by the pair (row,column) it is in. Let Alice paint in red the fields

$$(1,1),(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,5),(6,6),(7,7),(8,8),$$

as in the following figure.



Suppose that Bob has managed to paint all red fields in black. The cells $(6,6)$, $(7,7)$, $(8,8)$ are painted black by three different lines (rows or columns) containing no other red felds, so the remaining 10 red fields have to be painted black by the remaining 5 lines. As no line contains more than 2 red fields, each red field has to be contained in exactly one of these lines. Assume that $(1,1)$ is painted black by a row, that is, row 1 is painted black. Let k be the least positive integer such that row k has not been painted black, where $2 \leq k \leq 5$. Then field $(k, k-1)$ should be painted black by column $k-1$. However, in this column there is another red field $(j, k-1)$ contained in the painted row with number $j < k$, which is a contradiction. Similar reasoning works if $(1,1)$ is painted black by a column.
□

Comment by PSC. Here is another reasoning to conclude that if Alice paints the table as above, then she wins.
Let A be the 5×5 square defined by the corners $(1,1)$ and $(5,5)$.

Case 1: If Bob chooses 3 rows to paint black the cells $(6,6)$, $(7,7)$ and $(8,8)$ then he has to use 1 row and 4 columns to paint in black the remaining 10 red squares in A. Then no matter which 4 columns

Bob select, the remaining 1 column in A contains two red squares which cannot be painted in black using only 1 row. Similar reasoning stands if Bob chooses 3 columns.

Case 2: If Bob chooses 2 rows and 1 column to paint black the cells $(6,6)$, $(7,7)$ and $(8,8)$, then he has to use 3 columns and 2 rows to paint in black the remaining 10 red squares in A. Then, no matter which 3 columns Bob select, the remaining 2 columns in A contain 4 red squares in 3 different rows which cannot be painted in black using only 2 rows. Similar reasoning stands if Bob chooses 1 row and 2 columns.

GEOMETRY

G 1. Let H be the orthocentre of an acute triangle ABC with $BC > AC$, inscribed in a circle Γ. The circle with centre C and radius CB intersects Γ at the point D, which is on the arc AB not containing C. The circle with centre C and radius CA intersects the segment CD at the point K. The line parallel to BD through K, intersects AB at point L. If M is the midpoint of AB and N is the foot of the perpendicular from H to CL, prove that the line MN bisects the segment CH.

Solution. We use standard notation for the angles of triangle ABC. Let P be the midpoint of CH and O the centre of Γ. As

$$\alpha = \angle BAC = \angle BDC = \angle DKL,$$

the quadrilateral $ACKL$ is cyclic. From the relation $CB = CD$ we get $\angle BCD = 180^\circ - 2\alpha$, so

$$\angle ACK = \gamma + 2\alpha - 180^\circ,$$

where $\gamma = \angle ACB$. From the relation $CK = CA$ we get

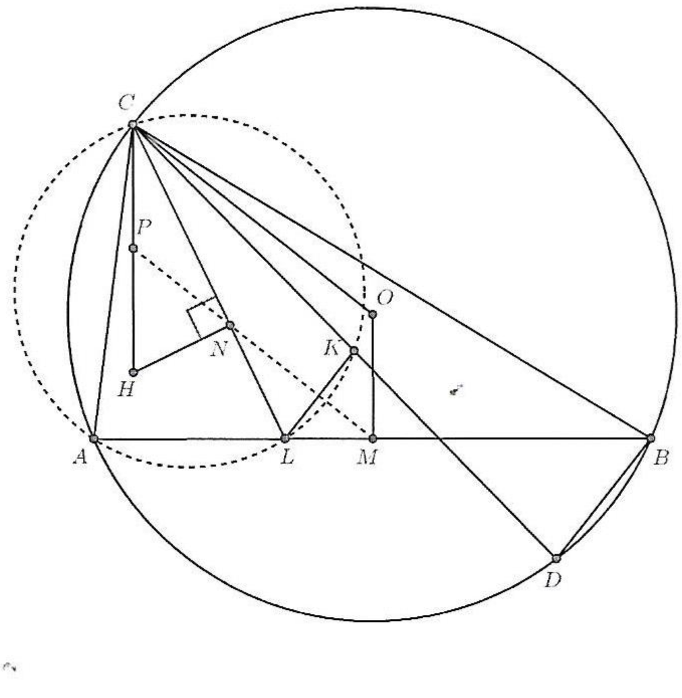
$$\angle ALC = \angle AKC = 180^\circ - \alpha - \frac{\gamma}{2}$$

and thus from the triangle ACL we obtain

$$\angle ACL = 180^\circ - \alpha - \angle ALC = \frac{\gamma}{2},$$

which means that CL is the angle bisector of $\angle ACB$, thus $\angle ACL = \angle BCL$. Moreover, from the fact that $CH \perp AB$ and the isosceles triangle BOC has $\angle BOC = 2\alpha$, we get $\angle ACH = \angle BCO = 90^\circ - \alpha$. It follows that,

$$\angle NPH = 2\angle NCH = \angle OCH. \tag{3}$$



On the other hand, it is known that $2CP = CH = 2OM$ and $CP \parallel OM$, so $CPMO$ is a parallelogram and

$$\angle MPH = \angle OCH. \tag{4}$$

Now from (3) and (4) we obtain that

$$\angle MPH = \angle NPH,$$

which means that the points M, N, P are collinear.

□

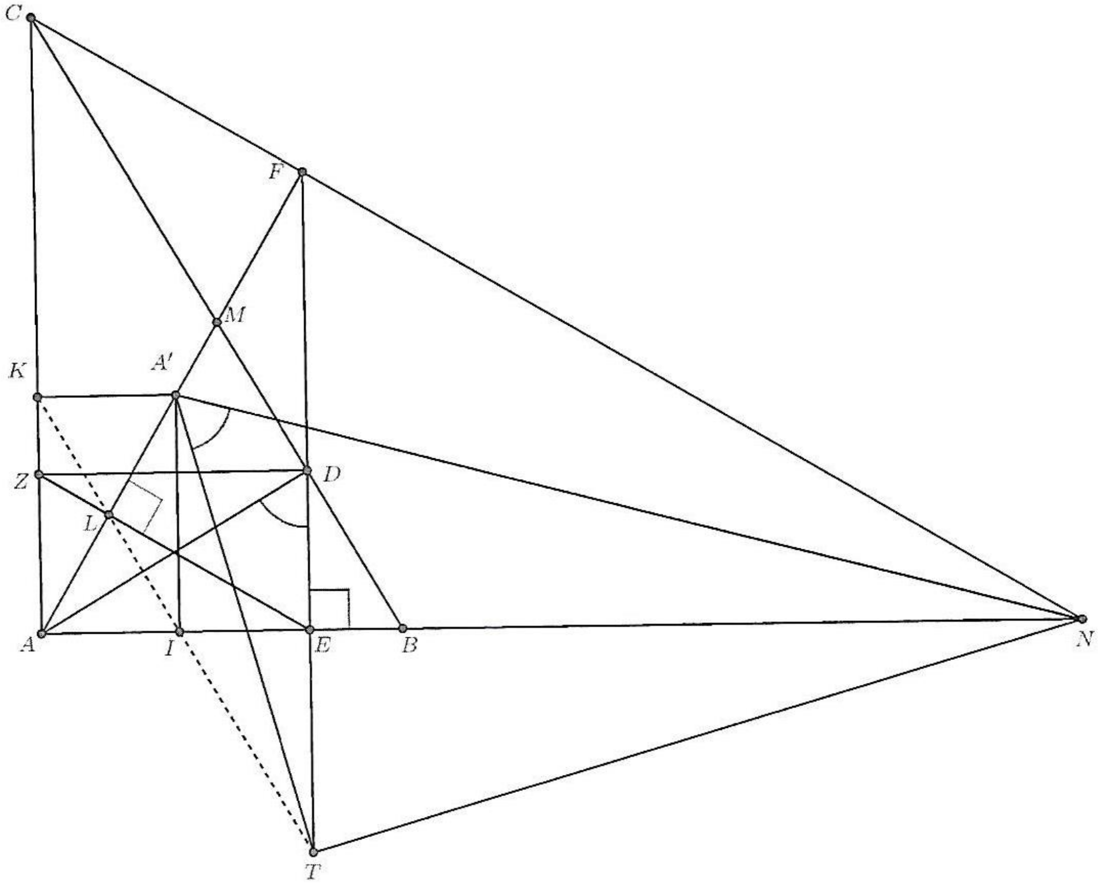
Alternative formulation of the statement by PSC.

Let H be the orthocentre of an acute triangle ABC with $BC > AC$, inscribed in a circle Γ. A point D on Γ, which is on the arc AB not containing C, is chosen such that $CB = CD$. A point K is chosen on the segment CD such that $CA = CK$. The line parallel to BD through K, intersects AB at point L. If M is the midpoint of AB and N is the foot of the perpendicular from H to CL, prove that the line MN bisects the segment CH.

G 2. Let ABC be a right angled triangle with $\angle A = 90^\circ$ and AD its altitude. We draw parallel lines from D to the vertical sides of the triangle and we call E, Z their points of intersection with AB and AC respectively. The parallel line from C to EZ intersects the line AB at the point N Let A' be the symmetric of A with respect to the line EZ and I, K the projections of A' onto AB and AC respectively. If T is the point of intersection of the lines IK and DE, prove that $\angle NA'T = \angle ADT$.

Solution. Suppose that the line AA' intersects the lines EZ, BC and CN at the points L, M, F respectively. The line IK being diagonal of the rectangle $KA'IA$ passes through L, which by construction of A', is the middle of the other diagonal AA'. The triangles ZAL, ALE are similar, so $\angle ZAL = \angle AEZ$. By the similarity of the triangles ABC, DAB, we get $\angle ACB = \angle BAD$. We have also that $\angle AEZ = \angle BAD$, therefore

$$\angle ZAL = \angle CAM = \angle ACB = \angle ACM.$$



Since $AF \perp CN$, we have that the right triangles AFC and CDA are equal. Thus the altitudes from the vertices F, D of the triangles AFC, CDA respectively are equal. It follows that $FD \parallel AC$ and since $DE \parallel AC$ we get that the points E, D, F are collinear.

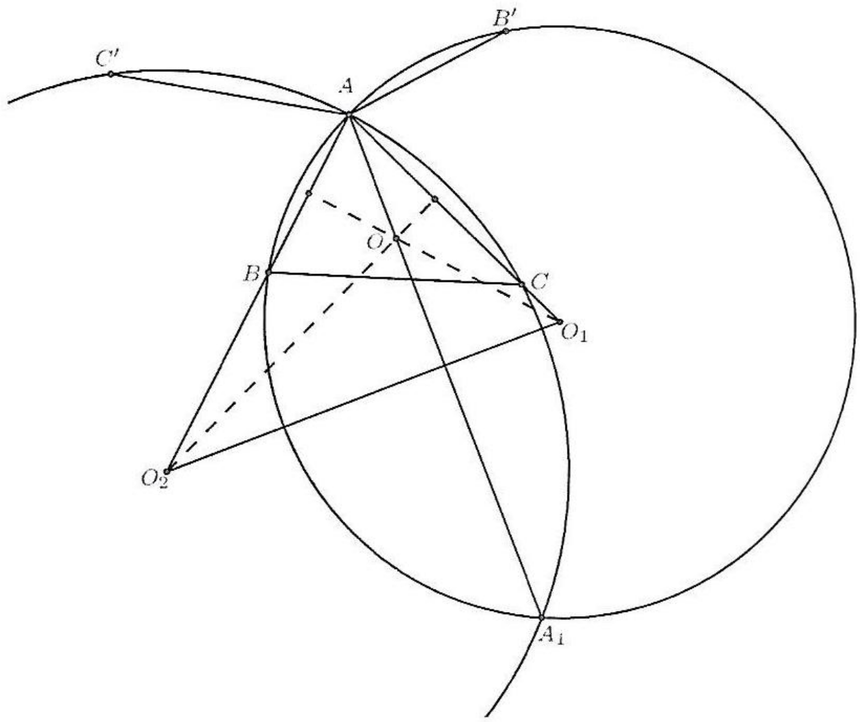
In the triangle LFT we have, $A'I \parallel FT$ and $\angle LA'I = \angle LIA'$, so $\angle LFT = \angle LTF$. Therefore the points F, A', I, T belong to the same circle. Also, $\angle A'IN = \angle A'FN = 90^\circ$ so the quadrilateral $IA'FN$ is cyclic. Thus, the points F, A', I, T, N all lie on a circle. From the above, we infer that

$$\angle NA'T = \angle TFN = \angle ACF = \angle FEZ = \angle ADT.$$

□

G 3. Let ABC be an acute triangle, A', B', C' the reflexions of the vertices A, B and C with respect to BC, CA, and AB, respectively, and let the circumcircles of triangles ABB' and ACC' meet again at A_1. Points B_1 and C_1 are defined similarly. Prove that the lines AA_1, BB_1, and CC_1 have a common point.

Solution. Let O_1, O_2 and O be the circumcenters of triangles ABB', ACC' and ABC respectively. As AB is the perpendicular bisector of the line segment CC', O_2 is the intersection of the perpendicular bisector of AC with AB. Similarly, O_1 is the intersection of the perpendicular bisector of AB with AC. It follows that O is the orthocenter of triangle AO_1O_2. This means that AO is perpendicular to O_1O_2. On the other hand, the segment AA_1 is the common chord of the two circles, thus it is perpendicular to O_1O_2. As a result, AA_1 passes through O. Similarly, BB_1 and CC_1 pass through O, so the three lines are concurrent at O. □



Comment by PSC. We present here a different approach.
We first prove that A_1, B and C' are collinear. Indeed, since $\angle BAB' = \angle CAC' = 2\angle BAC$, then from the circles (ABB'), (ACC') we get

$$\angle AA_1B = 90^\circ - \angle BAC = \angle AA_1C'.$$

It follows that

$$\angle A_1AC = \angle A_1C'C = \angle BC'C = 90^\circ - \angle ABC \tag{1}$$

On the other hand, if O is the circumcenter of ABC, then

$$\angle OAC = 90^\circ - \angle ABC. \tag{2}$$

From (1) and (2) we conclude that A_1, A and O are collinear. Similarly, BB_1 and CC_1 pass through O, so the three lines are concurrent in O.

G 4. Let ABC be a triangle with side-lengths a, b, c, inscribed in a circle with radius R and let I be it's incenter. Let P_1, P_2 and P_3 be the areas of the triangles ABI, BCI and CAI, respectively. Prove that

$$\frac{R^4}{P_1^2}+\frac{R^4}{P_2^2}+\frac{R^4}{P_3^2}\geq 16.$$

Solution. Let r be the radius of the inscribed circle of the triangle ABC. We have that

$$P_1=\frac{rc}{2},\quad P_2=\frac{ra}{2},\quad P_3=\frac{rb}{2}.$$

It follows that

$$\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}=\frac{4}{r^2}\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right).$$

From Leibniz's relation we have that if H is the orthocenter, then

$$OH^2=9R^2-a^2-b^2-c^2.$$

It follows that

$$9R^2\geq a^2+b^2+c^2. \tag{1}$$

Therefore, using the AM-HM inequality and then (1), we get

$$\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\geq\frac{9}{a^2+b^2+c^2}\geq\frac{1}{R^2}.$$

Finally, using Euler's inequality, namely that $R\geq 2r$, we get

$$\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}\geq\frac{4}{r^2R^2}\geq\frac{16}{R^4}.$$

□

Comment by PSC. We can avoid using Leibniz's relation as follows:
as in the above solution we have that

$$\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}=\frac{4}{r^2}\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right).$$

Let $a+b+c=2\tau$, $E=(ABC)$ and using the inequality $x^2+y^2+z^2\geq xy+yz+zx$ we get

$$\begin{aligned}\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}&\geq\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{2\tau}{abc}\\&=\frac{\tau}{2RE}=\frac{1}{2Rr},\end{aligned}$$

where we used the area formulas $E=\dfrac{abc}{4R}=\tau r$. Finally, using Euler's inequality, namely that $R\geq 2r$, we get

$$\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}\geq\frac{2}{r^3R}\geq\frac{16}{R^4}.$$

G 5. Given a rectangle $ABCD$ such that $AB = b > 2a = BC$, let E be the midpoint of AD. On a line parallel to AB through point E, a point G is chosen such that the area of GCE is

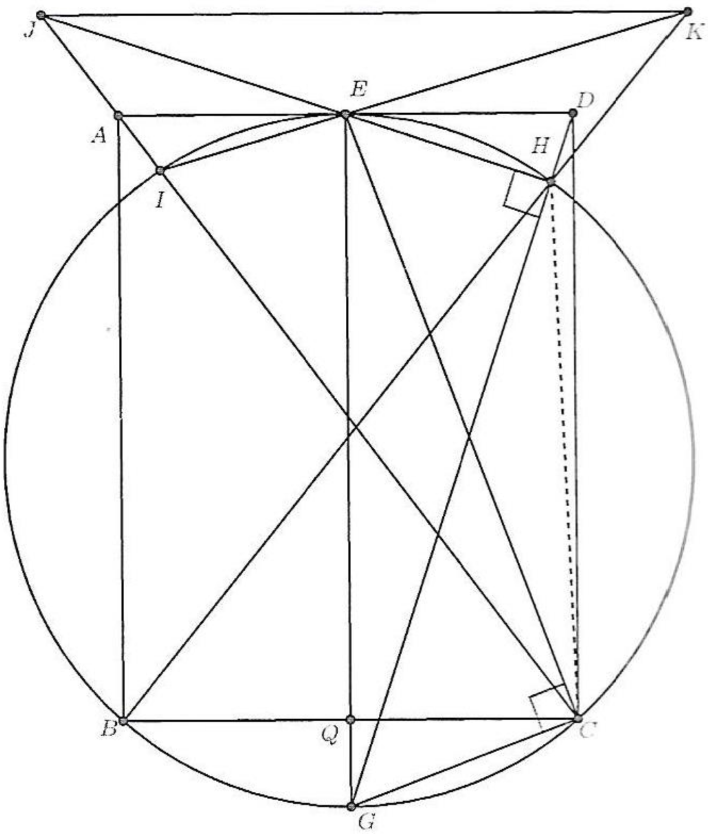
$$(GCE) = \frac{1}{2}\left(\frac{a^3}{b} + ab\right).$$

Point H is the foot of the perpendicular from E to GD and a point I is taken on the diagonal AC such that the triangles ACE and AEI are similar. The lines BH and IE intersect at K and the lines CA and EH intersect at J. Prove that $KJ \perp AB$.

Solution. Let L be the foot of the perpendicular from G to EC and let Q the point of intersection of the lines EG and BC. Then,

$$(GCE) = \frac{1}{2}EC \cdot GL = \frac{1}{2}\sqrt{a^2+b^2} \cdot GL.$$

So, $GL = \frac{a}{b}\sqrt{a^2+b^2}$.



Observing that the triangles QCE and ELG are similar, we have $\frac{a}{b} = \frac{GL}{EL}$, which implies that $EL = \sqrt{a^2+b^2}$, or in other words $L \equiv C$.
Consider the circumcircle ω of the triangle EBC. Since

$$\angle EBG = \angle ECG = \angle EHG = 90^\circ,$$

the points H and G lie on ω.
From the given similarity of the triangles ACE and AEI, we have that

$$\angle AIE = \angle AEC = 90^\circ + \angle GEC = 90^\circ + \angle GHC = \angle EHC,$$

therefore $EHCI$ is cyclic, thus I lies on ω.
Since $EB = EC$, we get that $\angle EIC = \angle EHB$, thus $\angle JIE = \angle EHK$. We conclude that $JIHK$ is cyclic, therefore

$$\angle JKH = \angle HIC = \angle HBC.$$

It follows that $KJ \parallel BC$, so $KJ \perp AB$.

□

Comment. The proposer suggests a different way to finish the proof after proving that I lies on ω: We apply Pascal's Theorem to the degenerated hexagon $EEHBCI$. Since BC and EE intersect at infinity, this implies that KJ, which is the line through the intersections of the other two opposite pairs of sides of the hexagon, has to go through this point at infinity, thus it is parallel to BC, and so $KJ \perp AB$.

G 6. Let XY be a chord of a circle Ω, with center O, which is not a diameter. Let P, Q be two distinct points inside the segment XY, where Q lies between P and X. Let ℓ the perpendicular line dropped from P to the diameter which passes through Q. Let M be the intersection point of ℓ and Ω, which is closer to P. Prove that

$$MP \cdot XY \geq 2 \cdot QX \cdot PY.$$

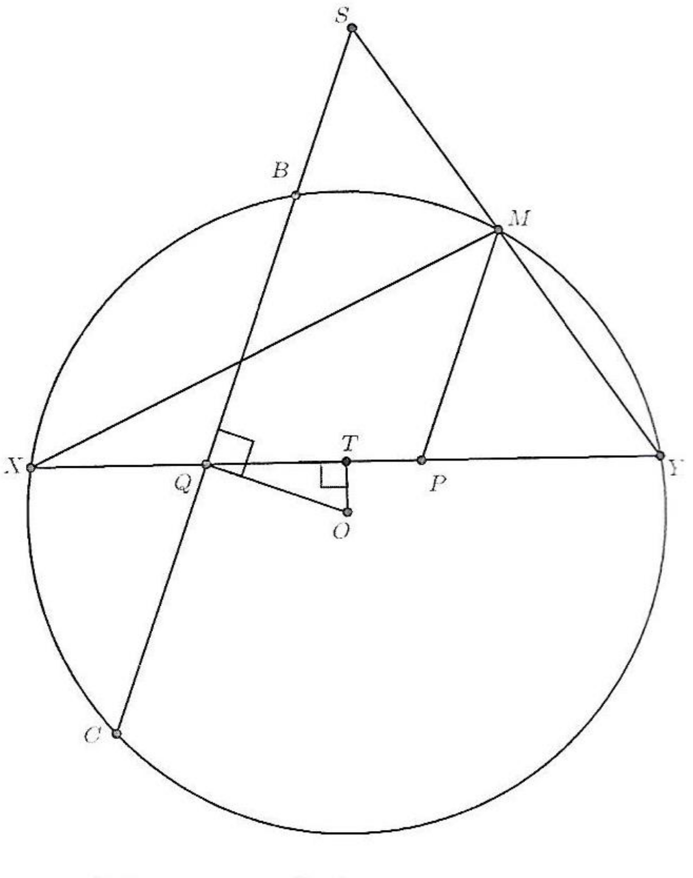
Solution by PSC. At first, we will allow P and Q to coincide, and we will prove the inequality in this case. Let the perpendicular from Q to OQ meet Ω at B and C. Then, we have that $QB = QC$. We will show that

$$BQ \cdot XY \geq 2QX \cdot QY. \qquad (1)$$

By the power of a point Theorem we have that

$$QX \cdot QY = QB \cdot QC = QB^2,$$

therefore it is enough to prove that $XY \geq 2BQ$ or $XY \geq BC$. Let T be the foot of the perpendicular from O to XY. Then, from the right-angled triangle OTQ we have that $OT \leq OQ$, so the distance from O to the chord XY is smaller or equal to the distance from O to the chord BC. This means that $XY \geq BC$, so (1) holds.



Back to the initial problem, we have to prove that

$$MP \cdot XY \geq 2QX \cdot PY \iff \frac{XY}{2QX} \geq \frac{PY}{PM}.$$

By (1) we have that

$$\frac{XY}{2QX} \geq \frac{QY}{QB},$$

so it is enough to prove that

$$\frac{QY}{QB} \geq \frac{PY}{PM}.$$

If CB meets YM at S, then from $MP \parallel QS$ we get

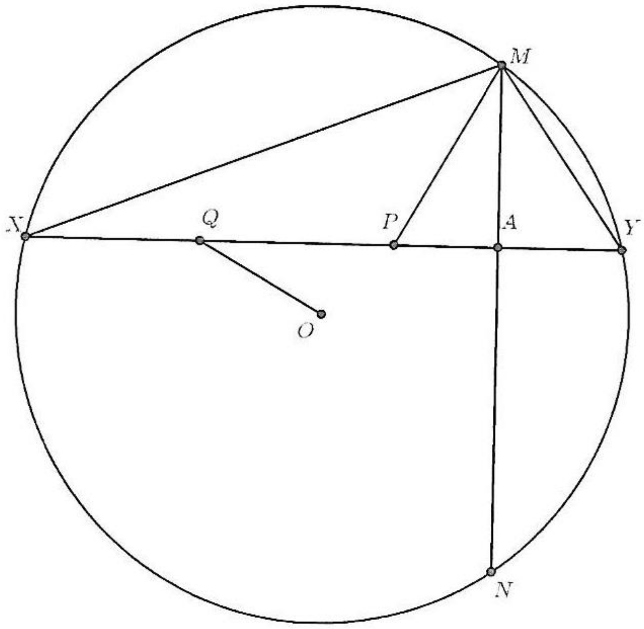
$$\frac{QY}{QB} \geq \frac{QY}{QS} = \frac{PY}{PM},$$

which is the desired.

□

Comment. The proposer's solution uses analytic geometry and it is the following.
We will show that $(QM - QP) \cdot XY \geq 2 \cdot QX \cdot PY$. Since $MP \geq QM - QP$, our inequality follows directly. Let A the intersection point of ℓ with the diameter which passes through Q. Like in the following picture, choose a coordinative system centered at O and such that $Q = (a, 0)$, $A = (c, 0)$, $P = (c, h)$ and denote the lengths $QX = x$, $PQ = t$, $PY = y$, $OP = d$, $QM = z$.
Let $\lambda_Q = r^2 - a^2$ and $\lambda_P = r^2 - d^2$ respectively the power of Q and P qith respect to our circle Ω.
We will show that:

$$(z - t)(t + x + y) \geq 2xy \tag{1}$$



Adding and multiplying respectively the relations $x(t + y) = \lambda_Q$ and $y(t + x) = \lambda_P$, we will have

$$t(x + y) + 2xy = \lambda_P + \lambda_Q \tag{2}$$

and

$$xy(t + x)(t + y) = \lambda_P \lambda_Q. \tag{3}$$

Using these two equations, it's easy to deduce that:

$$(xy)^2 - xy(t^2 + \lambda_P + \lambda_Q) + \lambda_P \lambda_Q = 0 \tag{4}$$

So, $w_1 = xy$ is a zero of the second degree polynomial:

$$p(w) = w^2 - w(t^2 + \lambda_P + \lambda_Q) + \lambda_P \lambda_Q$$

But $w_1 = xy < x(t+y) = \lambda_Q$ and

$$\begin{aligned}
p(\lambda_Q) &= (r^2-a^2)^2 - (r^2-a^2)(t^2+\lambda_P+\lambda_Q) + \lambda_P\lambda_Q \\
&= (r^2-a^2)^2 - (r^2-a^2)(t^2+r^2-d^2+r^2-a^2) + (r^2-d^2)(r^2-a^2) \\
&= (r^2-a^2)^2 - (r^2-a^2)t^2 - (r^2-d^2)(r^2-a^2) - (r^2-a^2)^4 + (r^2-d^2)(r^2-a^2) \\
&= -t^2(r^2-a^2) = -t^2\lambda_Q < 0
\end{aligned}$$

This implies that λ_Q lies (strictly) between the two (positive) zeros w_1, w_2 of $p(w)$ and $w_1 = xy$ is the smaller one.

After using (2) and (3), inequality (1) can be rewritten as:

$$(xy)^2 \le \left(\frac{z-t}{z+t}\right)\lambda_P\lambda_Q \tag{5}$$

In order to show this, it is enough to show that

$$p\left(\sqrt{\frac{z-t}{z+t}\lambda_P\lambda_Q}\right) \le 0, \tag{6}$$

because this will imply $\sqrt{\frac{z-t}{z+t}\lambda_P\lambda_Q} \in [w_1, w_2]$. After some manipulations, inequality (6) can be equivalently transformed to:

$$4z^2\lambda_P\lambda_Q \le (z^2-t^2)(t^2+\lambda_P+\lambda_Q)^2 \tag{7}$$

Since $z^2 - t^2 = r^2 - d^2 = \lambda_P$, this is equivalent to:

$$4z^2\lambda_Q \le (t^2+\lambda_P+\lambda_Q)^2 \tag{8}$$

But $t^2 = (a-c)^2+h^2 = a^2+d^2-2ac$, $z^2 = t^2+r^2-d^2 = a^2-2ac+r^2$ and $t^2+\lambda_P+\lambda_Q = \ldots = 2(r^2-ac)$. Hence, (8) is equivalent with:

$$(a^2-2ac+r^2)(r^2-a^2) \le (r^2-ac)^2 \Leftrightarrow \cdots \Leftrightarrow 0 \le a^2(a-c)^2,$$

which is clearly true.

NUMBER THEORY

NT 1. Find all the integers pairs (x, y) which satisfy the equation

$$x^5 - y^5 = 16xy.$$

Solution. If one of x, y is 0, the other has to be 0 too, and $(x, y) = (0, 0)$ is one solution. If $xy \neq 0$, let $d = \gcd(x, y)$ and we write $x = da$, $y = db$, $a, b \in \mathbb{Z}$ with $(a, b) = 1$. Then, the given equation is transformed into

$$d^3a^5 - d^3b^5 = 16ab \tag{1}$$

So, by the above equation, we conclude that $a \mid d^3b^5$ and thus $a \mid d^3$. Similarly $b \mid d^3$. Since $(a, b) = 1$, we get that $ab \mid d^3$, so we can write $d^3 = abr$ with $r \in \mathbb{Z}$. Then, equation (1) becomes

$$\begin{aligned} abra^5 - abrb^5 &= 16ab \Rightarrow \\ r(a^5 - b^5) &= 16. \end{aligned}$$

Therefore, the difference $a^5 - b^5$ must divide 16. Therefore, the difference $a^5 - b^5$ must divide 16. This means that

$$a^5 - b^5 = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

The smaller values of $|a^5 - b^5|$ are 1 or 2. Indeed, if $|a^5 - b^5| = 1$ then $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$, a contradiction. If $|a^5 - b^5| = 2$, then $a = 1$ and $b = -1$ or $a = -1$ and $b = 1$. Then $r = -8$, and $d^3 = -8$ or $d = -2$. Therefore, $(x, y) = (-2, 2)$. If $|a^5 - b^5| > 2$ then, without loss of generality, let $a > b$ and $a \geq 2$. Putting $a = x + 1$ with $x \geq 1$, we have

$$\begin{aligned} |a^5 - b^5| &= |(x+1)^5 - b^5| \\ &\geq |(x+1)^5 - x^5| \\ &= |5x^4 + 10x^3 + 10x^2 + 5x + 1| \geq 31 \end{aligned}$$

which is impossible. Thus, the only solutions are $(x, y) = (0, 0)$ or $(-2, 2)$.

□

NT 2. Find all pairs (m, n) of positive integers such that

$$125 \cdot 2^n - 3^m = 271.$$

Solution. Considering the equation mod5 we get

$$3^m \equiv -1 \pmod 5,$$

so $m = 4k + 2$ for some positive integer k. Then, considering the equation mod7 we get

$$-2^n - 9^{2k+1} \equiv 5 \pmod 7 \Rightarrow$$
$$2^n + 2^{2k+1} \equiv 2 \pmod 7.$$

Since $2^s \equiv 1, 2, 4 (\bmod 7)$ for $s \equiv 0, 1, 2 (\bmod 3)$, respectively, the only possibility is $2^n \equiv 2^{2k+1} \equiv 1 (\bmod 7)$, so $3 \mid n$ and $3 \mid 2k+1$. From the last one we get $3 \mid m$, so we can write $n = 3x$ and $m = 3y$. Therefore, the given equation takes the form

$$5^3 \cdot 2^{3x} - 3^{3y} = 271, \tag{2}$$

or

$$(5 \cdot 2^x - 3^y)(25 \cdot 2^{2x} + 5 \cdot 2^x \cdot 3^y + 3^{2y}) = 271.$$

It follows that $25 \cdot 2^{2x} + 5 \cdot 2^x \cdot 3^y + 3^{2y} \leq 271$, and so $25 \cdot 2^{2x} \leq 271$, or $x < 2$. We conclude that $x = 1$ and from (2) we get $y = 2$. Thus, the only solution is $(m, n) = (6, 3)$.

□

Alternative solution by PSC. Considering the equation mod5 we get

$$3^m \equiv -1 \pmod 5,$$

so $m = 4k + 2$ for some positive integer k. For $n \geq 4$, considering the equation mod16 we get

$$-3^{4k+2} \equiv -1 \pmod{16} \Rightarrow$$
$$9 \cdot 81^k \equiv 1 \pmod{16},$$

which is impossible since $81 \equiv 1 \bmod 16$. Therefore, $n \leq 3$.
We can readily check that $n = 1$ and $n = 2$ give no solution for m, and $n = 3$ gives $m = 6$. Thus, the only solution is $(m, n) = (6, 3)$.

Comment by PSC. Note that first solution works if 271 is replaced by any number A of the form 1 mod 5 and at the same time 5 mod 7, which gives $A \equiv 26 \bmod 35$, while the second solution works if 271 is replaced by any number B of the form 1 mod 5 and which is not 7 mod 16, which gives that B is not of the form 71 mod 80. This means, for example, that if 271 is replaced by 551, then the first solution works, while the second doesn't.

NT 3. Find all four-digit positive integers $\overline{abcd} = 10^3a + 10^2b + 10c + d$, with $a \neq 0$, such that

$$\overline{abcd} = a^{a+b+c+d} - a^{-a+b-c+d} + a.$$

Solution. It is obvious that $a \neq 1$ and $-a+b-c+d \geq 0$. It follows that $b+d \geq c+a \geq a$. Then,

$$\begin{aligned}
10000 > \overline{abcd} &= a^{a+b+c+d} - a^{-a+b-c+d} + a \\
&> a^{a+b+c+d} - a^{a+b+c+d-2} \\
&= a^{a+b+c+d-2}(a^2-1) \\
&\geq a^{2a-2}(a^2-1).
\end{aligned}$$

For $a \geq 4$, we have

$$a^{2a-2}(a^2-1) = 4^6 \cdot 15 > 4^5 \cdot 10 = 10240 > 10000,$$

a contradiction. This means that $a = 2$ or $a = 3$.

Case 1: If a $= 3$, then since $3^7 = 2187 < 3000$, we conclude that $a+b+c+d \geq 8$ and like in the previous paragraph we get

$$3^{a+b+c+d} - 3^{-a+b-c+d} + 3 > 3^{a+b+c+d-2} \cdot 8 \geq 729 \cdot 8 > 4000,$$

which is again a contradiction.

Case 2: If $a = 2$ then $2^{10} = 1024 < 2000$, thus $a+b+c+d \geq 11$. If $a+b+c+d \geq 12$, we have again as above that

$$\overline{abcd} > 2^{a+b+c+d-2} \cdot 3 = 1024 \cdot 3 > 3000,$$

which is absurd and we conclude that $a+b+c+d = 11$. Then $\overline{abcd} < 2^{11} + 2 = 2050$, so $b = 0$. Moreover, from

$$2050 - 2^{d-c-2} \geq 2000 \iff 2^{d-c-2} \leq 50,$$

we get $d-c-2 \leq 5$. However, from $d+c = 9$ we have that $d-c-2$ is odd, so $d-c-2 \in \{1,3,5\}$. This means that

$$\overline{abcd} = 2050 - 2^{d-c-2} \in \{2048, 2042, 2018\}.$$

The only number that satisfies $a+b+c+d = 11$ is 2018, so it is the only solution.

□

Comment by PSC. After proving $b+d \geq a+c \geq 2$, we can alternatively conclude that $a \leq 3$, as follows. If $a \geq 4$, then

$$\begin{aligned}
\overline{abcd} &= a^{a+b+c+d} - a^{-a+b-c+d} + a \\
&> a^{b+d-a-c}\left(a^{2a+2c} - 1\right) \\
&> a^{2a+2c} - 1 \\
&\geq 4^8 - 1 = 65535,
\end{aligned}$$

a contradiction.

NT 4. Show that there exist infinitely many positive integers n such that

$$\frac{4^n + 2^n + 1}{n^2 + n + 1}$$

is an integer.

Solution. Let $f(n) = n^2 + n + 1$. Note that

$$f(n^2) = n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1).$$

This means that $f(n) \mid f(n^2)$ for every positive integer n. By induction on k, one can easily see that $f(n) \mid f(n^{2^k})$ for every positive integers n and k. Note that the required condition is equivalent to $f(n) \mid f(2^n)$. From the discussion above, if there exists a positive integer n so that 2^n can be written as n^{2^k}, for some positive integer k, then $f(n) \mid f(2^n)$. If we choose $n = 2^{2^m}$ and $k = 2^m - m$ for some positive integer m, then $2^n = n^{2^k}$ and since there are infinitely many positive integers of the form $n = 2^{2^m}$, we have the desired result.

□