SEEMOUS

South Eastern European Mathematical Olympiad for University Students

CYPRUS 2026

Language: **English**

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Problem 1. Determine all differentiable functions $f : \mathbb{R} \to [0, \infty)$ with continuous derivative such that $f^2(x) \leqslant f'(x)$ for all $x \in \mathbb{R}$.

Problem 2. Let A and B be $n \times n$ matrices with complex entries for which

$$A^2 + B^2 = 2(AB - BA).$$

Prove that $A^2 + B^2$ is a nilpotent matrix.

An $n \times n$ matrix M is called *nilpotent*, if there is a positive integer k such that $M^k = O_n$.

Problem 3. Let A and B be $n \times n$ matrices with complex entries for which $A^2 = AB - BA$ and $\operatorname{rank}(A - B) = 1$. Prove that $ABA = O_n$.

Problem 4. Find all bijective functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x^n f(x) + y^n f(y) \geqslant 2x^n f(y)$$

for all $x, y \in \mathbb{R}$, where n is a fixed positive integer.

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Time: 5 hours

Each problem is worth 10 points

Problem 1. Determine all differentiable functions $f : \mathbb{R} \to [0, \infty)$ with continuous derivative such that $f^2(x) \leqslant f'(x)$ for all $x \in \mathbb{R}$.

Solution 1. We will show that the only such function is the trivial one, $f(x) = 0$. Suppose instead that there is some x_0 with $f(x_0) > 0$. As $f'(x) \geqslant f^2(x) \geqslant 0$ for every x, the function f is increasing. Hence, for every $x \geqslant x_0$, we have $f(x) \geqslant f(x_0) > 0$.

The function $g(x) = f'(x)/f^2(x)$ is well-defined, continuous and satisfies $g(x) \geqslant 1$ on $[x_0, \infty)$. Therefore

$$x - x_0 = \int_{x_0}^{x} 1 \,\mathrm{d}t \leqslant \int_{x_0}^{x} \frac{f'(t)}{f^2(t)} \,\mathrm{d}t$$

for every $x \geqslant x_0$.

Since $G(x) = -\dfrac{1}{f(x)}$ is continuous on $[x_0, \infty)$ and continuously differentiable on (x_0, ∞) with $G'(x) = g(x)$ for every $x > x_0$, by the Fundamental Theorem of Calculus we get

$$\int_{x_0}^{x} \frac{f'(t)}{f^2(t)} \,\mathrm{d}t = G(x) - G(x_0) = \frac{1}{f(x_0)} - \frac{1}{f(x)}$$

for every $x \geqslant x_0$. Therefore

$$0 \leqslant \frac{1}{f(x)} \leqslant \frac{1}{f(x_0)} + x_0 - x, \tag{1}$$

for every $x \geqslant x_0$, which is impossible, as the right hand side eventually becomes negative.

Solution 2. We define x_0 in the same way as in Solution 1. Define $h(x) = x + \dfrac{1}{f(x)}$ for $x \geqslant x_0$ and note that it is continuous on $[x_0, \infty)$ and differentiable on (x_0, ∞) with $h'(x) = 1 - f'(x)/f^2(x) \leqslant 0$ for every $x \in (x_0, \infty)$. Therefore h is decreasing which also leads to (1).

Solution 3.

Suppose that there is some x_0 such that $f(x_0) > 0$. We will prove the following lemma.

Lemma. *For every $x \geqslant x_0$ and every non-negative integer n, we have*

$$f(x) \geqslant 2^{n+2} \left(\frac{f(x_0)}{4} \right)^{2^n} (x - x_0)^{2^n - 1}$$

Proof. We have that $f'(x) \geqslant f^2(x) \geqslant 0$, and therefore $f(x)$ is increasing. Thus, we have that, for every $x \geqslant x_0$,

$$f(x) \geqslant f(x_0) = 2^{0+2} \left(\frac{f(x_0)}{4} \right)^{2^0} (x - x_0)^{2^0 - 1},$$

i.e., the Lemma is true for $n = 0$.

Suppose that the Lemma is true for $n = k$. We have that, for every $x \geqslant x_0$,

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(u)\,\mathrm{d}u \geqslant \int_{x_0}^{x} f^2(u)\,\mathrm{d}u.$$

Hence, by applying the induction hypothesis, we deduce that

$$\begin{aligned} f(x) &\geqslant 2^{2k+4}\left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \int_{x_0}^{x} (u - x_0)^{2^{k+1}-2}\,\mathrm{d}u \\ &= 2^{2k+4}\left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \left[\frac{(u - x_0)^{2^{k+1}-1}}{2^{k+1} - 1}\right]_{x_0}^{x} \\ &> 2^{2k+4}\left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \frac{(x - x_0)^{2^{k+1}-1}}{2^{k+1}} \\ &= 2^{(k+1)+2}\left(\frac{f(x_0)}{4}\right)^{2^{k+1}} (x - x_0)^{2^{k+1}-1}. \end{aligned}$$

□

Now choose $x = x_0 + \frac{4}{f(x_0)}$. By the Lemma, we have

$$f(x) \geqslant 2^n f(x_0)$$

for every n. Taking $n \to \infty$, we arrive at a contradiction. Hence, f must be identically zero.

Problem 2. Let A and B be $n \times n$ matrices with complex entries for which

$$A^2 + B^2 = 2(AB - BA).$$

Prove that $A^2 + B^2$ is a nilpotent matrix.

An $n \times n$ matrix M is called *nilpotent*, if there is a positive integer k such that $M^k = O_n$.

Solution.

Let $C = AB - BA$. The given equation is $A^2 + B^2 = 2C$. Define

$$L = A - iB, \quad \text{and} \quad M = A + iB$$

Then

$$LM = (A - iB)(A + iB) = A^2 + B^2 + i(AB - BA) = 2C + iC = (2 + i)C.$$

and

$$ML = (A + iB)(A - iB) = A^2 + B^2 - i(AB - BA) = 2C - iC = (2 - i)C.$$

This gives

$$ML = \frac{2 - i}{2 + i} LM = \frac{(2 - i)^2}{5} LM = \frac{3 - 4i}{5} LM.$$

Let $\alpha = \frac{3-4i}{5}$. Note that $|\alpha| = 1$. So $\alpha = e^{i\vartheta}$ where $\cos\vartheta = \frac{3}{5}$. By Niven's Theorem, since $\cos\vartheta \in \mathbb{Q} \setminus \{0, \pm\frac{1}{2}, \pm 1\}$, then ϑ is not a rational multiple of π. Thus for every positive integer k we have $\alpha^k \neq 1$.

For any $k \geqslant 1$, the relation $ML = \alpha LM$ implies $(ML)^k = \alpha^k (LM)^k$. Taking the trace of both sides and using the cyclic property $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ we get

$$\operatorname{tr}((LM)^k) = \operatorname{tr}((ML)^k) = \alpha^k \operatorname{tr}((LM)^k).$$

Then

$$(1 - \alpha^k) \operatorname{tr}((LM)^k) = 0$$

and since $\alpha^k \neq 1$, then $\operatorname{tr}((LM)^k) = 0$.

Since this holds for every $k \in \{1, 2, \ldots, n\}$, then LM is nilpotent. Since

$$A^2 + B^2 = 2C = \frac{2}{2 + i} LM,$$

then $A^2 + B^2$ is also nilpotent.

Solution 2 After arriving at $ML = \alpha LM$, and proving that a is not a root of unity, we can also proceed as follows.

Suppose that LM is not nilpotent. Then, it will have an eigenvalue $\lambda \neq 0$. Let v be a corresponding eigenvector, so that $LMv = \lambda v$. Then, $MLv = \alpha\lambda v$, giving that $\alpha\lambda$ is an eigenvalue for ML. On the other hand, we know that ML has the same spectrum as LM. Hence, $\alpha\lambda$ is an eigenvalue of LM. Inductively, $\alpha^n\lambda$ is an eigenvalue for every n. As $\lambda, \alpha \neq 0$, and α is not a root of unity, this gives us infinitely many eigenvalues, which is a contradiction.

Remark. We note the following two alternative ways to deduce that $\alpha = \frac{3-4i}{5}$ is not a root of unity without using Niven's Theorem:

(a) We have $(5\alpha - 3)^2 = (-4i)^2$, which gives that $p(x) = 5x^2 - 6x + 5$ is the minimal polynomial of α. In order for α to be a root of unity, it must be a root of $x^m - 1$ for some m. Hence, $p(x)$ must divide $x^m - 1$. As $p(x)$ is irreducible and non-monic, while $x^m - 1$ is monic, this contradicts Gauss's Lemma.

(b) Note that $(3-4i)^2 = -7-24i \equiv 3-4i \bmod 5$ and inductively we get $(3-4i)^k \equiv 3-4i$. So $(5\alpha)^k \neq 5^k$.

Problem 3. Let A and B be $n \times n$ matrices with complex entries for which $A^2 = AB - BA$ and $\text{rank}(A - B) = 1$. Prove that $ABA = O_n$.

Solution 1.

Since $AB - BA = A^2$, then $AB - BA$ commutes with A and so (by Jacobson's Lemma) it is nilpotent. Thus A^2 is nilpotent and therefore A is nilpotent.

Assume $\text{Im}(A-B) = \langle x \rangle$. We have $A(A-B) = -BA$ so it is enough to show that $A^2x = 0$.

Let $K = A - B$ and observe that

$$KA - AK = A^2 - BA - (A^2 - AB) = AB - BA = A^2.$$

Since A is nilpotent, there is a minimal k such that $A^k x = 0$. We have

$$A^{k+1} = A^{k-1}A^2 = A^{k-1}(KA - AK) = A^{k-1}KA - A^k K$$

Since $Kx \in \langle x \rangle$ then

$$A^{k-1}KAx = A^{k+1}x - A^k Kx = 0$$

Since $KAx \in \langle x \rangle$, and $A^{k-1}(KAx) = 0$, by definition of k we get $KAx = 0$. We have $Kx = \lambda x$ for some $\lambda \in \mathbb{C}$. Thus

$$A^2 x = (KA - AK)x = -AKx = -\lambda Ax.$$

So either $Ax = 0$, or Ax is an eigenvector of A which together with nilpotency implies $\lambda = 0$.

In both cases we have $A^2 x = 0$.

Solution 2.

Assume $\text{Im}(A-B) = \langle x \rangle$. We have $A(A-B) = -BA$ so it is enough to show that $A^2x = 0$.

Let $Ax = y$ and pick $\lambda, \mu \in \mathbb{C}$ such that $(A - B)x = \lambda x$ and $(A - B)y = \mu x$. We have

$$A(A-B)x = -BAx \implies \lambda y = -By \implies By = -\lambda y.$$

This implies that $Ay = \mu x + By = \mu x - \lambda y$

$$A(A-B)y = -BAy \implies \mu y = -B(\mu x - \lambda y) = -\mu(y - \lambda x) - \lambda^2 y$$

This leads to

$$(\lambda^2 + 2\mu)y = \lambda \mu x.$$

So either $\mu = -\lambda^2/2$, or $y = \alpha x$ for some $\alpha \in \mathbb{C}$.

In the first case we get $-(\lambda^3/2)x = 0$, and since $x \neq 0$, then $\lambda = 0$. But then $Ax = Bx = y$ and so

$$A^2 x = (AB - BA)x = (A - B)y = \mu x = 0.$$

In the second case $Ax = y = \alpha x$ and $Bx = y - \lambda x = (\alpha - \lambda)x$ so

$$A^2 x = (AB - BA)x = A(\alpha - \lambda)x - B\alpha x = (\alpha - \lambda)\alpha x - \alpha(\alpha - \lambda)x = 0.$$

Solution 3.

As in Solution 1, A is nilpotent. Writing $K = A - B$ we have

$$A^2 = KA - AK$$

and so A^2 is the sum of two rank-1 matrices. Then $\operatorname{rank}(A^2) \leqslant 2$ and since A is nilpotent, then $\operatorname{rank}(A^3) \leqslant 1$ and therefore $A^4 = O_n$.

Note that

$$ABA = A(A - K)A = A^3 - AKA$$

and

$$A^3 = A(KA - AK) = AKA - A^2K.$$

So $ABA = -A^2K$.

Since $\operatorname{rank}(K) = 1$, then $K = uv^*$ for some non-zero vectors $u, v \in \mathbb{C}^n$.

Case 1. Suppose $A^3 = 0$. Then

$$0 = A^4 = A^2(KA - AK) = A^2KA = A^2uv^*A.$$

If $v^*A = 0$, then $KA = 0$, so $A^2 = -AK$ and therefore $-A^2K = A^3 = 0$. Otherwise, $A^2u = 0$ and so again $A^2K = A^2uv^* = 0$.

Case 2. Suppose $A^4 = 0$ but $A^3 \neq 0$ and pick w such that $A^3w \neq 0$. Then

$$0 = A^5w = A^2(A^3w) = (KA - AK)A^3w = -AKA^3w = -Au(v^*A^3w).$$

If $Au = 0$, then $A^2K = 0$ as in Case 1. So we can assume that $v^*A^3w = 0$. Now

$$0 = A^4w = (KA - AK)A^2w = uv^*A^3w - Auv^*A^2w = -Auv^*A^2w.$$

With a similar reasoning as before, we may assume $v^*A^2w = 0$. Now

$$0 \neq A^3w = (KA - AK)Aw = uv^*A^2w - Auv^*Aw = -Auv^*Aw$$

which guarantees that $v^*Aw \neq 0$. But now

$$0 = A^4w = A(A^3w) = -A^2uv^*Aw$$

and since $v^*Aw \neq 0$, then $A^2u = 0$. As in Case 1, we get $A^2K = 0$.

Solution 4.

As in Solution 3, we have $A^4 = 0_n$ and $ABA = -A^2K$. We may assume that $A^2K \neq O_n$ and therefore $\operatorname{rank}(A^2K) \geqslant 1$. We apply Frobenious rank inequality

$$\operatorname{rank}(XY) + \operatorname{rank}(YZ) \leqslant \operatorname{rank}(Y) + \operatorname{rank}(XYZ)$$

to get

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA^3) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA^3).$$

Since $A^2KA^3 = (-ABA)A^3 = -ABA^4 = 0$ and $\operatorname{rank}(A^2K) \geqslant 1$, we get that $\operatorname{rank}(KA^3) = 0$ and therefore $KA^3 = O_n$.

Thus $(A - B)A^3 = O_n$ which leads to $BA^3 = O_n$. Thus $A^2KA^2 = (-ABA)A^2 = O_n$ and so

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA^2) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA^2)$$

gives that $KA^2 = O_n$ and $A^3 = BA^2$. Thus $A^2KA = -ABA^2 = -A^4 = O_n$ and so

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA)$$

gives that $KA = O_n$.

From this we get $A^2 = KA - AK = -AK$ and therefore $A^3 = -AKA = O_n$ and we can conclude as in Solution 3.

Alternatively, since $KA = O_n$, then $A^2 = BA$ and so

$$A^2K^2 = A(AK)K = A(-A^2)K = A(ABA) = A^4 = O_n$$

and $AK^2 = (-A^2)K = ABA$. Then

$$\operatorname{rank}(A^2K) + \operatorname{rank}(AK^2) \leqslant \operatorname{rank}(AK) + \operatorname{rank}(A^2K^2).$$

implies that $2\operatorname{rank}(ABA) \leqslant \operatorname{rank}(AK) \leqslant 1$ which leads to $ABA = O_n$.

Problem 4. Find all bijective functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x^n f(x) + y^n f(y) \geqslant 2x^n f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$, where n is a fixed positive integer.

Solution 1.

Setting $x = 0$ in (1) gives $y^n f(y) \geqslant 0$, for all $y \in \mathbb{R}$. If n is even, then $f(y) \geqslant 0$ for all y, hence f cannot be surjective and there are no solutions. Therefore, we only have to consider the case where n is odd.

Coming back to $y^n f(y) \geqslant 0$, for all $y \in \mathbb{R}$, since n is odd, this implies that $f(y) \geqslant 0$ for $y > 0$, and $f(y) \leqslant 0$ for $y < 0$. We claim that $f(0) = 0$. Assume for contradiction $f(0) = c \neq 0$. Setting $y = 0$ in (1) gives $x^n f(x) \geqslant 2x^n f(0)$. If $c > 0$, then we get $f(x) > 2c$ for every $x > 0$ and so $\mathrm{Im}(f)$ does not contain any value in $(0, 2c) \setminus \{c\}$. A similar contradiction arises if $c < 0$.

From (1), we have that

$$\frac{f(x)}{f(y)} \geqslant 2 - \left(\frac{y}{x}\right)^n$$

for every $x, y > 0$. Hence, for any fixed positive real number $r < \sqrt[n]{2}$, and any natural number k, we will have

$$\frac{f(x)}{f(rx)} = \frac{f(x)}{f(r^{\frac{1}{k}}x)} \cdot \frac{f(r^{\frac{1}{k}}x)}{f(r^{\frac{2}{k}}x)} \cdot \ldots \cdot \frac{f(r^{\frac{k-1}{k}}x)}{f(rx)} \geqslant \left(2 - r^{n/k}\right)^k.$$

Here, we used the fact that, for $r < \sqrt[n]{2}$, we have $2 - r^{n/k} > 2 - 2^{1/k} \geqslant 0$. Taking the limit as k tends to infinity, we have

$$\frac{f(x)}{f(rx)} \geqslant \lim_{k \to \infty} \left(2 - r^{n/k}\right)^k.$$

To evaluate the limit, we write

$$r^{n/k} = e^{(n \log r)/k} = 1 + \frac{n \log r}{k} + O\left(k^{-2}\right),$$

which gives

$$2 - r^{n/k} = 2 - e^{(n \log r)/k} = 1 - \frac{n \log r}{k} + O\left(k^{-2}\right).$$

Hence,

$$\left(2 - r^{n/k}\right)^k \to e^{-n \log r} = r^{-n},$$

which gives

$$\frac{f(x)}{f(rx)} \geqslant r^{-n}.$$

Similarly, for $r > 1/\sqrt[n]{2}$, we get

$$\frac{f(rx)}{f(x)} \geqslant r^n.$$

Therefore, for $r \in \left(1/\sqrt[n]{2}, \sqrt[n]{2}\right)$, we have $f(rx) = r^n f(x)$. Letting $a = f(1)$, and applying the latter equation repeatedly, we get

$$f(x) = ax^n$$

for every $x > 0$. Similarly, we can show that there exists a constant b such that

$$f(x) = bx^n$$

for every $x < 0$.

In conclusion, if n is even, then there are no solutions. If n is odd, then the solutions are the functions

$$f(x) = \begin{cases} ax^n, & x \geqslant 0, \\ bx^n, & x < 0, \end{cases}$$

where $a, b > 0$ are arbitrary constants. Note that these functions verify (1). Indeed, if $x, y < 0$ or $x, y \geqslant 0$, (1) becomes

$$x^{2n} + y^{2n} \geqslant 2x^n y^n \iff (x^n - y^n)^2 \geqslant 0.$$

If $x < 0$ and $y \geqslant 0$, then (1) becomes $bx^{2n} + ay^{2n} \geqslant 2ax^n y^n$, which is true since $bx^{2n} + ay^{2n} \geqslant 0$ and $2ax^n y^n \leqslant 0$. Finally, if $x \geqslant 0$ and $y < 0$, then (1) becomes $ax^{2n} + by^{2n} \geqslant 2bx^n y^n$, which is true for similar reasons.

Solution 2.

As in Solution 1, we have that n is odd and that the function f preserves signs.

Interchanging x and y in (1) yields

$$x^n f(x) + y^n f(y) \geqslant 2y^n f(x) \tag{2}$$

for all $x, y \in \mathbb{R}$. Adding (1) and (2), we obtain

$$2(x^n f(x) + y^n f(y)) \geqslant 2(x^n f(y) + y^n f(x)),$$

hence

$$(x^n - y^n)(f(x) - f(y)) \geqslant 0 \tag{3}$$

for all $x, y \in \mathbb{R}$. Thus the functions $\varphi(x) = x^n$ and f have the same monotonicity and hence f is increasing. Since f is injective, it is strictly increasing and therefore admits one-sided limits at every point,

$$L^- = \lim_{x \to x_0^-} f(x) \leqslant f(x_0) \leqslant \lim_{x \to x_0^+} f(x) = L^+.$$

Monotonicity together with surjectivity implies $L^- = L^+$, so f is continuous on $\mathbb{R}$.

Write

$$f(x) = \begin{cases} f_1(x), & x > 0, \\ 0, & x = 0, \\ f_2(x), & x < 0, \end{cases}$$

where $f_1 : (0, \infty) \to (0, \infty)$ and $f_2 : (-\infty, 0) \to (-\infty, 0)$ are continuous bijections.

For $0 < y < x$, inequalities (1) and (2) can be rewritten as

$$\frac{f_1(y)}{x^n} \leqslant \frac{f_1(x) - f_1(y)}{x^n - y^n} \leqslant \frac{f_1(x)}{y^n}.$$

Similarly, for $0 < x < y$, it holds that

$$\frac{f_1(x)}{y^n} \leqslant \frac{f_1(x) - f_1(y)}{x^n - y^n} \leqslant \frac{f_1(y)}{x^n}.$$

Letting y tend to x gives

$$\frac{f_1(x)}{x^n} = \lim_{y \to x} \frac{f_1(x) - f_1(y)}{x - y} \cdot \frac{x - y}{x^n - y^n} = \frac{1}{nx^{n-1}} \lim_{y \to x} \frac{f_1(x) - f_1(y)}{x - y},$$

hence f_1 is differentiable and satisfies $f_1'(x) = \dfrac{n}{x} f_1(x)$, for all $x > 0$. Thus

$$\frac{f_1'(x)}{f_1(x)} = \frac{n}{x},$$

which yields $\log(f_1(x)) = n \log x + c$. It follows that $f_1(x) = c_1 x^n$, for all $x > 0$, with $c_1 > 0$ constant ($c_1 = \mathrm{e}^c$). Similarly, $f_2(x) = c_2 x^n$, for all $x < 0$, for some constant $c_2 > 0$.

Therefore, we get the same conclusion as in the first solution.