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25-th Balkan Mathematical Olympiad
Ohrid, 04-10 May 2008
Republic of Macedonia

Union of Mathematicinas of Macedonia

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Издавач:Сојуз на математичари на Македонија
Бул.Александар Маледонски бб
1000 Скопје, Република Македонија

CIP-Каталогизација и публикација
Национална и универзитетска библиотека “Св. Климент Охридски”

51(079.1)

MALCHESKI Aleksa
25-th Balkan mathematical Olympiad:Ohrid, 04-10 May 2008, Republic of Macedonia/
[Aleksa Malcheski, Vesna Manova Erakovik].-Skopje:Macedonian mathematical society, 2011,
-41 стр. илустр.; 24 см.

ISBN 978-9989-646-13-3
1.Manova-Erakovik,Vesna [автор]
а)Математика-Задачи од натпревари
COBISS.MK-ID 88095498

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Foreword

The 25-th Balkan Mathematical Olympiad (BMO 2008) for high-school students took place from 04.05.2008 until 10.05.2008 in Ohrid, Republic of Macedonia. In some way, the manifestations of this kind are forgotten after a short period after they are held, despite the existence of numerous electronic versions and copies of both the problems and the results.

Nevertheless, this and all other Balkan Olympiads deserve to be more decently marked, for the benefit of both the students and the leaders, deputy-leaders and hosts, who, as a rule, selflessly operate for the taking place of the Olympiads and the preservation of the tradition.

Shortlisted -Problems

Contributing Countries
Moldova, Serbia, Bulgaria,
Monte Negro, Romania,Albania,
Greece, Cyprus

Number theory

NT1.

Prove that for every natural number a there exists a natural number that has the number a (the sequence of digits that constitute a) at its beginning, and which decreases a times when a is moved from its beginning to its end (any number zeros that appear in the beginning of the number obtained in this way are to be removed).

For example, for $a=4$ we have $\underline{4}10256=4\cdot 10256\underline{4}$; for $a=46$ we have

$$\underline{46}0100021743857360295716=46\cdot 10002174385736029571\underline{46};$$

for $a=58$ we have

$$\underline{58}0100017244352474564=58\cdot 100017244352474564\underline{58}.$$

(Serbia)

NT2.

Let a be a positive integer. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1=a$, $a_{k+1}=a_k^2+a_k+a^3$ for every positive integer $k\ge 1$. Find all values of a for which there exists a positive integer n such that $a_n^2+a^3$ is a m-th power, $m\ge 2$, $m\in\mathbb{N}$, of a positive integer.

(Bulgaria)

NT3.

The sequence $(x_n)_{n=1}^{\infty}$ is given by

$$x_{n+1}=x_n+x_{\lceil n/2\rceil},\ x_1=1.$$

Proof that none of the members of the sequence is divisible by 4.

(Crna Gora)

NT4.

Solve in the prime numbers the equation

$$xyz+1=2^{y^2+1}.$$

(Albania)

NT5.

Let $\{a_n\}$ be the sequence with $a_1=0$ and $a_{n+1}=2+a_n$ for odd n and $a_{n+1}=2a_n$ for even *n*. Prove that for each prime $p>3$ the number $b=\frac{2^{2p}-1}{3}$ divides a_n for the infinitely many values of *n*.

(Albania)

NT6.

Let $(x_n), n=1,2,3,...$ be a sequence defined by $x_1 = 2008$ and

$$x_1 + x_2 + \cdots + x_{n-1} = (n^2 - 1)x_n \quad \text{, for every } n \geq 2 .$$

Let, also, the sequence $a_n = x_n + \frac{1}{n}S_n$, $n=1,2,3,...$ where $S_n = x_1 + x_2 + \cdots + x_n$.

Determine the values of n for which the terms of the sequence a_n are perfect squares of an integer.

(Greece)

Algebra and Inequalities

A1.

For all positive real numbers $\alpha_1, \alpha_2, \alpha_3$ prove that

$$\frac{1}{2\nu\alpha_1 + \alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + 2\nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \alpha_2 + 2\nu\alpha_3} >$$

$$> \frac{2\nu}{2\nu+1}\left(\frac{1}{\nu\alpha_1 + \nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \nu\alpha_2 + \nu\alpha_3} + \frac{1}{\nu\alpha_1 + \alpha_2 + \nu\alpha_3}\right),$$

for every positive real number ν.

(Greece)

A2.

Is there a sequence $a_1, a_2, ...$ of positive real numbers such that $\sum_{i=1}^{n} a_i \leq n^2$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2008$ for any positive integer n?

(Bulgaria)

A3.

Let (a_m) be a sequence satisfaing

$a_n \geq 0$, $n = 0,1,2,3,...$

$\exists A > 0$, $a_m - a_{m+1} \geq A a_m^2$, $\quad m \geq 0$, $m \in \mathbb{N}$.

Prove that there exists $B > 0$ such that

$a_n \leq \frac{B}{n}$, $n = 1,2,3,...$

(Crna Gora)

A4.

We consider the set

$$\mathbb{C}^\nu = \{(z_1, z_2, ..., z_\nu) : z_1, z_2, ..., z_\nu \in \mathbb{C}\}, \nu \geq 2,$$

and the function $\varphi:\mathbb{C}^{\nu}\to\mathbb{C}^{\nu}$ mapping every element $(z_1,z_2,...,z_{\nu})\in\mathbb{C}^{\nu}$ to

$$\varphi(z_1,z_2,...,z_{\nu})=(z_1-z_2,z_2-z_3,...,z_{\nu}-z_1).$$

We also consider the ν-tuple $(w_0,w_1,w_2,...,w_{\nu-1})\in\mathbb{C}^{\nu}$ of the n-th roots of -1, where

$$w_{\mu}=\cos\left(\frac{\pi+2\mu\pi}{\nu}\right)+i\sin\left(\frac{\pi+2\mu\pi}{\nu}\right),\ \mu=0,1,2,...\nu-1.$$

Let after κ, $\kappa\in\mathbb{N}^*$ successive applications of the function φ to the element $(w_0,w_1,w_2,...,w_{\nu-1})$, we obtain the element

$$\varphi^{(\kappa)}(w_0,w_1,w_2,...,w_{\nu-1})\equiv$$
$$\equiv\Big(\underbrace{\varphi\circ\varphi\circ...\circ\varphi}_{\kappa-times}\Big)(w_0,w_1,w_2,...,w_{\nu-1})=(Z_{\kappa 1},Z_{\kappa 2},...,Z_{\kappa\nu})$$

Determine:

(i) the values of ν for which all coordinates of $\varphi^{(\kappa)}(w_0,w_1,w_2,...,w_{\nu-1})$ have measure less or equal to 1,

(ii) for $\nu=4$, the minimal value of $\kappa\in\mathbb{N}^*$ for which: $|Z_{\kappa i}|\geq 2^{100}$, for every $i=1,2,3,4$.

(Greece)

A5.

Consider an ingteger $n\geq 1$, $a_1,a_2,...,a_n$ real numbers in $[-1,1]$ satisfying $a_1+a_2+...+a_n=0$ and a funcion $f:[-1,1]\to\mathbb{R}$ such that

$$|f(x)-f(y)|\leq|x-y|$$

for every x,y in $[-1,1]$. Prove that

$$\left|f(x)-\frac{f(a_1)+f(a_2)+...+f(a_n)}{n}\right|\leq 1$$

for every x in the interval $[-1,1]$. For a given sequence $a_1,a_2,...,a_n$, find f and x so that equality holds.

(Romania)

A6.

Prove that if *x*, *y*, *z* are positive real numbers such that *xy*, *yz* and *zx* are lengths of the side of the triangle and $k\in\,]-1,1[$

then the inequality

$$\frac{\sqrt{xy}}{\sqrt{xz+yz+kxy}}+\frac{\sqrt{yz}}{\sqrt{xy+xz+kyz}}+\frac{\sqrt{zx}}{\sqrt{xy+yz+kzx}}\geq 2\sqrt{1-k}$$

is true. In which conditions the equality is hold.

(Albania)

A7.

Let x, y, z, t be non-negative reals. Show that

$$\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+\sqrt{yz}+\sqrt{yt}+\sqrt{zt}\geq 3\sqrt[3]{xyz+xyt+xzt+yzt}\ .$$

Find all cases when equality holds.

(Romania)

Combinatorics

C1.

All $n+3$ offices of University of Somewhere are numbered with numbers $0,1,2,\ldots,n+1,n+2$, for some $n\in\mathbb{N}$. One day, Profesor D came up with a polynomial with real coefficients and power n. Then, on the door of every office he wrote the value of that polynomial evaluated in the number assigned to that office. On the i-th office, for $i\in\{0,1,2,\ldots,n+1\}$, he wrote 2^i, and on the $(n+2)$ nd office he wrote $2^{n+2}-n-3$.

a) Prove that Professor D made a calculation error.

b) Assuming that Professor D made a calculation error, what is the smallest number of errors he made? Prove that in this case the errors are uniquelly determined, find them and correct them!

(Srbija)

C2.

In one of the countries there are $n\geq 5$ cities operated by two airline companies. Every two cities are operated in both directions by at most one of the companies. The government introduced a restriction that all round trips that a company can offer should have at least six cities. Prove that there no more than $\left[\frac{n^2}{3}\right]$ flights offered by these companies.

(Moldova)

C3.

Let n be positive integer. The rectangle $ABCD$ with sides $AB=90n+1$ and $BC=90n+5$ is divided into unit squares by lines which are parallel to its sides. Prove that the number of the different lines which pass through at least two vertices of the unit squares is divisible to 4.

(Bulgaria)

C4.

An array $n\times n$ is given, consisting of n^2 unit squares. A pawn is placed arbitrarily on an unit square. A *move* of the pawn means a jump from a square of the k-th column to any square of the k-th row. Show that the exists a sequence of n^2 moves of the pawn so that all the unit squares of the array are visited once and, in the end, the pawn returns to the original position.

(Romania)

Geometry

G1.

In acute angled triangle ABC we denote by a,b,c the side lengths, by m_a, m_b, m_c the median lengths and by r_{bc}, r_{ca}, r_{ab} the radii of the circles tangents to two sides and to circumscribed circle of the triangle, respectively. Prove that

$$\frac{m_a^2}{r_{bc}}+\frac{m_b^2}{r_{ac}}+\frac{m_c^2}{r_{ab}}\geq\frac{27\sqrt{3}}{8}\sqrt[3]{abc}\ .$$

(Moldova)

G2.

A non-iscosceles acute triangle ABC is given with $AC > BC$ and H the point of intersection of the heights AZ and CM. We call point P on AB such that $AM=PM$ and N the midpoint of AC. If O the circumcentre of the triangle ABC and $K \equiv PH \cap BC$, $X \equiv ON \cap MK$, $T \equiv OM \cap AC$, prove that the points M, N, T, X are lie on the same circumference.

(Cyprus)

G3.

We draw two lines (ℓ_1), (ℓ_2) through the orthocenter H of the triangle ABC such that each one is dividing the triangle into two figures of equal area and equal perimeters. Find the angles of the triangle.

(Cyprus)

G4.

A triangle ABC is given with barycentre G and circumcentre O. The perpendicular bisectors of GA, GB meet at C_1, of GB, GC meet at A_1 and GC, GA meet at B_1. Prove that O is the barycenter of the triangle $A_1B_1C_1$.

(Greece)

G5.

The circle k_a touches the extensions of sides AB and BC, as well as the circumscribed circle of the triangle ABC (from the outside). We denote the intersection of k_a with the circumscribed circle of the triangle ABC by A'.

Analogously, we define points B' and C'. Prove that the lines AA', BB' and CC' intersect in one point.

(Srbija)

G6.

On triangle ABC the AM $(M \in BC)$ is mediane and BB_1 and CC_1 $(B_1 \in AC, C_1 \in AB)$ are altitudes. The stright line *d* is perpendicular to *AM* at the point *A* and intersect the lines BB_1 and CC_1 at the points *E* and *F* respectively. Let denoted with ω the circle passing through the points *E, M* and *F* and with ω_1 and with ω_2 the circles that are tangent to segment *EF* and with ω at the arc *EF* which is not contain the point *M*. If the points *P* and *Q* are intersections points for ω_1 and ω_2 then prove that the points *P, Q* and *M* are collinear.

(Albania)

G7.

In the non-isosceles triangle ABC consider the points X on $[AB]$ and Y on $[AC]$ such that $[BX]=[CY]$. M and N are the midpoints of the segments $[BC]$, respectively $[XY]$, and the straight lines XY and BC meet in K. Prove that the circumcircle of triangle KMN contains a point, different from M, which is independent of the position of the points X and Y.

(Romania)

G8.

Let P be a point in the interior of a triangle ABC and let d_a, d_b, d_c be its distances to BC, CA, AB respectively. Prove that

$$\max(AP, BP, CP) \geq \sqrt{d_a^2 + d_b^2 + d_c^2}.$$

(Moldova)

Shortlisted -Solutions

Number theory

NT1. **Medium**

Prove that for every natural number a there exists a natural number that has the number a (the sequence of digits that constitute a) at its beginning, and which decreases a times when a is moved from its beginning to its end (any number zeros that appear in the beginning of the number obtained in this way are to be removed).

For example, for $a=4$ we have $\underline{4}10256=4\cdot 10256\underline{4}$; for $a=46$ we have

$$\underline{46}0100021743857360295716=46\cdot 100021743857360295716\underline{46};$$

for $a=58$ we have

$$\underline{58}0100017244352474564=58\cdot 100017244352474564\underline{58}.$$

Solution. Let a be a natural number, and let k be the number digits of a. We want to prove that there exists a number b (with some zeros possibly added at its beginning) such that $\overline{ab}=a\cdot\overline{ba}$. If b has l digits, then we have

$$a\cdot 10^l+b=a\cdot b\cdot 10^k+a^2 \iff a(10^l-a)=b(a\cdot 10^k-1),$$

i.e., it is enough to prove that there exists $l\ge k$ such that

$$(a\cdot 10^k-1)\,|\,(10^l-a),$$

since $(a,a\cdot 10^l-1)=1$ and $10^l>a\dfrac{10^l-a}{a\cdot 10^k-1}=b\ge 0$ ($l\ge k$ implies $10^l\ge a$). Knowing $(a,a\cdot 10^k-1)=1$, we get

$$(a\cdot 10^k-1)\,|\,(10^l-a) \text{ if and only if } (a\cdot 10^k-1)\,|\,(a^s 10^l-a^{s+1}),$$

and fixing $s=\varphi(a\cdot 10^k-1)-1$, from Euler Theorem we get that it is enough to find $l\ge k$ such that $(a\cdot 10^k-1)\,|\,(a^s 10^l-1)$. Since $a\cdot 10^k\equiv 1(\mathrm{mod}\,(a\cdot 10^k-1))$ setting $l=sk$ we get

$$a^s\cdot 10^{sk}=(a\cdot 10^k)^s\equiv 1\ (\mathrm{mod}\ (a\cdot 10^k-1)),$$

and with $l\ge k$ the statement is proved.

(Srbija)

NT2. **Medium**

Let a be a positive integer. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1=a$, $a_{k+1}=a_k^2+a_k+a^3$ for every positive integer $k\ge 1$. Find all values of a for which there exists a positive integer n such that $a_n^2+a^3$ is a m-th power, $m\ge 2$, $m\in\mathbb{N}$, of a positive integer.

Solution. We firstly prove by induction that $4a^3+1$ is coprime with $2a_n+1$, for every $n\ge 1$.

Let $n=1$ and p be a prime divisor of $4a^3+1$ and $2a_1+1=2a+1$. Then p divides $2(4a^3+1)=(2a+1)(4a^2-2a+1)+1$, whence p divides 1, a contradiction. Assume now that $(4a^3+1,2a_n+1)=1$ for some $n\ge 1$ and the prime p divides $4a^3+1$ and $2a_{n+1}+1$. Then p divides $4a_{n+1}+2=(2a_n+1)^2+4a^3+1$, which gives a contradiction.

Assume that for some $n\ge 1$ the number

$$a_{n+1}^2 + a^3 = (a_n^2 + a_n + a^3)^2 + a^3 = (a_n^2 + a^3)(a_n^2 + 2a_n + 1 + a^3)$$

is a power. It follows from the above that $a_n^2 + a^3$ and $a_n^2 + 2a_n + 1 + a^3$ are coprime. This means that $a_n^2 + a^3$ is a power as well. The same argument can be further applied giving that $a_1^2 + a^3 = a^2 + a^3 = a^2(a+1)$ is a power.

If $a^2(a+1) = t^k$ with odd $k \ge 3$, then $a = t_1^k$ and $a+1 = t_2^k$, which is impossible. If $a^2(a+1) = t^{2k}$ with $k \ge 2$, then $a = t_1^k$ and $a+1 = t_2^k$, which is impossible. Therefore $a^2(a+1) = t^2$ whence we obtain the solutions $a = s^2 - 1$, $s \ge 2$, $s \in \mathbb{N}$.

(Bulgaria)

NT3. Medium

The sequence $(x_n)_{n=1}^{\infty}$ is given by

$$x_{n+1} = x_n + x_{\lceil n/2 \rceil}, \quad x_1 = 1.$$

Proof that none of the members of the sequence is divisible by 4.

Solution. From the recurence, we get

$$x_{2n+1} = x_{2n} + x_n = x_{2n-1} + 2x_n, \qquad x_1 = 1.$$

Therefore, every odd-indexedmember is odd, hence not divisible by 4.

Next, let prove that $x_{4n} - x_n$ is divisible by 4. For $n = 1$,

$$x_4 = x_3 + x_2 = x_2 + x_1 + x_2 = 2x_2 + x_1 = 4x_1 + x_1,$$

hence $x_4 - x_1$ is divisible by 4.

In a similar member:

$$x_{4(n+1)} = x_{4n+4} = x_{4n} + 4x_{2n+1} + x_{n+1} - x_n,$$

and

$$x_{4(n+1)} - x_{n+1} = (x_{4n} - x_n) + 4x_{2n+1}.$$

By induction, it follows that $4 \mid x_{4n} - 1$ and $x_1 = 1$. Hence, the members $x_{4n}, n \in \mathbb{N}$ are not divisible by 4. It remains to be proved that evey x_{4n+2} is not divisible by 4:

$$x_{4n+2} = x_{4n+1} + x_{2n+1} = x_{4n} + x_{2n} + x_{2n+1} = (x_{4n} - x_n) + 2x_{2n+1}.$$

This gives $x_{4n+2} \equiv 2 \pmod 4$.

(Crna Gora)

NT4. Easy

Solve in the prime numbers the equation

$$xyz + 1 = 2^{y^2+1}.$$

Solution. It is clear that *x*, *y* and *z* are odd prime and that $2^{y^2+1} \equiv 1 \pmod y$. From the little Fermat theorem we have $2^{y-1} \equiv 1 \pmod y$. If we denote with *d* the primitive root of the congruence $2^m \equiv 1 \pmod y$ then it is clear that *d* divides y^2+1 and $y-1$, so *d* divides $y^2 - y + 2$ and at the end , *d* divides 2. Since *y* is prime then it is clear that *d*=2 and *y*=3. Now it is easy to show that solutions are (11,3,31) and (31,3,11).

(Albania)

NT5. **Easy**

Let $\{a_n\}$ be the sequence with $a_1 = 0$ and $a_{n+1} = 2 + a_n$ for odd n and $a_{n+1} = 2a_n$ for even n. Prove that for each prime $p>3$ the number $b = \frac{2^{2p}-1}{3}$ divides a_n for the infinitely many values of n.

Solution. It is very easy to show that $a_{2k} = 2^{k+1} - 2$ and $a_{2k+1} = 2^{k+2} - 4$. If we take $n = \frac{2^{2p+1}-8}{3}$ then $a_n = 2^{\frac{2^{2p}-1}{3}} - 2$. From the Fermat theorem it is clear that *2p* divides number $b-1$. Now, on the system with base two, we have $6b = 111.....110$ (with *2p* unity) and $a_n = 2^b - 2 = 111.....110$ (with $b-1$ unity) and the result is clear.

Another solution. First we show that

$$a_{2k} = 2^{k+1} - 2, \qquad a_{2k+1} = 2^{k+1} - 4.$$

From the recurence

$$a_{2(n+1)} = 2 + a_{2n+1} + 2 + 2a_{2n}$$

so

$$a_{2(n+1)} + 2 = 2(2 + a_{2n}).$$

By induction, having in mind that $a_2 = 2$, we obtain

$$a_{2n} + 2 = 2^{n+1}.$$

Then

$$a_{2n+1} = 2a_{2n} = 2^{n+2} - 4.$$

For $2p \mid k$, $2^{2p} - 1 \mid 2^k - 1$. So for any $n = 4ps$, $s \in \mathbb{N}$, $b \mid a_n$.

(Albania)

NT6. **Easy**

Let $(x_n), n = 1,2,3,...$ be a sequence defined by $x_1 = 2008$ and

$$x_1 + x_2 + \cdots + x_{n-1} = (n^2 - 1)x_n \quad \text{, for every } n \ge 2. \qquad (1)$$

Let, also, the sequence $a_n = x_n + \frac{1}{n}S_n$, $n = 1,2,3,...$ where $S_n = x_1 + x_2 + \cdots + x_n$. Determine the values of n for which the terms of the sequence a_n are perfect squares of an integer.

Solution. The given relation (1) can be written as

$$x_1 + x_2 + \cdots + x_{n-1} + x_n = n^2 x_n$$

$$\Leftrightarrow (n-1)^2 x_{n-1} = (n^2 - 1)x_n \Leftrightarrow (n-1)x_{n-1} = (n+1)x_n.$$

Therefore we have the relations

$$\left.\begin{array}{l} \frac{x_n}{x_{n-1}} = \frac{n-1}{n+1} \\ \frac{x_{n-1}}{x_{n-2}} = \frac{n-2}{n} \\ \vdots \\ \frac{x_3}{x_2} = \frac{2}{4} \\ \frac{x_2}{x_1} = \frac{1}{3} \end{array}\right\}$$

Multiplying by parts the above relations we obtain

$$x_n = \frac{2x_1}{n(n+1)},\ n = 1,2,3,\dots,$$

Since $S_n = x_1 + x_2 + \dots + x_n = n^2 x_n$ we have

$$S_{n+1} = (n+1)^2 x_{n+1} = (n+1)^2 (S_{n+1} - S_n) \Rightarrow \frac{S_{n+1}}{S_n} = \frac{(n+1)^2}{n(n+2)}$$

and as in the previous case we find

$$\boxed{S_n = \frac{2S_1 n}{(n+1)} = \frac{2x_1 n}{(n+1)}},\ n = 1,2,3,\dots.$$

So, we have

$$a_n = x_n + \frac{1}{n} S_n = \frac{2x_1}{n(n+1)} + \frac{2x_1 n}{n(n+1)} = \frac{2x_1}{n} = \frac{2^4 \cdot 251}{n}.$$

The term a_n is a perfect square of an integer, when $n = 251,\ 2^2 \cdot 251,\ 2^4 \cdot 251$.

(Greece)

Algebra and Inequalities

A1. Hard

For all positive real numbers $\alpha_1, \alpha_2, \alpha_3$ prove that

$$\frac{1}{2\nu\alpha_1 + \alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + 2\nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \alpha_2 + 2\nu\alpha_3} >$$
$$> \frac{2\nu}{2\nu+1}\left(\frac{1}{\nu\alpha_1 + \nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \nu\alpha_2 + \nu\alpha_3} + \frac{1}{\nu\alpha_1 + \alpha_2 + \nu\alpha_3} \right),$$

for every positive real number ν.

Solution. It is enough to prove that

$$\frac{1}{2\nu\alpha_1 + \alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + 2\nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \alpha_2 + 2\nu\alpha_3} >$$
$$> \frac{2}{\frac{2\nu+1}{\nu}(\nu(\alpha_1 + \alpha_2) + \alpha_3)} + \frac{2}{\frac{2\nu+1}{\nu}(\alpha_1 + \nu(\alpha_2 + \alpha_3))} + \frac{2}{\frac{2\nu+1}{\nu}(\alpha_2 + \nu(\alpha_1 + \alpha_3))}$$

or

$$\frac{1}{2\nu\alpha_1 + \alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + 2\nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \alpha_2 + 2\nu\alpha_3} >$$
$$> \frac{2}{(2\nu+1)(\alpha_1 + \alpha_2) + \left(2 + \frac{1}{\nu}\right)\alpha_3} + \frac{2}{(2\nu+1)(\alpha_1 + \alpha_3) + \left(2 + \frac{1}{\nu}\right)\alpha_2} + \frac{2}{(2\nu+1)(\alpha_2 + \alpha_3) + \left(2 + \frac{1}{\nu}\right)\alpha_1}$$

or, it is enough to prove that

$$\frac{1}{2\nu\alpha_1 + \alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + 2\nu\alpha_2 + \alpha_3} + \frac{1}{\alpha_1 + \alpha_2 + 2\nu\alpha_3} \geq$$
$$\geq \frac{2}{(2\nu+1)(\alpha_1 + \alpha_2) + 2\alpha_3} + \frac{2}{(2\nu+1)(\alpha_1 + \alpha_3) + 2\alpha_2} + \frac{2}{(2\nu+1)(\alpha_2 + \alpha_3) + 2\alpha_1} \qquad (1).$$

We put

$$x = 2\nu\alpha_1 + \alpha_2 + \alpha_3,\ y = \alpha_1 + 2\nu\alpha_2 + \alpha_3 \text{ and } z = \alpha_1 + \alpha_2 + 2\nu\alpha_3,$$

and then we put:

$$a = x + y = (2\nu + 1)(\alpha_1 + \alpha_2) + 2\alpha_3$$
$$b = y + z = (2\nu + 1)(\alpha_2 + \alpha_3) + 2\alpha_1$$
$$c = x + z = (2\nu + 1)(\alpha_1 + \alpha_3) + 2\alpha_2 .$$

Now we observe that:

$$\left.\begin{array}{l} a + b = x + 2y + z > x + z = c \\ b + c = x + y + 2z > x + y = a \\ a + c = 2x + y + z > y + z = b \end{array}\right\} .$$

Therefore the numbers a,b,c are measures of the sides of a triangle and we have:

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} = \frac{2c}{c^2 - (a-b)^2} \geq \frac{2}{c} \quad (*)$$

$$\frac{1}{c+a-b} + \frac{1}{a+b-c} \geq \frac{2}{a}$$

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} \geq \frac{2}{b} .$$

Adding the last three inequalities by parts and putting $\tau = \frac{a+b+c}{2}$ we obtain:

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \qquad \text{or}$$

$$\frac{1}{2(\tau-a)} + \frac{1}{2(\tau-b)} + \frac{1}{2(\tau-c)} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \qquad \text{or}$$

$$\frac{1}{\tau\text{-a}} + \frac{1}{\tau\text{-b}} + \frac{1}{\tau\text{-c}} \geq \frac{2}{a} + \frac{2}{b} + \frac{2}{c} . \qquad (2)$$

However, we have

$$\tau = \frac{a+b+c}{2} \quad \text{and} \quad \begin{cases} a = x + y = (2\nu + 1)(\alpha_1 + \alpha_2) + 2\alpha_3 \\ b = y + z = (2\nu + 1)(\alpha_2 + \alpha_3) + 2\alpha_1 \\ c = x + z = (2\nu + 1)(\alpha_1 + \alpha_3) + 2\alpha_2 \end{cases}$$

and hence:

$$\tau = \frac{a+b+c}{2} = 2(\nu + 1) \cdot \alpha_1 + 2(\nu + 1) \cdot \alpha_2 + 2(\nu + 1) \cdot \alpha_3$$
$$\tau - a = \alpha_1 + \alpha_2 + 2\nu\alpha_3$$
$$\tau - b = 2\nu\alpha_1 + \alpha_2 + \alpha_3$$
$$\tau - c = \alpha_1 + 2\nu\alpha_2 + \alpha_3 .$$

Substituting the last three equalities to inequality (2) we obtain inequality (1).

Comment

(*) In order to prove the inequality $\frac{2c}{c^2 - (a-b)^2} \geq \frac{2}{c}$, it is enough to prove that:

$$2c^2 \geq 2c^2 - 2(a-b)^2 \text{ (which is clear).}$$

(Greece)

A2. Medium

Is there a sequence $a_1, a_2, \ldots$ of positive real numbers such that $\sum_{i=1}^{n} a_i \leq n^2$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 2008$ for any positive integer n?

Solution. The answer is no. It is enough to show that if $\sum_{i=1}^{n} a_i \le n^2$ for any n, then $\sum_{i=2}^{2^n} \frac{1}{a_i} > \frac{n}{4}$. For this, we use that $\sum_{i=2^k+1}^{2^{k+1}} a_i \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} \ge 2^{2k}$ for any $k \ge 0$ by the power mean inequality. Since $\sum_{i=2^k+1}^{2^{k+1}} a_i < \sum_{i=1}^{2^{k+1}} a_i \le 2^{2k+2}$, it follows that $\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} > \frac{1}{4}$ and hence

$$\sum_{i=2}^{2^n} \frac{1}{a_i} > \sum_{k=0}^{n-1} \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} > \frac{n}{4}.$$

(Bulgaria)

A3. **Easy → medium**

Let (a_m) be a sequence satisfaing

$$a_n \ge 0,\ n = 0,1,2,3,...$$

$$\exists A > 0,\ a_m - a_{m+1} \ge A a_m^2, \qquad m \ge 0,\ m \in \mathbb{N}.$$

Prove that there exists $B > 0$ such that

$$a_n \le \frac{B}{n},\ n = 1,2,3,...$$

Solution. If $a_{m_\circ} = 0$ for some $m_\circ$, then from the condition $a_n - a_{n+1} \ge A a_n^2$, $a_n \ge 0$ we get $a_m = 0$ for every $m \ge m_\circ$.

So, we can take $B = m_\circ \max\{a_1, a_2, ..., a_m\}$.

Let suppose that $a_m > 0$, $\forall m \ge 0$. Then, dividing the given recurent inequality $a_m - a_{m+1} \ge A a_m^2$ by $a_m a_{m+1}$ we get:

$$\frac{1}{a_{m+1}} - \frac{1}{a_m} = \frac{a_m - a_{m+1}}{a_m a_{m+1}} \ge \frac{a_m}{a_{m+n}} A \ge A > 0,\ n \ge n_\circ.$$

Summing up from 0 to $n-1 \ge 0$, we get

$$\frac{1}{a_n} - \frac{1}{a_\circ} \ge nA.$$

It follows that

$$a_n \le \frac{1}{nA} = \frac{\frac{1}{A}}{n} = \frac{B}{n},$$

where $B = \frac{1}{A}$. **(Crna Gora)**

A4. **Easy → medium**

We consider the set

$$\mathbb{C}^\nu = \left\{ (z_1, z_2, ..., z_\nu) : z_1, z_2, ..., z_\nu \in \mathbb{C} \right\}, \nu \ge 2,$$

and the function $\varphi : \mathbb{C}^\nu \to \mathbb{C}^\nu$ mapping every element $(z_1, z_2, ..., z_\nu) \in \mathbb{C}^\nu$ to

$$\varphi(z_1, z_2, ..., z_\nu) = (z_1 - z_2, z_2 - z_3, ..., z_\nu - z_1).$$

We also consider the ν-tuple $(w_0, w_1, w_2, ..., w_{\nu-1}) \in \mathbb{C}^\nu$ of the n-th roots of -1, where

$$w_\mu = \cos\left(\frac{\pi + 2\mu\pi}{\nu}\right) + i \sin\left(\frac{\pi + 2\mu\pi}{\nu}\right),\ \mu = 0,1,2,...\nu - 1.$$

Let after κ, $\kappa \in \mathbb{N}^*$ successive applications of the function φ to the element $(w_0, w_1, w_2, ..., w_{\nu-1})$, we obtain the element

$$\varphi^{(\kappa)}(w_0, w_1, w_2, ..., w_{\nu-1}) \equiv \left(\underbrace{\varphi \circ \varphi \circ ... \circ \varphi}_{\kappa - times} \right)(w_0, w_1, w_2, ..., w_{\nu-1}) =$$
$$= (Z_{\kappa 1}, Z_{\kappa 2}, ..., Z_{\kappa \nu})$$

(i) the values of ν for which all coordinates of $\varphi^{(\kappa)}(w_0, w_1, w_2, ..., w_{\nu-1})$ have measure less or equal to 1,

(ii) for $\nu = 4$, the minimal value of $\kappa \in \mathbb{N}^*$ for which: $|Z_{\kappa i}| \geq 2^{100}$, for every $i = 1, 2, 3, 4$.

Solution. **(i)** The n-th roots of -1 can be written as

$$w_0 = \cos\frac{\pi}{\nu} + i\sin\frac{\pi}{\nu},\ w_1 = w_0\omega,\ w_2 = w_0\omega^2, ...,\ w_{\nu-1} = w_0\omega^{\nu-1},$$

where $\omega = \cos\frac{2\pi}{\nu} + i\sin\frac{2\pi}{\nu}$ is such that $\omega^\nu = 1$.

We have

$$\varphi(w_0, w_1, w_2, ..., w_{\nu-1}) =$$
$$= \left(w_0(1-\omega), w_0\omega(1-\omega), w_0\omega^2(1-\omega), ..., w_0\omega^{\nu-1}(1-\omega)\right)$$

and using induction we obtain

$$\varphi^{(\kappa)}(w_0, w_1, w_2, ..., w_{\nu-1}) =$$
$$= \left(w_0(1-\omega)^\kappa, w_0\omega(1-\omega)^\kappa, w_0\omega^2(1-\omega)^\kappa, ..., w_0\omega^{\nu-1}(1-\omega)^\kappa\right),$$

for every $\kappa \in \mathbb{N}^*$. Therefore we have

$$Z_{\kappa i} = w_0\omega^{i-1}(1-\omega)^\kappa, i = 1, 2, ..., \nu,\ \kappa \in \mathbb{N}^*.$$

Since $|w_0| = 1, |\omega| = 1$ and $|\omega^{i-1}| = |\omega|^{i-1} = 1,$ for every $i = 1, 2, ..., \nu$, we have

$$|Z_{\kappa i}| = \left|w_0\omega^{i-1}(1-\omega)^\kappa\right| = |1-\omega|^\kappa = \left|\left(1 - \sigma\upsilon\nu\frac{2\pi}{\nu}\right) - i\eta\mu\frac{2\pi}{\nu}\right|^\kappa$$

$$\Rightarrow |Z_{\kappa i}| = \left[\left(1 - \sigma\upsilon\nu\frac{2\pi}{\nu}\right)^2 + \eta\mu^2\frac{2\pi}{\nu}\right]^{\frac{\kappa}{2}} = \left[2\left(1 - \sigma\upsilon\nu\frac{2\pi}{\nu}\right)\right]^{\frac{\kappa}{2}}$$

$$\Rightarrow |Z_{\kappa i}| = \left(4\eta\mu^2\frac{\pi}{\nu}\right)^{\frac{\kappa}{2}} = \left(2\eta\mu\frac{\pi}{\nu}\right)^\kappa,\ \kappa \in \mathbb{N}^*.$$

Hence all the coordinates of $\varphi^{(\kappa)}(w_0, w_1, w_2, ..., w_{\nu-1})$ have measure $\left(2\eta\mu\frac{\pi}{\nu}\right)^\kappa$ and having in mind that for every $\nu \geq 2$, we have $0 < \frac{\pi}{\nu} \leq \frac{\pi}{2}$, we obtain

$$|Z_{\kappa i}| = \left(2\eta\mu\frac{\pi}{\nu}\right)^\kappa \leq 1 \Leftrightarrow 2\eta\mu\frac{\pi}{\nu} \leq 1 \Leftrightarrow \eta\mu\frac{\pi}{\nu} \leq \frac{1}{2} \Leftrightarrow \frac{\pi}{\nu} \leq \frac{\pi}{6} \Leftrightarrow \nu \geq 6.$$

(ii) We have

$$|Z_{\kappa i}| = \left(2\eta\mu\frac{\pi}{4}\right)^\kappa \geq 2^{100} \Leftrightarrow \left(\sqrt{2}\right)^\kappa \geq 2^{100} \Leftrightarrow 2^{\frac{\kappa}{2}} \geq 2^{100}$$
$$\Leftrightarrow \kappa \geq 200.$$

Hence the minimal value of κ is 200.

(Greece)

A5. **Easy $\to$ medium**

Consider an ingteger $n \geq 1$, $a_1, a_2, ..., a_n$ real numbers in $[-1,1]$ satisfying $a_1 + a_2 + ... + a_n = 0$ and a funcion $f : [-1,1] \to \mathbb{R}$ such that

$$|f(x) - f(y)| \leq |x - y|$$

for every x, y in $[-1,1]$.

Prove that

$$\left| f(x) - \frac{f(a_1) + f(a_2) + ... + f(a_n)}{n} \right| \leq 1$$

for every x in the interval $[-1,1]$.

For a given sequence $a_1, a_2, ..., a_n$, find f and x so that equality holds.

Solution. Using the Lipschitz condition we easily get

$$\left| f(x) - \frac{f(a_1) + f(a_2) + ... + f(a_n)}{n} \right| = \left| \frac{nf(x) - (f(a_1) + f(a_2) + ... + f(a_n))}{n} \right| =$$

$$= \left| \frac{(f(x) - f(a_1)) + (f(x) - f(a_2)) + ... + (f(x) - f(a_n))}{n} \right| \leq$$

$$= \frac{|f(x) - f(a_1)| + |f(x) - f(a_2)| + ... + |f(x) - f(a_n)|}{n} \leq$$

$$\leq \frac{|x - a_1| + |x - a_2| + ... + |x - a_n|}{n}$$

Thus, it is enough to show that

$$g(x) = |x - a_1| + |x - a_2| + ... + |x - a_n| \leq n,$$

on the interval $[-1,1]$. WLOG we may suppose that the numbers are ordered, i.e. $a_1 \leq a_2 \leq ... \leq a_n$. For $x \in [a_{k-1}, a_k)$, $k = 2,3,...,n$ we have

$$g(x) = (2k - n)x + a_1 + a_2 + ... + a_k - a_{k+1} - ... - a_n,$$

implying that the function g is decreasing on the interval $\left[-1, a_{\left[\frac{n+1}{2}\right]}\right]$ and increasing on $\left[a_{\left[\frac{n+1}{2}\right]}, 1\right]$. So, to prove the inequality $g(x) \leq n$, it suffices to verify it at -1 and 1, where it is obvious.

It is clear that the equality in the last part of the proof is attaned only when $x = \pm 1$. In this case we also need the equalities $|f(1) - f(x)| = 1 - x$ for all $x \in [-1,1]$ (and the similar in -1). This implies $f(x) = -x + 1 + f(1)$ or $f(x) = x - 1 + f(1)$. In both situations are possible, say $f(x_1) = -x_1 + 1 + f(1)$ and $f(x_2) = x_2 - 1 + f(1)$ we get $|f(x_1) - f(x_2)| = |x_1 - x_2 + 2|$, in contradiction with the given condition. Thus, the equality is possible if and only if $f(x) = x + k$ for all x or $f(x) = -x + k$ for all x and the value of x in the problem is ± 1.

(Romania)

A6. **Easy**

Prove that if *x*, *y*, *z* are positive real numbers such that *xy*, *yz* and *zx* are lengths of the side of the triangle and $k \in \left]-1,1\right[$

then the inequality

$$\frac{\sqrt{xy}}{\sqrt{xz + yz + kxy}} + \frac{\sqrt{yz}}{\sqrt{xy + xz + kyz}} + \frac{\sqrt{zx}}{\sqrt{xy + yz + kzx}} \geq 2\sqrt{1-k}$$

is true. In which conditions the equality is hold.

Solution. We have $\sqrt{(1-k)xy(xz+yz+kxy)} \le \frac{xy+yz+zx}{2}$ and it follows that

$$\frac{\sqrt{xy}}{\sqrt{xz+yz+kxy}} \ge 2\sqrt{1-k}\,\frac{xy}{xy+yz+zx}.$$

Now the result is clear. The conditions of equality are

$$1-2k = \frac{z}{x}+\frac{z}{y} = \frac{y}{x}+\frac{y}{z} = \frac{x}{y}+\frac{x}{z}$$

and it follows that $3-6k = (\frac{z}{x}+\frac{x}{z})+(\frac{z}{y}+\frac{y}{z})+(\frac{y}{x}+\frac{x}{y}) \ge 6$ and equality holds for $x=y=z$ and $k=-\frac{1}{2}$.

(Albania)

A7. Easy

Let x, y, z, t be non-negative reals. Show that

$$\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+\sqrt{yz}+\sqrt{yt}+\sqrt{zt} \ge 3\sqrt[3]{xyz+xyt+xzt+yzt}.$$

Find all cases when equality holds.

Solution. By the AGM inequality, we have

$$(\sqrt{xy}+\sqrt{zt})+(\sqrt{xz}+\sqrt{yt})+(\sqrt{xt}+\sqrt{yz}) \ge 3\sqrt[3]{(\sqrt{xy}+\sqrt{zt})(\sqrt{xz}+\sqrt{yt})(\sqrt{xt}+\sqrt{yz})}.$$

Remark that

$$(\sqrt{xy}+\sqrt{zt})(\sqrt{xz}+\sqrt{yt})(\sqrt{xt}+\sqrt{yz}) =$$
$$= xyz+xyt+xzt+yzt+\sqrt{xyzt}(x+y+z+t) \ge,$$
$$\ge xyz+xyt+xzt+yzt$$

thus

$$\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+\sqrt{yz}+\sqrt{yt}+\sqrt{zt} \ge 3\sqrt[3]{xyz+xyt+xzt+yzt}.$$

By the above, equality implies $xyzt=0$, so one of the numbers is zero. Then, the other numbers must be equal.

(Romania)

Combinatorics

C1. Hard

All $n+3$ offices of University of Somewhere are numbered with numbers $0,1,2,\ldots,n+1,n+2$, for some $n\in\mathbb{N}$. One day, Profesor D came up with a polynomial with real coefficients and power n. Then, on the door of every office he wrote the value of that polynomial evaluated in the number assigned to that office. On the i-th office, for $i\in\{0,1,2,\ldots,n+1\}$, he wrote 2^i, and on the $(n+2)$ nd office he wrote $2^{n+2}-n-3$.

a) Prove that Professor D made a calculation error.

b) Assuming that Professor D made a calculation error, what is the smallest number of errors he made? Prove that in this case the errors are uniquely determined, find them and correct them!

Solution. (a) Assume for a contradiction that Professor D did not make any errors. Denote by P the polynomial that he came up with.

We define

$$Q(x)=\binom{x}{0}+\binom{x}{1}+\ldots+\binom{x}{n},$$

where $\binom{x}{k}=\frac{x\cdot(x-1)\cdot(x-2)\cdot\ldots\cdot(x-k+1)}{k!}$, for every $x\in\mathbb{R}$ and $k\in\mathbb{N}$, and $\binom{x}{0}=1$, for every $x\in\mathbb{R}$. Hence, for every $0\le i\le n$ we have

$$Q(i)=\binom{i}{0}+\binom{i}{1}+\ldots+\binom{i}{n}=\binom{i}{0}+\binom{i}{1}+\ldots+\binom{i}{i}=(1+1)^i=2^i.$$

Both polynomials P and Q are of power n, and they have the same values in $n+1$ points, meaning that $Q\equiv P$. However, that means that

$$2^{n+1}=P(n+1)=Q(n+1)=\binom{n+1}{0}+\binom{n+1}{1}+\ldots+\binom{n+1}{n}= =(1+1)^{n+1}-\binom{n+1}{n+1}=2^{n+1}-1,$$

which is obviously not true, a contradiction.

(b) We will prove that the minimal number of errors Professor D made is one. Since

$$Q(n+2)=\binom{n+2}{0}+\binom{n+2}{1}+\ldots+\binom{n+2}{n}= =(1+1)^{n+1}-\binom{n+2}{n+1}-\binom{n+2}{n+2}=2^{n+2}-n-3,$$

if Professor D made exactly one error, he must have came up with the polynomial Q. Then, the only error would be the value of the polynomial evaluated in $(n+1)$-instead of 2^{n+1}, it should be $2^{n+1}-1$.

Let us prove that this is the only possibility for making one error. Assume for a contradiction that there is another way of making one error. Part (a) implies that the error has not been made in evaluation of the polynomial in $n+2$. By j, $0\le j\le n$, we denote the point in which the polynomial has been evaluated wrongly.

Let $P(j)=2^j+g$, instead of initially evaluated $P(j)=2^j$, for some $g\in\mathbb{R}$. Then, Lagrange's Interpolation Formula implies

$$P(x)=\sum_{k=0}^{n}P(k)\frac{(x-0)(x-1)\cdot\ldots\cdot(x-(k-1))(x-(k+1))\cdot\ldots\cdot(x-n)}{(k-0)(k-1)\cdot\ldots\cdot(k-(k-1))(k-(k+1))\cdot\ldots\cdot(k-n)}.$$

It follows that

$$2^{n+1}=P(n+1)=\sum_{k=0}^{n}2^k\frac{(n+1-0)(n+1-1)\cdot\ldots\cdot(n+1-(k-1))(n+1-(k+1))\cdot\ldots\cdot(n+1-n)}{(k-0)(k-1)\cdot\ldots\cdot(k-(k-1))(k-(k+1))\cdot\ldots\cdot(k-n)}+$$
$$+g\frac{(n+1-0)(n+1-1)\cdot\ldots\cdot(n+1-(j-1))(n+1-(j+1))\cdot\ldots\cdot(n+1-n)}{(j-0)(j-1)\cdot\ldots\cdot(j-(j-1))(j-(j+1))\cdot\ldots\cdot(j-n)}$$
$$=g\binom{n+1}{j}(-1)^{n-j}+\sum_{k=0}^{n}2^k\binom{n+1}{k}(-1)^{n-k}=g\binom{n+1}{j}(-1)^{n-j}-1+2^{n+1},$$

i.e.,

$$g\binom{n+1}{j}(-1)^{n-j}=1. \qquad (1)$$

Similarly, when we use $P(n+1)=2^{n+1}$ for the polynomial evaluation using Lagrange's Interpolation Formula, we get

$$2^{n+2}-n-3=g\binom{n+2}{j}(-1)^{n+1-j}-1+2^{n+2},$$

and

$$g\binom{n+2}{j}(-1)^{n-j}=n+2\,. \qquad (2)$$

Dividing (2) by (1) we obtain

$$\binom{n+2}{j}=(n+2)\binom{n+1}{j},$$

and

$$\binom{n+2}{j}=\binom{n+2}{j}(n-j+2),$$

i.e., $j=n+1$. This, is however in contradiction with the assumption that $j\le n$.

(Srbija)

C2. Hard

In one of the countries there are $n\ge 5$ cities operated by two airline companies. Every two cities are operated in both directions by at most one of the companies. The government introduced a restriction that all round trips that a company can offer should have at least six cities. Prove that there no more than $\left[\frac{n^2}{3}\right]$ flights offered by these companies.

Solution. Consider the graph G with n vertices representing cities with edges colored in two colors(blue and red), representing connections between them operated by two companies. The condition of the problem is equivalent that there does not exist circuit subgraphs C_3, C_4, C_5.

Assume to the contrary that are at least $\left[\frac{n^2}{3}\right]+1$ edges in the graph G. Using the Turan's Theorem we conclude there exist a complete subgraph $K_4=\{A_1,A_2,A_3,A_4\}$ of the graph G with all its edges colored in two colors (blue and red). As there no circuit subgraphs C_3, C_4, C_5 in G, the only possible coloring is the following: the edges A_1A_2, A_2A_3, A_3A_4 are colored blue and A_1A_3, A_1A_4, A_2A_4 are colored red.

First of all we prove that we get contradiction for $n=5,6,7,8$. Extract from the graph G four vertices A_1,A_2,A_3,A_4 of the subgraph K_4 and observe that each of the remaining $n-4$ vertices has at most 2 connections with these 4 vertices. If there will be three connections that two of them will be of the same color and they together with vertices of K_4 will form the subgraphs C_2, C_3 or C_5. There are at most $\frac{(n-4)(n-5)}{2}$ edges between $n-4$ remaining vertices. Thus there are totally at most $6+2(n-4)+\dfrac{(n-4)(n-5)}{2}$ edges in the graph G. But

$$6+2(n-4)+\frac{(n-4)(n-5)}{2}\le\frac{n^2}{3} \quad \text{for } 5\le n\le 8,$$

because

$$12n-12+3(n-4)(n-5)\le 2n^2 \quad \text{or} \quad n^2-15n+48\le 0\,.$$

So, the statement of the problem is true for $n=5,6,7,8$.

Now we prove it using mathematical induction, using the above result as a base case. We apply the same idea. We assume the contrary and find the subgraph K_4 whose existence is ensured by Thuran's Theorem. By the induction hypothesis there will be no more than $\frac{(n-4)^2}{3}$ edges between the remaining $n-4$ vertices. Thus in the graph G there will be at most $6+2(n-4)+\frac{(n-4)^2}{2}$ edges. But

$$6+2(n-4)+\frac{(n-4)^2}{3}\le\frac{n^2}{3},$$

because

$$6n-12+(n-4)^2\le n^2 \text{ or } 4\le 2n,$$

a contradiction and we are done. The problem is solved.

(Moldova)

C3. Eesy-medium

Let n be positive integer. The rectangle $ABCD$ with sides $AB=90n+1$ and $BC=90n+5$ is divided into unit squares by lines which are parallel to its sides. Prove that the number of the different lines which pass through at least two vertices of the unit squares is divisible to 4.

Solution. Denote $90n+1=m$. We investigate the number of the lines modulo 4 consecutively reducing different types of lines.

The vertical and horizontal lines are $(m+5)+(m+1)=2(m+3)$ which is divisible to 4. Moreover, every line which makes an acute angle to the axe Ox (i.e. that line has a positive angular coefficient) corresponds to unique line with an obtuse angle (consider the symmetry with respect to the line through the midpoints of AB and CD). Therefore it is enough to prove that the lines with acute angles are an even number.

Every line which does not pass through the center O of the rectangle corresponds to another line with the same angular coefficient (consider the symmetry with respect to O). Therefore it is enough to consider the lines through O.

Every line through O has an angular coefficient $\frac{p}{q}$, where $(p,q)=1$, p and q are odd positive integers. (To see this, consider the two nearest, from the two sides, to O points of the line). If $p\ne1$, $q\ne1$, $p\le m$ and $q\le m$, the line with angular coefficient $\frac{p}{q}$, uniquely corresponds to the line with angular coefficient $\frac{p}{q}$. It remains to prove that the number of the remaining lines is even.

The last number is

$$1+\frac{\varphi(m+2)}{2}+\frac{\varphi(m+4)}{2}-1=\frac{\varphi(m+2)+\varphi(m+4)}{2}$$

because we have:

1) one line with $p=q=1$;
2) $\frac{\varphi(m+2)}{2}$ lines with angular coefficient $\frac{p}{m+2}$, $p\le m$ is odd and $(p,m+2)=1$;
3) $\frac{\varphi(m+4)}{2}-1$ lines with angular coefficient $\frac{p}{m+4}$, $p\le m$ is odd and $(p,m+4)=1$.

Now the assertion follows from the fact that the number

$$\varphi(m+2)+\varphi(m+4)=\varphi(90n+3)+\varphi(90n+5)$$

is divisible to 4.

(Bulgaria)

C4. **Easy**

An array $n\times n$ is given, consisting of n^2 unit squares. A pawn is placed arbitrarily on an unit square. A *move* of the pawn means a jump from a square of the k-th column to any square of the k-th row. Show that the exists a sequence of n^2 moves of the pawn so that all the unit squares of the array are visited once and, in the end, the pawn returns to the original position.

Solution. Label the unit squares (i,j). A move will be denoted $(i,k)\to(k,t)$.

We use induction on n; the base case is trivial.

Consider now a $(n+1)\times(n+1)$ array and assume that the pawn starts in the $n\times n$ array A situated in the upper left corner and that a circuit exists inside A. Let (i,a_i) be the next square visited after (i,i) in this circuit. Now replace the move $(n,n)\to(n,a_n)$ with the moves

$$(n,n)\to(n,n+1)\to(n+1,n+1)\to(n+1,n)\to(n,a_n),$$

and replace $(i,i)\to(i,a_i)$, $i=1,2,...,n-1$ by the sequence

$$(i,i)\to(i,n+1)\to(n+1,i)\to(i,a_i).$$

With the rest of the moves unaltered, notice that all the $3+2(n-1)=2n+1=(n+1)^2-n^2$ new squares of the $(n+1)\times(n+1)$ array are visited, so we are done.

If the pawn starts from an unit square S situated the $(n+1)$-th row or column, take any circuit which covers the $(n+1)\times(n+1)$ array and starts in a square P of A, then rearrange the sequence of moves $P\to...\to S\to...\to P$ in the form $S\to...\to P\to...\to S$, to get a circuit at S.

(Romania)

Geometry

G1. **Medium**

In acute angled triangle ABC we denote by a,b,c the side lengths, by m_a,m_b,m_c the median lengths and by r_{bc},r_{ca},r_{ab} the radii of the circles tangents to two sides and to circumscribed circle of the triangle, respectively. Prove that

$$\frac{m_a^2}{r_{bc}}+\frac{m_b^2}{r_{ac}}+\frac{m_c^2}{r_{ab}}\geq\frac{27\sqrt{3}}{8}\sqrt[3]{abc}.$$

Solution. Let the circle with center O_1 is tangent to the sides AC and AB at the points B_1 and C_1 respectively and the point P is the common point of this circle and circumscribed circle of the triangle ABC. If O is the circumcenter, then the points P,O,I_1 are collinear and

$$OA=R,\ OI_1=R-r_{bc},\ AI_1=\frac{r_{bc}}{\sin\frac{A}{2}},\ \cos(\angle OAI_1)=\cos\frac{B-C}{2}.$$

By applying the cosinus law in the triangle OAI_1 we obtain

$$4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=r_{bc}\cos^2\frac{A}{2}=r,$$

where r is the radius of the incircle of the triangle ABC. So, we have

$$r_{bc}=\frac{r}{\cos^2\frac{A}{2}},\ r_{ca}=\frac{r}{\cos^2\frac{B}{2}},\ r_{ab}=\frac{r}{\cos^2\frac{C}{2}}.$$

Let $2p=a+b+c$, and R is the radius of the circumcircle of the triangle ABC. Then

$$abc=4pRr,\ \ p\ge 3\sqrt{3}r,\ \ \frac{1}{r_{bc}r_{ca}r_{ab}}=\frac{p^2}{16R^2r^3}.$$

Suppose WLOG that in the acute angled triangle ABC we have $a\ge b\ge c$. Then

$$m_a^2\le m_b^2\le m_c^2,\qquad \cos^2\frac{A}{2}\le\cos^2\frac{B}{2}\le\cos^2\frac{C}{2},\qquad \frac{1}{r_{bc}}\le\frac{1}{r_{ca}}\le\frac{1}{r_{ab}}.$$

Thus, the triples (m_a^2,m_b^2,m_c^2) and $\left(\frac{1}{r_{bc}},\frac{1}{r_{ca}},\frac{1}{r_{ab}}\right)$ have the same ordering. By applying Chebyshev inequality we obtain

$$\frac{m_a^2}{r_{bc}}+\frac{m_b^2}{r_{ac}}+\frac{m_c^2}{r_{ab}}\ge\frac{1}{3}(m_a^2+m_b^2+m_c^2)\left(\frac{1}{r_{bc}}+\frac{1}{r_{ca}}+\frac{1}{r_{ab}}\right)=\frac{1}{3}\cdot\frac{3}{4}\cdot(a^2+b^2+c^2)\left(\frac{1}{r_{bc}}+\frac{1}{r_{ca}}+\frac{1}{r_{ab}}\right)\ge$$

$$\ge\frac{1}{12}(a+b+c)^2\left(\frac{1}{r_{bc}}+\frac{1}{r_{ca}}+\frac{1}{r_{ab}}\right)\ge\frac{1}{4}(a+b+c)^2\sqrt[3]{\frac{1}{r_{bc}r_{ca}r_{ab}}}=\frac{1}{4}(a+b+c)^2\sqrt[3]{\frac{p^2}{16R^2r^3}}=$$

$$=\frac{1}{4}(a+b+c)^2\cdot\sqrt[3]{\frac{p^2}{16r}\frac{1}{(Rr)^2}}=\frac{1}{4}(a+b+c)^2\sqrt[3]{\frac{1}{r}\cdot\frac{p^4}{(abc)^2}}\ge\frac{1}{4}(a+b+c)^2\sqrt[3]{\frac{3\sqrt{3}}{p}\cdot\frac{p^4}{(abc)^2}}=$$

$$=\frac{\sqrt{3}}{4}\cdot p\cdot(a+b+c)^2\cdot\sqrt[3]{\frac{1}{(abc)^2}}=\frac{\sqrt{3}}{8}(a+b+c)^3\frac{1}{\sqrt[3]{(abc)^2}}\ge\frac{\sqrt{3}}{8}(3\sqrt[3]{abc})^3\frac{1}{\sqrt[3]{(abc)^2}}=\frac{27\sqrt{3}}{8}\sqrt[3]{abc}$$

The equality holds for the equilateral triangle ABC. The problem is solved.

(Moldova)

G2. Easy-medium

A non-iscosceles acute triangle *ABC* is given with $AC>BC$ and H the point of intersection of the heights *AZ* and *CM*. We call point *P* on *AB* such that *AM*=*PM* and *N* the midpoint of *AC*. If O the circumcentre of the triangle *ABC* and $K\equiv PH\cap BC$, $X\equiv ON\cap MK$, $T\equiv OM\cap AC$, prove that the points *M*, *N*, *T*, *X* are lie on the same circumference.

Solution. It is enough to show that $OM\perp MK$.

Let $OE\perp AB$, then it is trivial that :

$CH=2OE$. (1)

Since from the hypothesis we have $PM=AM$ then we take $PB=PM-BM$ or

$PM=AM-BM$ (2)

Also, $\angle KPB=\angle HAP$ and $\angle HAP=\angle HCK$ since *AMZC* in inscribable, so $\angle KPB=\angle HCK$ and since $\angle BKP=\angle HKC$, the triangles *KHC* and *KBP* are similar.

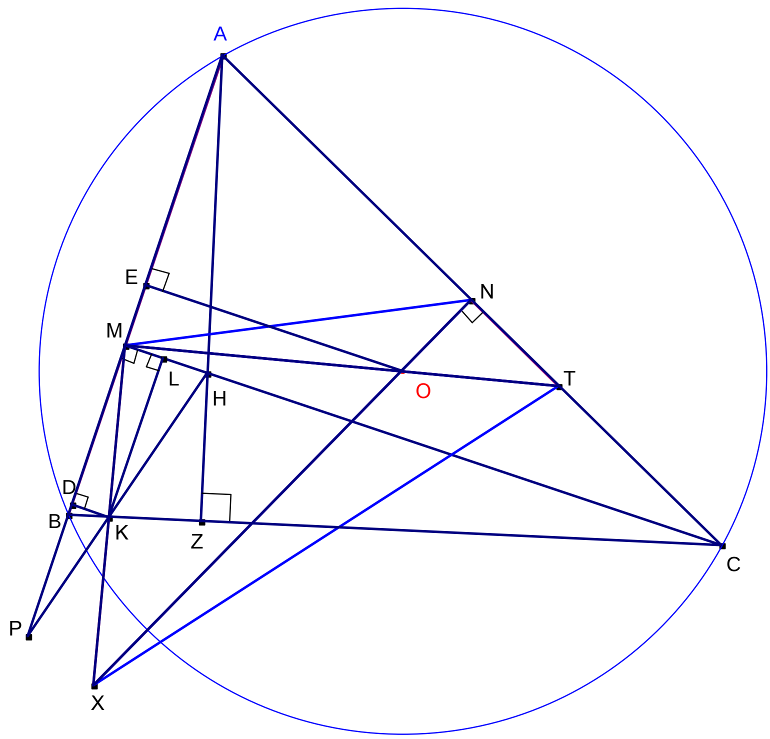
If *KL* and *KD* are respectively the heights of the triangles *KHC* and *KBP* we have: $\frac{KD}{KL}=\frac{PB}{CH}$, and from (1) and (2) we get:

$$\frac{KD}{KL}=\frac{AM-BM}{2OE}=\frac{ME}{OE}\Rightarrow\frac{KD}{MD}=\frac{ME}{OE}$$

Therefore the triangles *KMD*, *OEM* are similar and we get:

$$\angle OMK=\angle OMC+\angle LMK=\angle MOE+\angle MKD=\angle KMD+\angle MKD=90^\circ,$$

so $OM\perp MK$.



(Cyprus)

G3. **Easy**

We draw two lines (ℓ_1), (ℓ_2) through the orthocenter H of the triangle ABC such that each one is dividing the triangle into two figures of equal area and equal perimeters. Find the angles of the triangle.

Solution. Lemma: If a line divides a triangle into two equal area figures with equal perimeters then this line passes through the incentre I of the triangle.

Proof of Lemma. Let in triangle ABC the line (ℓ) intersects the sides AB, AC at the points D, E respectively. Then $area(ADE) = area(BDEC)$ and

$$AD + DE + EA = BD + DE + EC + CB \Rightarrow AD + EA = BD + EC + CB \qquad (1)$$

We observe that if r the radius of the inscribed circle of the triangle ABC, then

$$area(ADIE) = \frac{1}{2}(AD + EA)r \qquad (2)$$

and

$$area(DBCEI) = \frac{1}{2}(DB + BC + CE)r\ . \qquad (3)$$

From $(1),(2),(3)$ we obtain,

$$area(ADIE) = area(DBCEI) \Rightarrow area(ADE) + area(DIE) = area(DBCE) - area(DIE),$$

through which $area(DIE) = 0$, so the line (ℓ) passes through the incentre I of the triangle.

According to the lemma and through the data of the problem, the lines (ℓ_1), (ℓ_2) pass through the incentre I and the orthocenter of the H of the triangle ABC.

Therefore the triangle is equilateral.

(Cyprus)

G4. Medium

A triangle ABC is given with barycentre G and circumcentre O. The perpendicular bisectors of GA, GB meet at C_1, of GB, GC meet at A_1 and GC, GA meet at B_1. Prove that O is the barycenter of the triangle $A_1B_1C_1$.

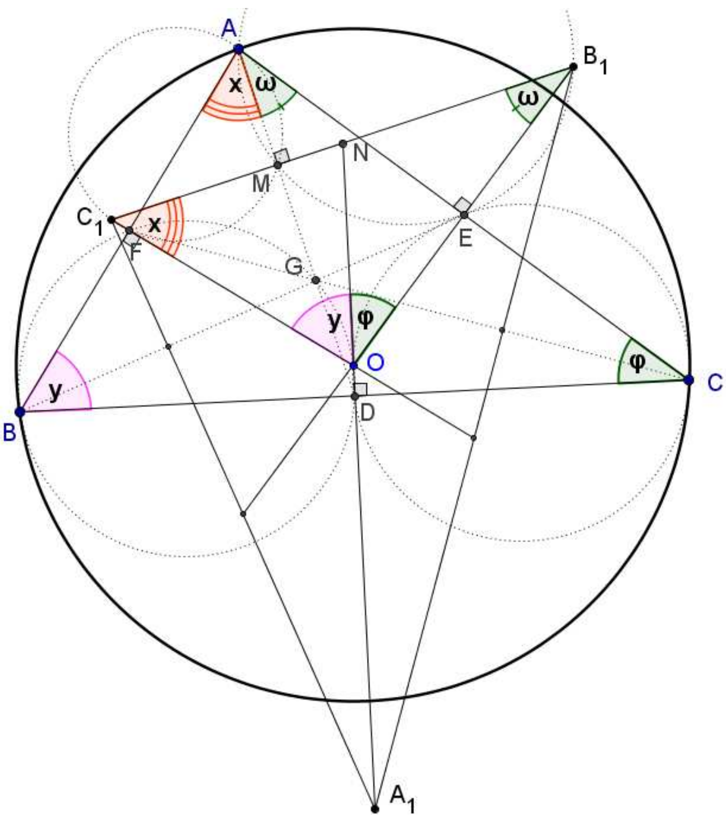
Solution. Let D,E,F be the midpoints of the sides BC,AC,AB of the triangle ABC, respectively. Let, also B_1C_1, A_1C_1 and A_1B_1 the perpendicular bisectors of the line segments GA, GB and GC.

Then the points A_1, B_1 and C_1 are the circumcenters of the triangles GBC, GAC and GAB, respectively. (They are the points of intersection of the perpendicular bisectors of their sides).

Therefore A_1D, B_1E and C_1F are the perpendicular bisectors of the sides BC, AC and AB, respectively, and hence they are passing through the circumcenter O of the triangle ABC.

We shall prove that A_1D, B_1E and C_1F are the medians of the triangle $A_1B_1C_1$.

Let the extension of A_1D intersects B_1C_1 at the point N. We shall prove that N is the midpoint of the segment B_1C_1.



From the cyclic quadrilateral $AMEB_1$ ($\measuredangle M = \measuredangle E = 90^\circ$), we get.

$$\measuredangle MAE = \measuredangle MB_1E = \measuredangle \omega \tag{1}$$

From the cyclic quadrilateral $DOEC$ ($\measuredangle D = \measuredangle E = 90^\circ$), we get

$$\measuredangle ECD = \measuredangle EON = \measuredangle \varphi . \tag{2}$$

From (1) and (2) we conclude that the triangles ADC and B_1NO are similar and therefore

$$\frac{NB_1}{NO} = \frac{AD}{CD}. \tag{3}$$

From the cyclic quadrilateral $AMFC_1$ ($\measuredangle M = \measuredangle F = 90^\circ$), we get

$$\measuredangle MAF = \measuredangle MC_1F = \measuredangle x . \tag{4}$$

From the cyclic quadrilateral $DOFB$ ($\measuredangle D = \measuredangle F = 90^\circ$), we get

$$\measuredangle FBD = \measuredangle FON = \measuredangle y . \tag{5}$$

From (4) and (5) we conclude that the triangles ADB and C_1NO are similar, and so:

$$\frac{NC_1}{NO} = \frac{AD}{BD}. \tag{6}$$

From the relations (3) and (6) we find:

$$NB_1 = NC_1 .$$

Similarly, we prove that B_1E, C_1F are medians of the triangle $A_1B_1C_1$.

(Greece)

G5. Medium

The circle k_a touches the extensions of sides AB and BC, as well as the circumscribed circle of the triangle ABC (from the outside). We denote the intersection of k_a with the circumscribed circle of the triangle ABC by A'. Analogously, we define points B' and C'. Prove that the lines AA', BB' and CC' intersect in one point.

Solution. Let R and r be the radii of the circumscribed and inscribed circle of ΔABC, respectively, let r_a, r_b, r_c be the radii of the escribed circles of ΔABC touching BC, CA, AB, respectively, and let ρ_a, ρ_b, ρ_c be the radii of circles k_a, k_b, k_c, respectively.

Let $\measuredangle BAA' = \alpha_1$, $\measuredangle CAA' = \alpha_2$ and let O be the center of the circumscribed circle of ΔABC. Let O_a be the center of the circle k_a and let k_a touch the extensions of AB and AC in D and E, respectively. We have $\rho_a = \frac{r_a}{\cos^2 \frac{\alpha}{2}}$. The points O_a, A' and O are colinear. We have $\measuredangle BOO_a = \measuredangle BOA' = 2\alpha_1$, as they are the inscribed angle and the central angle of the same arc. As $BO = R$ and $OO_a = R + \rho_a$, applying law of cosines on ΔBOO_a we get

$$\begin{aligned} BO_a^2 &= R^2 + (R+\rho_a)^2 - 2R(R+\rho_a)\cos 2\alpha_1 = 2R^2 + \rho_a^2 + 2R\rho_a - 2R(R+\rho_a)(1-2\sin^2\alpha_1) = \\ &= \rho_a^2 + 4R(R+\rho_a)\sin^2\alpha_1 \end{aligned}.$$

Looking at ΔADO_a, we obtain $AD = \rho_a \operatorname{ctg}\frac{\alpha}{2}$, so

$$\rho_a = \frac{r_a}{\cos^2\frac{\alpha}{2}} = \frac{rs}{s-a}\frac{1}{\cos^2\frac{\alpha}{2}} .$$

This implies

$$\begin{aligned} AD &= \frac{rs}{(s-a)\cos^2\frac{\alpha}{2}}\operatorname{ctg}\frac{\alpha}{2} = \frac{rs}{s-a}\frac{1}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{2rs}{s-a}\frac{1}{\sin\alpha} = \\ &= \frac{4\mathrm{R}\,rs}{(s-a)a} = \frac{abc}{(s-a)a} = \frac{bc}{s-a} . \end{aligned}$$

Hence, $BD = AD - AB = \frac{bc}{s-a} - c = \frac{c(s-c)}{s-a}$.

Applying Pythagorean theorem on ΔBDO_a we obtain

$$BO^2 = BD^2 + DO_a^2 = \rho_a^2 + \frac{c^2(s-c)^2}{(s-a)^2}.$$

Thus,

$$\sin^2\alpha_1 = \frac{c^2(s-c)^2}{(s-a)^2 4R(R+\rho_a)}.$$

Analogously, we get

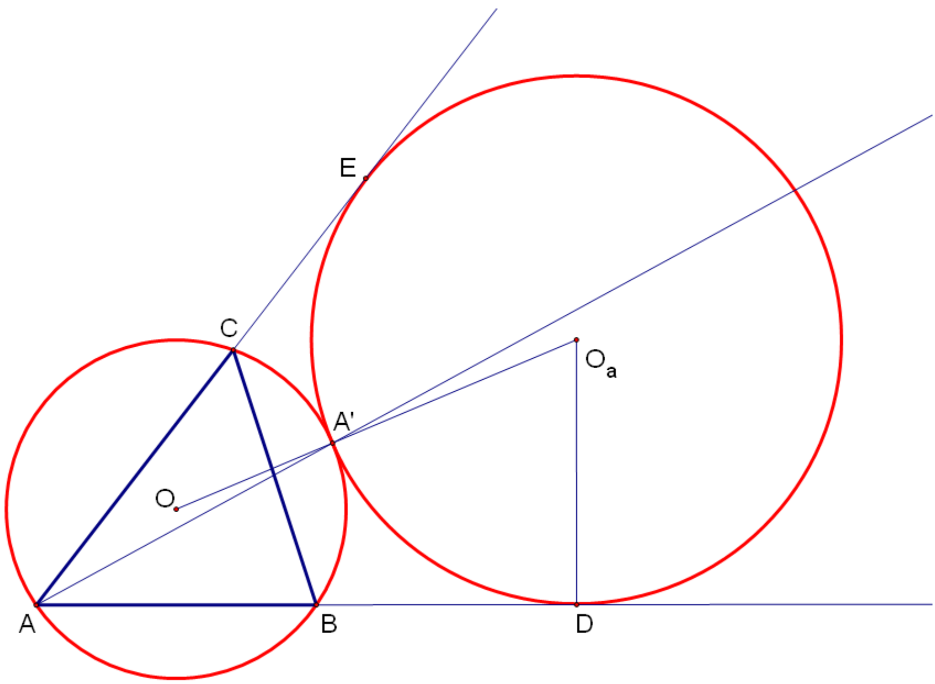
$$\sin^2\alpha_2 = \frac{b^2(s-b)^2}{(s-a)^2 4R(R+\rho_a)},$$

so $\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c(s-c)}{b(s-b)}$.

Similarly, we get $\frac{\sin\beta_1}{\sin\beta_2} = \frac{a(s-a)}{c(s-c)}$ and $\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{b(s-b)}{a(s-a)}$. Multiplying those three equalities, we obtain

$$\frac{\sin\alpha_1}{\sin\alpha_2}\frac{\sin\beta_1}{\sin\beta_2}\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{c(s-c)}{b(s-b)}\frac{a(s-a)}{c(s-c)}\frac{b(s-b)}{a(s-a)} = 1,$$

and the statement of the problem follows from Ceva's Theorem.



(Srbija)

G6. **Medium**

On triangle ABC the AM ($M \in BC$) is mediane and BB_1 and CC_1 ($B_1 \in AC, C_1 \in AB$) are altitudes. The stright line *d* is perpendicular to *AM* at the point *A* and intersect the lines BB_1 and CC_1 at the points *E* and *F* respectively. Let denoted with ω the circle passing through the points *E, M* and *F* and with ω_1 and with ω_2 the circles that are tangent to segment *EF* and with ω at the arc *EF* which is not contain the point *M*. If the points *P* and *Q* are intersections points for ω_1 and ω_2 then prove that the points *P, Q* and *M* are collinear.

Solution. Let *K* be the midpoint of BB_1 and *L* the midpoint of CC_1. It is clear that quadrilateral $EAKM$ is cyclic and that $\angle AME = \angle AKE$. In a similar way we can show that $\angle AMF = \angle ALF$. Since the triangles ABB_1 and ACC_1 are similar, it follows that $\angle AKE = \angle ALF$ and that *A* is the midpoint of *EF*. Now it is clear that the center *O* of ω lies in the line *AM* and that *M* is the midpoint of arc *EF*. From the generalized Ptolemy theorem in quadrilateral $MF\omega_1 E$, if we denote with *S* the tangent point of ω_1 with *EF* and with *T* the point of tangence from *M* to ω_1, we have $MF \cdot ES + MF \cdot FS = MT \cdot EF$ and consequentely that *MT* = *MF*. Now it is clear that if we denote with O_1 and O_2 centers of ω_1 and ω_2, respectively we have

$$MO_1^2 - O_1P^2 = MO_2^2 - O_2P^2$$

and the result is clear. **(Albania)**

G7. **Medium**

In the non-isosceles triangle ABC consider the points X on $[AB]$ and Y on $[AC]$ such that $[BX]=[CY]$. M and N are the midpoints of the segments $[BC]$, respectively $[XY]$, and the straight lines XY and BC meet in K. Prove that the circumcircle of triangle KMN contains a point, different from M, which is independent of the position of the points X and Y.

Solution. Let L be the midpoint of the arc $\overset{\frown}{BAC}$ belonging to the circumcircle of ABC. We shall prove that L is the fixed point we are looking for.

As $[BL]=[CL]$, $[BX]=[CY]$, and $\measuredangle ABL = \measuredangle ACL$ we have $\Delta LBX = \Delta LCY$. As a consequence, $\measuredangle AXL = \measuredangle AYL$ and $[XL]=[YL]$. Thus L is the midpoint of the arc $\overset{\frown}{XAY}$ of the circumscribed circle of XAY. This implies $\measuredangle LNK = \measuredangle LMK = 90^\circ$, which means that the point L belongs to the circumcircle of triangle KMN.

(Romania)

G8. **Easy**

Let P be a point in the interior of a triangle ABC and let d_a, d_b, d_c be its distances to BC, CA, AB respectively. Prove that

$$\max(AP, BP, CP) \ge \sqrt{d_a^2 + d_b^2 + d_c^2}\ .$$

Solution. Let $a = AP$, $b = BP$, $c = CP$ and denote

$x_1 = m(\measuredangle PAB)$, $x_2 = m(\measuredangle PAC)$, $y_1 = m(\measuredangle PBC)$,

$y_2 = m(\measuredangle PBA)$, $z_1 = m(\measuredangle PCA)$, $z_2 = m(\measuredangle PCB)$.

Because $(x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = \pi$, then WLOG assume that $x_1 + y_1 + z_1 \le \frac{\pi}{2}$.

Suppose that $\max(AP, BP, CP) = a$, and $a^2 < d_a^2 + d_b^2 + d_c^2$. Then

$$\sin^2 x_1 = \frac{d_c^2}{a^2} > \frac{d_c^2}{d_a^2 + d_b^2 + d_c^2},$$

$$\sin^2 y_1 = \frac{d_a^2}{b^2} \ge \frac{d_a^2}{a^2} > \frac{d_a^2}{d_a^2 + d_b^2 + d_c^2},$$

$$\sin^2 z_1 = \frac{d_b^2}{c^2} \ge \frac{d_b^2}{a^2} > \frac{d_b^2}{d_a^2 + d_b^2 + d_c^2}.$$

By ssuming these relations we obtain $\sin^2 x_1 + \sin^2 y_1 + \sin^2 z_1 > 1$. But this is false, because the following result holds: if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, $x + y < \frac{\pi}{2}$, then $\sin^2 x + \sin^2 y \le \sin^2(x + y)$, which is true because the triangle with angles $x, y, \pi - x - y$ is obtuse-angled or right-angled. Therefore

$$\sin^2 x_1 + \sin^2 y_1 + \sin^2 z_1 \le \sin^2(x_1 + y_1) + \sin^2 z_1 \le \sin^2(x_1 + y_1 + z_1) \le 1,$$

a contradiction. So, $a^2 \ge d_a^2 + d_b^2 + d_c^2$. The problem is solved.

(Moldova)

25th Balkan Mathematical Olympiad

Ohrid, 6th May, 2008

Problems

1. An acute-angled scalene triangle *ABC* is given, with $AC > BC$. Let *O* be its circumcentre, *H* its orthocentre, and *F* the foot of the altitude from *C*. Let *P* be the point (other than *A*) on the line *AB* such that *AF=PF*, and *M* be the midpoint of *AC*. We denote the intersection of *PH* and *BC* by *X*, the intersection of *OM* and *FX* by *Y*, and the intersection of *OF* and *AC* by *Z*. Prove that the points *F*, *M*, *Y* and *Z* are concyclic.

2. Does there exist a sequence $a_1, a_2, ..., a_n, ...$ of positive real numbers satisfying both of the following conditions:

(i) $\sum_{i=1}^{n} a_i \le n^2$, for every positive integer *n*;

(ii) $\sum_{i=1}^{n} \frac{1}{a_i} \le 2008$, for every positive integer n?

3. Let n be a positive integer. The rectangle *ABCD* with side lengths $AB = 90n + 1$ and $BC = 90n + 5$ is partitioned into unit squares with sides parallel to the sides of *ABCD*. Let *S* be the set of all points which are vertices of these unit squares. Prove that the number of lines which pass through at least two points from *S* is divisible by 4.

4. Let c be a positive integer. The sequence $a_1, a_2, ..., a_n, ...$ is defined by $a_1 = c$, and $a_{n+1} = a_n^2 + a_n + c^3$, for every positive integer n. Find all values of c for which there exist some integers $k \ge 1$ and $m \ge 2$, such that $a_k^2 + c^3$ is the m^{th} power of some positive integer.

Time allowed: 4.5 hours.
Each problem is worth 10 points.

National teams participating in BMO 2008

country	part.	name
ALBANIA	leader	Edmond Pisha
	deputy leader	Fatos Kopliku
	contestant	Redi Haderi
	contestant	Beniada Shabani
	contestant	Andi Reçi
	contestant	Manushaqe Muço
	contestant	Elona Hasa
	contestant	Erion Dervishi

country	part.	name
BOSNIA & HERZEGOVINA	leader	Vidan Govedarica
	deputy leader	Amer Krivošija
	contestant	Admir Beširević
	contestant	Vedran Karahodžić
	contestant	Salem Malikić
	contestant	Jelena Radović
	contestant	Franjo Šarčević
	contestant	Vlado Uljarević

country	part.	name
BULGARIA	leader	Nikolai Nikolov
	deputy leader	Peter Boyvalenkov
	observer	Oleg Mushkarov
	contestant	Nikolay Beluhov
	contestant	Lyuboslav Panchev
	contestant	Svetozar Stankov
	contestant	Aleksander Daskalov
	contestant	Evgeni Dimitrov
	contestant	Galin Statev

country	part.	Name
CYPRUS	leader	Andreas Philippou
	deputy leader	Theoklitos Paragyiou
	contestant	Anastos Michael
	contestant	Anastassiades Christos
	contestant	Assiotis Theodoros
	contestant	Demetriou Charis
	contestant	Makris Christos
	contestant	Katsamba Panagiota

country	part.	Name
GREECE	leader	Anargyros Felouris
	deputy leader	Evangelos Zotos
	contestant	Silouanos Brazitikos
	contestant	Ilias Giechaskiel
	contestant	Alkistis Mavroeidi
	contestant	Dimitrios Papadimitriou
	contestant	Nikolaos Rapanos
	contestant	Anastasios Vafeidis

country	part.	name
MACEDONIA 1	leader	Petar Sokoloski
	deputy leader	Ljupco Nastovski
	contestant	Bodan Arsovski
	contestant	Bojan Joveski
	contestant	Dimitar Trenevski
	contestant	Stefan Lozanovski
	contestant	Matej Dobrevski
	contestant	Kujtim Rahmani
MACEDONIA 2	contestant	Predrag Gruevski
	contestant	Zlatko Joveski
	contestant	Filip Talimdzioski
	contestant	Petar Filev
	contestant	Andrej Risteski
	contestant	Darko Domazetoski

country	part.	name
MOLDOVA	leader	Teleucă Marcel
	deputy leader	Bairac Radu
	contestant	Frimu Andrei
	contestant	Gramaţki Iulian
	contestant	Grecu Mircea
	contestant	Iliaşenco Andrei
	contestant	Sanduleanu Ştefan
	contestant	Zubarev Alexei

country	part.	name
MONTENEGRO	leader	Romeo Meštrović
	deputy leader	Velibor Bojković
	contestant	Marica Knežević
	contestant	Nikola Milinković
	contestant	Radovan Krtolica
	contestant	Bećo Merulić
	contestant	Tanja Ivošević
	contestant	Rastko Pajković

country	part.	name
ROMANIA	leader	Mihai Bălună
	deputy leader	Mariean Andronache
	observer	Dan Schwarz
	observer	Cristian Alexandrescu
	contestant	Mihail Eugen Dumitrescu
	contestant	Daniel Tiberiu Rimovecz
	contestant	Victor Pădureanu
	contestant	Mădălina Elena Persu
	contestant	Edgar Dobriban
	contestant	Eugenia Cristina Roşu

country	part.	name
SERBIA	leader	Miloš Stojaković
	deputy leader	Miloš Milosavljević
	contestant	Dušan Milijančević
	contestant	Luka Milićević
	contestant	Aleksandar Vasiljković
	contestant	Teodor fon Burg
	contestant	Vladimir Nikolić
	contestant	Marija Jelić

country	part.	name
TURKEY	leader	Ali Doğanaksoy
	deputy leader	Fatih Sulak
	contestant	Ömer Faruk Tekin
	contestant	Melih Üçer
	contestant	Alper İnecik
	contestant	Fehmi Emre Kadan
	contestant	Umut Varolgüneş
	contestant	Semih Yavuz

country	part.	name
AZERBAIJAN	leader	Fuad Garayev
	contestant	Sarkhan Badirli
	contestant	Ruslan Muslumov
	contestant	Farid Mammadov
	contestant	Eldar Babayev

country	part.	name
FRANCE	leader	Claude Deschamps
	contestant	Martin Clochard
	contestant	Juliette Fournier
	contestant	Ambroise Marigot
	contestant	Jean-François Martin
	contestant	Sergio Véga

country	part.	name
ITALY	leader	Massimo Gobbino
	deputy leader	Francesco Morandin
	observer	Ludovico Pernazza
	contestant	Andrea Fogari
	contestant	Mattia Francesko Galeotti
	contestant	Kirill Kuzmin
	contestant	Giovanni Paolini
	contestant	Leonardo Patimo
	contestant	Pietro Vertechi

country	part.	name
KAZAKHSTAN	leader	Assan Zholdassov
	deputy leader	Iskakova Aliya
	contestant	Yegor Klochkov
	contestant	Asset Daliyev
	contestant	Tussupbekov Yerken
	contestant	Nursultan Khajimuratov
	contestant	Sanzhar Orazbayev
	contestant	Yeskendir Kassenov

country	part.	name
TAJIKISTAN	leader	Erdal Eravcı
	contestant	Igor Korobeynikov
	contestant	Inomzhon Mirzaev

country	part.	name
TURKMENISTAN	leader	Erol Aslan
	contestant	Nazar Emirov
	contestant	Azat Meredov
	contestant	Merdan Artykov

country	part.	name
UNITED KINGDOM & IRELAND	leader	Adrian Sanders
	deputy leader	Jacqui Lewis
	contestant	Galin Ganchev
	contestant	Andrew Hyer
	contestant	Peter Leach
	contestant	Craig Newbold
	contestant	Hannah Roberts
	contestant	Rong Zhou

Complete results of BMO 2008

		contestant	problem 1	problem 2	problem 3	problem 4	total	medal
1	ROM6	Eugenia Cristina Roşu	10	10	9	10	39	gold
2	BUL1	Nikolay Beluhov	10	10	9	7	36	gold
3	ITA1	Andrea Fogari	10	10	9	5	34	gold
4	TUR2	Melih Üçer	10	10	3	10	33	gold
5	BUL3	Svetozar Stankov	6	10	5	10	31	gold
6	ROM3	Victor Pădureanu	10	10	10	0	30	gold
7	TUR1	Ömer Faruk Tekin	10	8	2	10	30	gold
8	BUL2	Lyuboslav Panchev	10	9	9	1	29	gold
9	MDA4	Iliaşenco Andrei	10	9	9	1	29	gold
10	MDA1	Frimu Andrei	10	2	9	7	28	silver
11	BUL4	Aleksander Daskalov	10	2	5	10	27	silver
12	SRB2	Luka Milićević	10	3	4	10	27	silver
13	SRB6	Marija Jelić	10	9	7	1	27	silver
14	ROM2	Daniel Tiberiu Rimovecz	10	10	3	3	26	silver
15	ROM4	Mădălina Elena Persu	10	1	8	6	25	silver
16	TUR5	Umut Varolgüneş	10	1	8	4	23	silver
17	ITA6	Pietro Vertechi	10	1	10	2	23	silver
18	BUL5	Evgeni Dimitrov	10	2	5	5	22	silver
19	KAZ1	Yegor Klochkov	10	2	10	0	22	silver
20	SRB4	Teodor fon Burg	10	0	8	3	21	silver
21	BUL6	Galin Statev	10	2	3	5	20	silver
22	KAZ5	Sanzhar Orazbayev	10	0	3	7	20	silver
23	KAZ6	Yeskendir Kassenov	10	0	8	2	20	silver
24	ALB1	Redi Haderi	10	9	0	0	19	silver
25	ITA3	Kirill Kuzmin	7	1	10	1	19	silver
26	UNK&IRL1	Galin Ganchev	10	0	2	7	19	silver
27	SRB1	Dušan Milijančević	10	0	7	1	18	silver
28	FRA5	Jean-François Martin	6	5	6	1	18	silver
29	GRE6	Anastasios Vafeidis	10	4	1	2	17	silver
30	MKD1A	Bodan Arsovski	10	3	4	0	17	silver
31	MNG1	Marica Knežević	8	2	3	4	17	silver
32	ROM1	Mihail Eugen Dumitrescu	5	3	9	0	17	silver
33	TUR6	Semih Yavuz	10	0	2	5	17	silver
34	ITA2	Mattia Francesko Galeotti	8	8	1	0	17	silver
35	KAZ2	Asset Daliyev	10	3	3	1	17	silver
36	BIH3	Salem Malikić	10	3	2	1	16	bronze
37	MDA5	Sanduleanu Ştefan	6	9	1	0	16	bronze
38	TUR3	Alper İnecik	10	0	3	3	16	bronze
39	ITA4	Giovanni Paolini	7	0	9	0	16	bronze
40	UNK&IRL2	Andrew Hyer	1	9	6	0	16	bronze
41	SRB5	Vladimir Nikolić	9	0	6	0	15	bronze

42	**AZE2**	**Ruslan Muslumov**	**4**	**1**	**0**	**10**	**15**	**bronze**
43	**FRA1**	**Martin Clochard**	**0**	**10**	**5**	**0**	**15**	**bronze**
44	**ROM5**	**Edgar Dobriban**	**8**	**0**	**6**	**0**	**14**	**bronze**
45	**SRB3**	**Aleksandar Vasiljković**	**10**	**0**	**2**	**2**	**14**	**bronze**
46	**KAZ3**	**Tussupbekov Yerken**	**10**	**3**	**1**	**0**	**14**	**bronze**
47	**UNK&IRL5**	**Hannah Roberts**	**10**	**4**	**0**	**0**	**14**	**bronze**
48	**TJK2**	**Inomzhon Mirzaev**	**10**	**0**	**0**	**3**	**13**	**bronze**
49	**ALB2**	**Beniada Shabani**	**10**	**0**	**2**	**0**	**12**	**bronze**
50	**TUR4**	**Fehmi Emre Kadan**	**10**	**0**	**0**	**2**	**12**	**bronze**
51	**AZE3**	**Farid Mammadov**	**10**	**1**	**1**	**0**	**12**	**bronze**
52	**KAZ4**	**Nursultan Khajimuratov**	**10**	**2**	**0**	**0**	**12**	**bronze**
53	**UNK&IRL3**	**Peter Leach**	**0**	**4**	**2**	**6**	**12**	**bronze**
54	**GRE4**	**Dimitrios Papadimitriou**	**10**	**1**	**0**	**0**	**11**	**bronze**
55	**ITA5**	**Leonardo Patimo**	**0**	**10**	**1**	**0**	**11**	**bronze**
56	**UNK&IRL4**	**Craig Newbold**	**10**	**0**	**1**	**0**	**11**	**bronze**
57	**GRE1**	**Silouanos Brazitikos**	**8**	**1**	**0**	**1**	**10**	**bronze**
58	**GRE2**	**Ilias Giechaskiel**	**1**	**6**	**2**	**1**	**10**	**bronze**
59	**MKD3B**	**Filip Talimdzioski**	**2**	**3**	**3**	**2**	**10**	**bronze**
60	**MDA3**	**Grecu Mircea**	**10**	**0**	**0**	**0**	**10**	**bronze**
61	**GRE5**	**Nikolaos Rapanos**	**9**	**0**	**0**	**0**	**9**	**bronze**
62	**MKD1B**	**Predrag Gruevski**	**2**	**2**	**5**	**0**	**9**	**bronze**
63	**CYP2**	**Anastassiades Christos**	**2**	**2**	**3**	**1**	**8**	**bronze**
63	**MDA6**	**Zubarev Alexei**	**0**	**2**	**6**	**0**	**8**	**bronze**
65	**MKD4A**	**Stefan Lozanovski**	**1**	**0**	**4**	**2**	**7**	**bronze**
66	**MDA2**	**Gramaţki Iulian**	**0**	**0**	**6**	**0**	**6**	**bronze**
67	**ALB3**	**Andi Reçi**	**1**	**1**	**2**	**1**	**5**	**bronze**
68	**BIH4**	**Jelena Radović**	**0**	**0**	**5**	**0**	**5**	**bronze**
69	**MKD3A**	**Dimitar Trenevski**	**4**	**0**	**1**	**0**	**5**	**bronze**
70	**BIH6**	**Vlado Uljarević**	**3**	**1**	**0**	**0**	**4**	
71	**CYP5**	**Makris Christos**	**1**	**0**	**2**	**1**	**4**	
72	**MKD6A**	**Kujtim Rahmani**	**3**	**0**	**0**	**1**	**4**	
73	**MKD4B**	**Petar Filev**	**0**	**1**	**1**	**2**	**4**	
74	**FRA3**	**Juliette Fournier**	**1**	**0**	**3**	**0**	**4**	
75	**BIH1**	**Admir Beširević**	**1**	**0**	**1**	**1**	**3**	
76	**BIH2**	**Vedran Karahodžić**	**1**	**1**	**0**	**1**	**3**	
77	**CYP1**	**Anastos Michael**	**0**	**0**	**3**	**0**	**3**	
78	**MKD2A**	**Bojan Joveski**	**1**	**0**	**2**	**0**	**3**	
79	**MNG2**	**Nikola Milinković**	**2**	**1**	**0**	**0**	**3**	
80	**CYP3**	**Assiotis Theodoros**	**0**	**0**	**2**	**0**	**2**	
81	**GRE3**	**Alkistis Mavroeidi**	**1**	**0**	**0**	**1**	**2**	
82	**MKD5A**	**Matej Dobrevski**	**0**	**0**	**2**	**0**	**2**	
83	**MKD2B**	**Zlatko Joveski**	**1**	**0**	**1**	**0**	**2**	
84	**TKM2**	**Azat Meredov**	**0**	**0**	**0**	**2**	**2**	
85	**TKM3**	**Merdan Artykov**	**1**	**0**	**0**	**1**	**2**	
86	**ALB4**	**Manushaqe Muço**	**1**	**0**	**0**	**0**	**1**	
87	**BIH5**	**Franjo Šarčević**	**0**	**0**	**1**	**0**	**1**	
88	**CYP6**	**Katsamba Panagiota**	**1**	**0**	**0**	**0**	**1**	

89	**MKD5B**	**Andrej Risteski**	**0**	**0**	**1**	**0**	**1**	
90	**MKD6B**	**Darko Domazetoski**	**0**	**0**	**0**	**1**	**1**	
91	**MNG3**	**Radovan Krtolica**	**0**	**0**	**1**	**0**	**1**	
92	**FRA6**	**Sergio Véga**	**0**	**0**	**1**	**0**	**1**	
93	**TJK1**	**Igor Korobeynikov**	**0**	**1**	**0**	**0**	**1**	
94	**UNK&IRL6**	**Rong Zhou**	**0**	**1**	**0**	**0**	**1**	
95	**ALB5**	**Elona Hasa**	**0**	**0**	**0**	**0**	**0**	
96	**ALB6**	**Erion Dervishi**	**0**	**0**	**0**	**0**	**0**	
97	**CYP4**	**Demetriou Charis**	**0**	**0**	**0**	**0**	**0**	
98	**MNG4**	**Bećo Merulić**	**0**	**0**	**0**	**0**	**0**	
99	**MNG5**	**Tanja Ivošević**	**0**	**0**	**0**	**0**	**0**	
100	**MNG6**	**Rastko Pajković**	**0**	**0**	**0**	**0**	**0**	
101	**AZE1**	**Sarkhan Badirli**	**0**	**0**	**0**	**0**	**0**	
102	**AZE4**	**Eldar Babayev**	**0**	**0**	**0**	**0**	**0**	
103	**FRA4**	**Ambroise Marigot**	**0**	**0**	**0**	**0**	**0**	
104	**TKM1**	**Nazar Emirov**	**0**	**0**	**0**	**0**	**0**	