The 15^{th} Romanian Master of Mathematics Competition

Day 1: Wednesday, February 28^{th}, 2024, Bucharest

Language: English

Problem 1. Let n be a positive integer. Initially, a bishop is placed in each square of the top row of a $2^n \times 2^n$ chessboard; those bishops are numbered from 1 to 2^n, from left to right. A *jump* is a simultaneous move made by all bishops such that the following conditions are satisfied:

- each bishop moves diagonally, in a straight line, some number of squares, and
- at the end of the jump, the bishops all stand in different squares of the same row.

Find the total number of permutations σ of the numbers $1, 2, \ldots, 2^n$ with the following property: There exists a sequence of jumps such that all bishops end up on the bottom row arranged in the order $\sigma(1), \sigma(2), \ldots, \sigma(2^n)$, from left to right.

Problem 2. Consider an odd prime p and a positive integer $N < 50p$. Let $a_1, a_2, \ldots, a_N$ be a list of positive integers less than p such that any specific value occurs at most $\frac{51}{100}N$ times and $a_1 + a_2 + \cdots + a_N$ is not divisible by p. Prove that there exists a permutation $b_1, b_2, \ldots, b_N$ of the a_i such that, for all $k = 1, 2, \ldots, N$, the sum $b_1 + b_2 + \cdots + b_k$ is not divisible by p.

Problem 3. Given a positive integer n, a set $\mathcal{S}$ is *n-admissible* if

- each element of $\mathcal{S}$ is an unordered triple of integers in $\{1, 2, \ldots, n\}$,
- $|\mathcal{S}| = n - 2$, and
- for each $1 \leq k \leq n - 2$ and each choice of k distinct $A_1, A_2, \ldots, A_k \in \mathcal{S}$,

$$|A_1 \cup A_2 \cup \cdots \cup A_k| \geq k + 2.$$

Is it true that, for all $n > 3$ and for each n-admissible set $\mathcal{S}$, there exist pairwise distinct points $P_1, \ldots, P_n$ in the plane such that the angles of the triangle $P_i P_j P_k$ are all less than $61°$ for any triple $\{i, j, k\}$ in $\mathcal{S}$?

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

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Day 2: Thursday, February 29$^{\text{th}}$, 2024, Bucharest

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Problem 4. Fix integers a and b greater than 1. For any positive integer n, let r_n be the (non-negative) remainder that b^n leaves upon division by a^n. Assume there exists a positive integer N such that $r_n < 2^n/n$ for all integers $n \geq N$. Prove that a divides b.

Problem 5. Let BC be a fixed segment in the plane, and let A be a variable point in the plane not on the line BC. Distinct points X and Y are chosen on the rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of ABC at B and C meet line XY at P and Q, respectively, such that the points X, P, Y, and Q are pairwise distinct and lie on the same side of BC. Let Ω_1 be the circle through X and P centred on BC. Similarly, let Ω_2 be the circle through Y and Q centred on BC. Prove that Ω_1 and Ω_2 intersect at two fixed points as A varies.

Problem 6. A polynomial P with integer coefficients is *square-free* if it is not expressible in the form $P = Q^2R$, where Q and R are polynomials with integer coefficients and Q is not constant. For a positive integer n, let $\mathcal{P}_n$ be the set of polynomials of the form

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer N so that, for all integers $n \geq N$, more than 99% of the polynomials in $\mathcal{P}_n$ are square-free.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.