APMO 2025 – Problems and Solutions

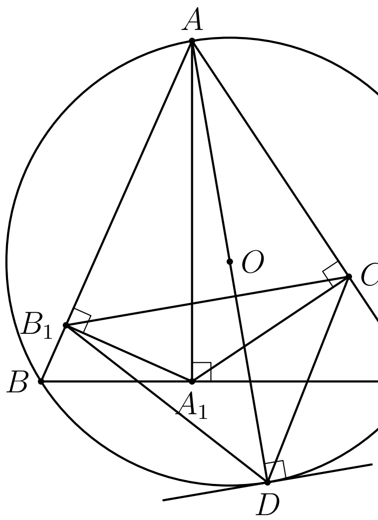
Problem 1

Let ABC be an acute triangle inscribed in a circle Γ. Let A_1 be the orthogonal projection of A onto BC so that AA_1 is an altitude. Let B_1 and C_1 be the orthogonal projections of A_1 onto AB and AC, respectively. Point P is such that quadrilateral AB_1PC_1 is convex and has the same area as triangle ABC. Is it possible that P strictly lies in the interior of circle Γ? Justify your answer.

Answer: No.

Solution

First notice that, since angles $\angle AA_1B_1$ and $\angle AA_1C_1$ are both right, the points B_1 and C_1 lie on the circle with AA_1 as a diameter. Therefore, $AC_1 = AA_1 \sin \angle AA_1C_1 = AA_1 \sin(90^\circ - \angle A_1AC) = AA_1 \sin \angle C$, similarly $AB_1 = AA_1 \sin \angle B$, and $B_1C_1 = AA_1 \sin \angle A$. Hence; triangles AC_1B_1 and ABC are similar.



Let O be the circumcenter of ABC and AD be one of its diameters. Since $\angle OAC = \frac{1}{2}(180^\circ - \angle AOC) = 90^\circ - \angle B = 90^\circ - \angle AC_1B_1$, it follows that AD is perpendicular to B_1C_1. Let $AD = 2R$; recall that, from the law of sines, $\frac{BC}{\sin \angle A} = 2R \iff BC = 2R \sin \angle A$. The area of quadrilateral AB_1DC_1 is

$$\frac{B_1C_1 \cdot AD}{2} = \frac{AA_1 \sin \angle A \cdot 2R}{2} = \frac{AA_1 \cdot BC}{2},$$

which is indeed the area of ABC.

Since B_1 and C_1 are fixed points, the loci of the points P such that AB_1PC_1 is a convex quadrilateral with the same area as ABC is a line parallel to B_1C_1. That is, perpendicular to AD. Since the area of AB_1DC_1 is the same as the area of ABC, this locus is the line perpendicular to AD through D, which is tangent to the circumcircle of ABC. Therefore, it is not possible that the point P lies inside the circumcircle of ABC.

Problem 2

Let α and β be positive real numbers. *Emerald* makes a trip in the coordinate plane, starting off from the origin $(0,0)$. Each minute she moves one unit up or one unit to the right, restricting herself to the region $|x-y| < 2025$, in the coordinate plane. By the time she visits a point (x,y) she writes down the integer $\lfloor x\alpha + y\beta \rfloor$ on it. It turns out that Emerald wrote each non-negative integer exactly once. Find all the possible pairs (α, β) for which such a trip would be possible.

Answer: (α,β) such that $\alpha+\beta = 2$.

Solution

Let (x_n, y_n) be the point that Emerald visits after n minutes. Then $(x_{n+1}, y_{n+1}) \in \{(x_n + 1, y_n), (x_n, y_n+1)\}$. Either way, $x_{n+1} + y_{n+1} = x_n + y_n + 1$, and since $x_0 + y_0 = 0 + 0 = 0$, $x_n + y_n = n$.

The n-th number would be then

$$z_n = \lfloor x_n\alpha + (n - x_n)\beta \rfloor \implies n\beta + x_n(\alpha-\beta) - 1 < z_n < n\beta + x_n(\alpha - \beta),$$

in which

$$-2025 < x_n - y_n < 2025 \iff \frac{n-2025}{2} < x_n < \frac{n+2025}{2}.$$

Suppose without loss of generality that $\alpha \geq \beta$. Then

$$n\beta + \frac{n-2025}{2}(\alpha-\beta) - 1 < z_n < n\beta + \frac{n+2025}{2}(\alpha-\beta),$$

which reduces to

$$\left| z_n - \frac{\alpha+\beta}{2}n \right| < \frac{2025}{2}(\alpha-\beta) + 1.$$

On the other hand, $z_{n+1} = \lfloor x_{n+1}\alpha + y_{n+1}\beta \rfloor \in \{\lfloor x_n\alpha + y_n\beta + \alpha \rfloor, \lfloor x_n\alpha + y_n\beta + \beta \rfloor\}$, which implies $z_{n+1} \geq z_n$. Since every non-negative integer appears exactly once, in increasing order, it follows that $z_n = n$.

Therefore, for all positive integers n,

$$\left| n - \frac{\alpha+\beta}{2}n \right| < \frac{2025}{2}(\alpha-\beta) + 1,$$

which can only be possible if $\alpha + \beta = 2$; otherwise, the left hand side would be unbounded.

If $\alpha+\beta = 2$, consider $x_n = \left\lceil \frac{n}{2} \right\rceil$ and $y_n = \left\lfloor \frac{n}{2} \right\rfloor$. If n is even,

$$z_n = \left\lfloor \frac{n}{2}\alpha + \frac{n}{2}\beta \right\rfloor = n;$$

if n is odd,

$$z_n = \left\lfloor \frac{n+1}{2}\alpha + \frac{n-1}{2}\beta \right\rfloor = n + \left\lfloor \frac{\alpha-\beta}{2} \right\rfloor,$$

which equals n because $0 < \beta \leq \alpha < \alpha + \beta = 2 \implies 0 \leq \alpha - \beta < 2$.

Problem 3
Let $P(x)$ be a non-constant polynomial with integer coefficients such that $P(0) \neq 0$. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of integers such that $P(i-j)$ divides $a_i - a_j$ for all distinct positive integers i, j. Prove that the sequence $a_1, a_2, a_3, \ldots$ must be constant, that is, a_n equals a constant c for all n positive integer.

Solution
Let $a_0 = P(0) \neq 0$ be the independent coefficient, i.e., the constant term of $P(x)$. Then there are infinitely many primes p such that p divides $P(k)$ but p does not divide k. In fact, since $P(k) - a_0$ is a multiple of k, $\gcd(P(k), k) = \gcd(k, a_0) \leq a_0$ is bounded, so pick, say, k with prime factors each larger than a_0.
Since $P(k)$ divides $a_{i+k} - a_i$, p divides $a_{i+k} - a_i$. Moreover, since $P(k+p) \equiv P(k) \equiv 0 \pmod{p}$, p also divides $a_{i+k+p} - a_i$. Therefore, $a_i \bmod p$ is periodic with periods $k+p$ and k. By *Bezout's theorem*, $\gcd(k+p, k) = 1$ is also a period, that is, p divides $a_{i+1} - a_i$ for all i and p such that $p \mid P(k)$ and $p \nmid k$ for some k. Since there are infinitely many such primes p, $a_{i+1} - a_i$ is divisible by infinitely many primes, which implies $a_{i+1} = a_i$, that is, the sequence is constant.

Problem 4
Let $n \geq 3$ be an integer. There are n cells on a circle, and each cell is assigned either 0 or 1 . There is a rooster on one of these cells, and it repeats the following operations:

- If the rooster is on a cell assigned 0, it changes the assigned number to 1 and moves to the next cell counterclockwise.
- If the rooster is on a cell assigned 1, it changes the assigned number to 0 and moves to the cell after the next cell counterclockwise.

Prove that the following statement holds true after sufficiently many operations:

> If the rooster is on a cell C, then the rooster would go around the circle exactly three times, stopping again at C. Moreover, every cell would be assigned the same number as it was assigned right before the rooster went around the circle 3 times.

Solution 1
Reformulate the problem as a n-string of numbers in $\{0, 1\}$ and a position at which the action described in the problem is performed, and add 1 or 2 modulo n to the position according to the action. Say that a *lap* is complete for each time the position resets to 0 or 1. We will prove that the statement claim holds after at most two laps, after which the n-tuple cycles every three laps.
Say the rooster *stops* at a position in a certain lap if it performs an action at that position on that lap; otherwise, the rooster *bypasses* that position. We start with some immediate claims:

- The rooster has to stop at at least one of each two consecutive positions.
- The rooster stops at every position preceded by a 0. Indeed, if the numbers preceding that position are 00 then the rooster will definitely stop at the second zero, and it the numbers preceding that position are 10 then the rooster will either stop at 1 and go directly to the position or bypass 1 and stop at the second zero, and then stop at the position.
- Therefore, if the rooster bypasses a position, then it is preceded by a 1, and that 1 must be changed to a 0. This means that the rooster never bypasses a position in two consecutive laps.
- The rooster bypasses every position preceded by 01. Indeed, the rooster stops at either 1 or at 0, after which it will move to 1; at any rate, it stops at 1 and bypasses the position.

Our goal is to prove that, eventually, for every three consecutive laps, each position is bypassed exactly once. Then each position changes states exactly twice, so it gets back to its initial state after three laps. The following two lemmata achieve this goal:

Lemma 1. *If the rooster stops at a certain position in two laps in a row, it bypasses it on the next lap, except for the n-string* $1010\ldots10$*, for which the problem statement holds.*

Proof. If the rooster stopped at a position in lap t, then it is preceded by either (A) a 0 that was changed to 1, (B) a 11 that was changed to 01, or (C) a 10 in which the rooster stopped at 1. In case (A), the position must be preceded by 11 in the lap $t + 1$, which becomes 01, so the rooster will bypass the position in the lap $t + 2$. In case (B), the position will be bypassed in lap $t + 1$.
Now we deal with case (C): suppose that the position was preceded by m occurrences of 10, that is, $(10)^m$, on lap t and take $m \leq \frac{n}{2}$ maximal. The rooster stopped at the 1 from each occurrence of 10, except possibly the first one.

First, suppose that $n \geq 2m+2$. After lap t, $(10)^m$ becomes either $(00)^m$ or $11(00)^{m-1}$ in lap $t+1$. In the latter case, initially we had $1(10)^m$, which became $011(00)^{m-1}$. It will then become $a01(11)^{m-1}$ in lap $t+2$, in which the rooster will bypass the position. In the former case, $(10)^m$ becomes $(00)^m$, so it was either $00(10)^m$ or $11(10)^m$ in lap t, which becomes respectively $a1(00)^m$ and $01(00)^m$ in lap $t+1$, respectively. In the second sub-case, it becomes $b001(11)^{m-1}$ in lap $t+2$, and the position will be bypassed. In the first sub-case, it must be $11(00)^m$ after which it becomes either $01(11)^m$ or $1001(11)^{m-1}$. In any case, the position will be bypassed in lap $t+2$.
If $n = 2m+1$, the possible configurations are

$$(10)^m0 \to (00)^m1 \to (11)^m0 \to (10)^m0,$$

the rooster stops at the 1 from the first 10 because it was preceded by a 0.

$$(10)^m1 \to (00)^m0 \to 0(11)^m \to 1(01)^m,$$

or

$$(10)^m1 \to 11(00)^{m-1}0 \to 01(11)^{m-1}1 \to (10)^m1,$$

In any case, the position is bypassed in lap $t+2$.
If $n = 2m$, the entire configuration is $(10)^m$, $m \geq 2$. If the rooster did not stop at the first 1, it becomes $11(00)^{m-1}$ in the lap $t+1$, then $01(11)^{m-1}$ in the lap $t+2$, so the position is bypassed in this last lap. If the rooster stopped at the first 1, it becomes $(00)^m$, then $(11)^m$, then $(01)^m$, then $10(00)^{m-1}$, then $(11)^m$, and then it cycles between $(11)^m$, $(01)^m$ and $10(00)^{m-1}$.
So, apart from this specific string, the rooster will stop at most two laps in a row at each position. ☐

Lemma 2. *If the rooster bypasses one position on a lap, then it stops at that position on the next two laps, with the same exception as lemma 1.*

Proof. The position must be preceded by 1 in lap t. If it is preceded by 11, it changes to 10 in lap $t+1$. Then it becomes 00 because the 1 was already skipped in the previous lap, and the rooster will stop at the position in the lap $t+2$.
Now suppose that the position was preceded by $(01)^m$ on lap t and take $m \leq \frac{n}{2}$ maximal. It becomes $10(00)^{m-1}$ or $(00)^m$ in lap $t+1$. In the former case, in lap $t+2$ it becomes either $00(11)^{m-1}$, after which the rooster stops at the position again, or $(11)^m$, which we'll study later. In the former case, $(00)^m$ becomes $(11)^m$ or $01(11)^{m-1}$. In the latter case, the 0 was bypassed, so it must be sopped in the next lap, becoming $(10)^m$. In the $(11)^m$ case, in order to bypass the position in lap $t+2$, it must become $(10)^m$. All in all, the preceding terms are $(01)^m$. Then, either $10(00)^{m-1}$ or $(00)^m$, then either $(11)^m$ or $01(11)^{m-1}$, then $(10)^m$. Then the second term in $(01)^m$ is 1, then 0, then 1, and then 0, that is, it changed three times. So we fall under the exception to lemma 1. ☐

The result then immediately follows from lemmata 1 and 2.

Solution 2
Define positions, laps, stoppings, and bypassing as in Solution 1. This other pair of lemmata also solves the problem.

Lemma 3. *There is a position and a lap in which the rooster stops twice and bypasses once (in some order) in the next three laps.*

Proof. There is a position j the rooster stops for infinitely many times. Each time it stops at j, it changes between stopping and bypassing $j+1$. So the rooster stops and bypasses $j+1$ infinitely many times. Then there is a lap in which the rooster stops at $j+1$, and bypasses it in the next. As in solution 1, it cannot bypass $j+1$ two times in a row, so it stops at $j+1$ in the next lap. ☐

Lemma 4. *If the rooster stops twice and bypasses once (in some order) at some position in three consecutive laps, it also stops twice and bypasses once at the next position (in some order) in the same laps (or in the next laps, in case the lap changes from one position to the other).*

Proof. When the rooster bypasses the position, it must stop at the next one. In the two times it stops at the position, the cell has different numbers on it, so the rooster will stop once and bypass once at the next position. □

By lemmata 3 and 4, there would be a moment that the rooster stops twice and bypasses once (in some order) at any position after this moment. After this moment, if the rooster is in a cell C in lap t, we know that it stopped; stopped-bypassed; and bypassed-stopped at C in laps $t, t+1, t+2$. Since it stopped twice and bypassed once (in some order) in laps $t+1, t+2, t+3$, it must stop at C in lap $t+3$. Moreover, the rooster stopped twice and bypassed once (in some order) each cell between stopping at C in laps t and $t+3$, so every cell has the same assigned number before and after going around the circle three times.

Solution 3

Let us reformulate the problem in terms of Graphs: we have a directed graph G with $V = \{v_1, v_2, \ldots, v_n\}$ representing positions and $E = \{v_i \to v_{i+1}, v_i \to v_{i+2} \mid 1 \le i \le n\}$ representing moves. Indices are taken mod n. Note that each vertex has in-degree and out-degree both equal to 2. We say that the edge $v_i \to v_{i+1}$ is *active* and $v_i \to v_{i+2}$ is *inactive* if the number on v_i is 0, or the edge $v_i \to v_{i+2}$ is *active* and $v_i \to v_{i+1}$ is *inactive* if the number on v_i is 1. The rooster then traces an infinite trail on the graph in the following manner;

- it starts at v_1;
- if the rooster is at v_i, it uses the active edge going out of v_i to continue the path, and changes the number on v_i.

Take the first vertex that appears 3 times on the rooster's trail, and suppose without any loss of generality that it is v_1 (otherwise, just ignore the path before the first time that this vertex appears). Consider the sub-trail from the first to the third time v_1 appears: $C = v_1 \to \cdots \to v_1 \to \cdots \to v_1$. Trail C induces a circuit C^* on G that passes through v_1 twice.

In between two occurrences of v_1, the rooster must have completed at least one lap. Vertex v_1 appears three times at C, so C contains at least two laps. Since (as per Solution 1) no vertex is bypassed twice in a row, C^* must contain every vertex. Moreover, v_1 is the first (and only) vertex that appeared three times on the rooster's path, so each vertex appear at most twice on C^*.

Lemma 5. *Take V' to be the subset of V containing the vertices that only appears once on C^* and E' to be the edges of G that don't appear on C^*. So $G' = (V', E')$ is a simple cycle or $V' = E' = \emptyset$.*

Proof. We will first prove that each edge in C^* appears exactly one. Suppose, for the sake of contradiction, that there is an edge $u \to v$ that appears twice on C^*. So u must appear times in C, because the active edge going out of u changes at each visit to it. So $u = v_1$, but the rooster only goes out of v_1 twice in C, which is a contradiction.

Now, since C^* does not have repeated edges, all four edges through v are in C^* if v is visited twice in this cycle. So the edges on E' cannot pass through vertices in $V \setminus V'$, then G' is well-defined.

Since $v \in V'$ is visited once on C^*, it has $\deg_{in}(v) = \deg_{out}(v) = 1$ in G', so G' is a union of disjoint simple cycles. Each of these cycles completes at least one lap, and v_1 is skipped on these laps, so all cycles must use the edge $v_n \to v_2$. But the cycles are disjoint, so there is at most one cycle. □

Lemma 6. *The rooster eventually traverses a contiguous* Eulerian *circuit of G.*

Proof. If $V' = E' = \emptyset$, C^* is already an Eulerian circuit traversed in C. If not, take u_1 to be the first vertex of C that is in V'. Let $U = u_1 \to \cdots \to u_k \to u_1$ be the cycle determined by G' by Lemma 5, $C_1 = v_1 \to \cdots \to u_1$ be the sub-path of C from the first v_1 to u_1, and $C_2 = u_1 \to \cdots \to v_1$ be the rest of C.
Each vertex on $V \setminus V'$ has been changed twice, so they would be in their initial states after C; every vertex on V' has been changed only once, and therefore are not on their initial states. Let us trace the rooster's trail after it traversed C. By the minimality of u_1, all the edges of C_1 are active after C, so the rooster traverses C_1 after C. Moreover, each u_i was visited exactly once in C and has not yet used the edge $u_i \to u_{i+1}$. Since all their states have changed, all edges of U are active after C. Therefore, the rooster traverse U after C_1, flipping all the vertices in U to their initial states. Then, since every state in U was changed, the rooster traverses C_2 instead of U. The trail is then $C \to C_1 \to U \to C_2$, and hen $C_1 \to U \to C_2$ is an Eulerian circuit. □

Having this in mind, after the rooster completes an Eulerian circuit, it has passed through each vertex twice and returned to its initial vertex, so the state of the rooster and of the edges are the same before and after the Eulerian cycle. The rooster will then traverse the Eulerian circuit repeatedly.
The edges $v_i \to v_{i+1}$ move forward by one position and the edges $v_i \to v_{i+2}$ move forward by two positions. Since all edges are used, each time the rooster traverses the Eulerian circuit, it moves forward a total of $n(1 + 2) = 3n$ positions, which corresponds to three laps. The proof is now complete.

Comment: Solution 1 presents a somewhat brute-force case analysis; the main ideas are

- Lemma 1: The rooster stops at one position in at most two consecutive laps, with one notable exception.
- Lemma 2: If the rooster bypasses one position then it stops at it in the next two laps, with the same notable exception.

Solution 2 has a recursive nature; the main ideas are

- Lemma 3: Proving that one position changes states exactly two out of three laps.
- Lemma 4: Given that this happens to one position, it also happens to the next position in the same laps, so it happens to every position.

Solution 3 cleverly identifies the moves the rooster performs within a graph theoretical framework. The main goal is to prove that the rooster traces an Eulerian circuit in a graph.

- Lemma 5: Analyzing what happens after the rooster first visits a position three times.
- Lemma 6: Proving that the rooster traces an Eulerian circuit.

Problem 5
Consider an infinite sequence $a_1, a_2, \ldots$ of positive integers such that

$$100!(a_m + a_{m+1} + \cdots + a_n) \quad \text{is a multiple of} \quad a_{n-m+1}a_{n+m}$$

for all positive integers m, n such that $m \leq n$.
Prove that the sequence is either bounded or linear.
Observation: A sequence of positive integers is *bounded* if there exists a constant N such that $a_n < N$ for all $n \in \mathbb{Z}_{>0}$. A sequence is *linear* if $a_n = n \cdot a_1$ for all $n \in \mathbb{Z}_{>0}$.

Solution
Let $c = 100!$. Suppose that $n \geq m+2$. Then $a_{m+n} = a_{(m+1)+(n-1)}$ divides both $c(a_m + a_{m+1} + \cdots + a_{n-1} + a_n)$ and $c(a_{m+1} + \cdots + a_{n-1})$, so it also divides the difference $c(a_m + a_n)$. Notice that if $n = m+1$ then a_{m+n} divides $c(a_m + a_{m+1}) = c(a_m + a_n)$, and if $n = m$ then a_{m+n} divides both ca_m and $2ca_m = c(a_m + a_n)$. In either cases; a_{m+n} divides $c(a_m + a_n)$.
Analogously, one can prove that if $m > n$, $a_{m-n} = a_{(m-1)-(n-1)}$ divides $c(a_m - a_n)$, as it divides both $c(a_{n+1} + \cdots + a_m)$ and $c(a_n + \cdots + a_{m-1})$.
From now on, drop the original divisibility statement and keep the statements "a_{m+n} divides $c(a_m + a_n)$" and "a_{m-n} divides $c(a_m - a_n)$." Now, all conditions are linear, and we can suppose without loss of generality that there is no integer $D > 1$ that divides every term of the sequence; if there is such an integer D, divide all terms by D.
Having this in mind, notice that $a_m = a_{m+n-n}$ divides $c(a_{m+n} - a_n)$ and also $c(a_{m+n} - a_m - a_n)$; analogously, a_n also divides $c(a_{m+n} - a_m - a_n)$, and since a_{m+n} divides $c(a_m + a_n)$, it also divides $c(a_{m+n} - a_m - a_n)$. Therefore, $c(a_{m+n} - a_m - a_n)$ is divisible by a_m, a_n, and a_{m+n}, and therefore also by $\operatorname{lcm}(a_m, a_n, a_{m+n})$. In particular, $ca_{m+n} \equiv c(a_m + a_n) \pmod{\operatorname{lcm}(a_m, a_n)}$.
Since a_{m+n} divides $c(a_m + a_n)$, $a_{m+n} \leq c(a_m + a_n)$.
From now on, we divide the problem in two cases.

Case 1: there exist m, n such that $\operatorname{lcm}(a_m, a_n) > c^2(a_m + a_n)$.
If $\operatorname{lcm}(a_m, a_n) > c^2(a_m + a_n) > c(a_m + a_n)$ then both $c(a_m + a_n)$ and ca_{m+n} are less than $ca_{m+n} \leq c^2(a_m + a_n) < \operatorname{lcm}(a_m, a_n)$. This implies $ca_{m+n} = c(a_m + a_n) \iff a_{m+n} = a_m + a_n$.
Now we can extend this further: since $\gcd(a_m, a_{m+n}) = \gcd(a_m, a_m + a_n) = \gcd(a_m, a_n)$, it follows that

$$\begin{aligned}\operatorname{lcm}(a_m, a_{m+n}) = \frac{a_m a_{m+n}}{\gcd(a_m, a_n)} &= \frac{a_{m+n}}{a_n}\operatorname{lcm}(a_m, a_n) > \frac{c^2(a_m + a_n)^2}{a_n} \\ &> \frac{c^2(2a_m a_n + a_n^2)}{a_n} = c^2(2a_m + a_n) = c^2(a_m + a_{m+n}).\end{aligned}$$

We can iterate this reasoning to obtain that $a_{km+n} = ka_m + a_n$, for all $k \in \mathbb{Z}_{>0}$. In fact, if the condition $\operatorname{lcm}(a_m, a_n) > c^2(a_m + a_n)$ holds for the pair (n, m), then it also holds for the pairs $(m+n, m), (2m+n, m), \ldots, ((k-1)m+n, m)$, which implies $a_{km+n} = a_{(k-1)m+n} + a_m = a_{(k-2)m+n} + 2a_m = \cdots = a_n + ka_m$.
Similarly, $a_{m+kn} = a_m + ka_n$.
Now, $a_{m+n+mn} = a_{n+(n+1)m} = a_{m+(m+1)n} \implies a_n + (n+1)a_m = a_m + (m+1)a_n \iff ma_n = na_m$. If $d = \gcd(m, n)$ then $\frac{n}{d}a_m = \frac{m}{d}a_n$.
Therefore, since $\gcd(\frac{m}{d}, \frac{n}{d}) = 1$, $\frac{m}{d}$ divides a_m and $\frac{n}{d}$ divides a_n, which means that

$$a_n = \frac{m}{d} \cdot t = \frac{t}{d}m \quad \text{and} \quad a_m = \frac{n}{d} \cdot t = \frac{t}{d}n, \quad \text{for some } t \in \mathbb{Z}_{>0},$$

which also implies

$$a_{km+n} = \frac{t}{d}(km+n) \quad \text{and} \quad a_{m+kn} = \frac{t}{d}(m+kn), \quad \text{for all } k \in \mathbb{Z}_{>0}.$$

Now let's prove that $a_{kd} = tk = \frac{t}{d}(kd)$ for all $k \in \mathbb{Z}_{>0}$. In fact, there exist arbitrarily large positive integers R, S such that $kd = Rm - Sn = (n+(R+1)m) - (m+(S+1)n)$ (for instance, let $u, v \in \mathbb{Z}$ such that $kd = mu - nv$ and take $R = u + Qn$ and $S = v + Qm$ for Q sufficiently large.)
Let $x = n + (R+1)m$ and $y = m + (S+1)n$. Then $kd = x - y \iff x = y + kd$, $a_x = \frac{t}{d}x$, and $a_y = \frac{t}{d}y = \frac{t}{d}(x - kd) = a_x - tk$. Thus a_x divides $c(a_y + a_{kd}) = c(a_{kd} + a_x - tk)$, and therefore also $c(a_{kd} - tk)$. Since $a_x = \frac{t}{d}x$ can be arbitrarily large, $a_{kd} = tk = \frac{t}{d}(kd)$. In particular, $a_d = t$, so $a_{kd} = ka_d$.
Since $ka_d = a_{kd} = a_{1+(kd-1)}$ divides $c(a_1 + a_{kd-1})$, $b_k = a_{kd-1}$ is unbounded. Pick $p > a_{kd-1}$ a large prime and consider $a_{pd} = pa_d$. Then

$$\operatorname{lcm}(a_{pd}, a_{kd-1}) \geq \operatorname{lcm}(p, a_{kd-1}) = pa_{kd-1}.$$

We can pick a_{kd-1} and p large enough so that their product is larger than a particular linear combination of them, that is,

$$\operatorname{lcm}(p, a_{kd-1}) = pa_{kd-1} > c^2(pa_d + a_{kd-1}) = c^2(a_{pd} + a_{kd-1}).$$

Then all the previous facts can be applied, and since $\gcd(pd, kd-1) = 1$, $a_k = a_{k\gcd(pd,kd-1)}$ $= ka_{\gcd(pd,kd-1)} = ka_1$, that is, the sequence is linear. Also, since a_1 divides all terms, $a_1 = 1$.

Case 2: $\operatorname{lcm}(a_m, a_n) \leq c^2(a_m + a_n)$ *for all* m, n.
Suppose that $a_m \leq a_n$; then $\operatorname{lcm}(a_m, a_n) = Ma_n \leq c^2(a_m + a_n) \leq 2c^2 a_n \implies M \leq 2c^2$, that is, the factor in the smaller term that is not in the larger term is at most $2c^2$.
We prove that in this case the sequence must be bounded. Suppose on the contrary; then there is a term a_m that is divisible by a large prime power p^d. Then every larger term a_n is divisible by a factor larger than $\frac{p^d}{2c^2}$. So we pick $p^d > (2c^2)^2$, so that every large term a_n is divisible by the prime power $p^e > 2c^2$. Finally, fix a_k for any k. It follows from $a_{k+n} \mid c(a_k + a_n)$ that (a_n) is unbounded, so we can pick a_n and a_{k+n} large enough such that both are divisible by p^e. Hence any a_k is divisible by p, which is a contradiction to the fact that there is no $D > 1$ that divides every term in the sequence.