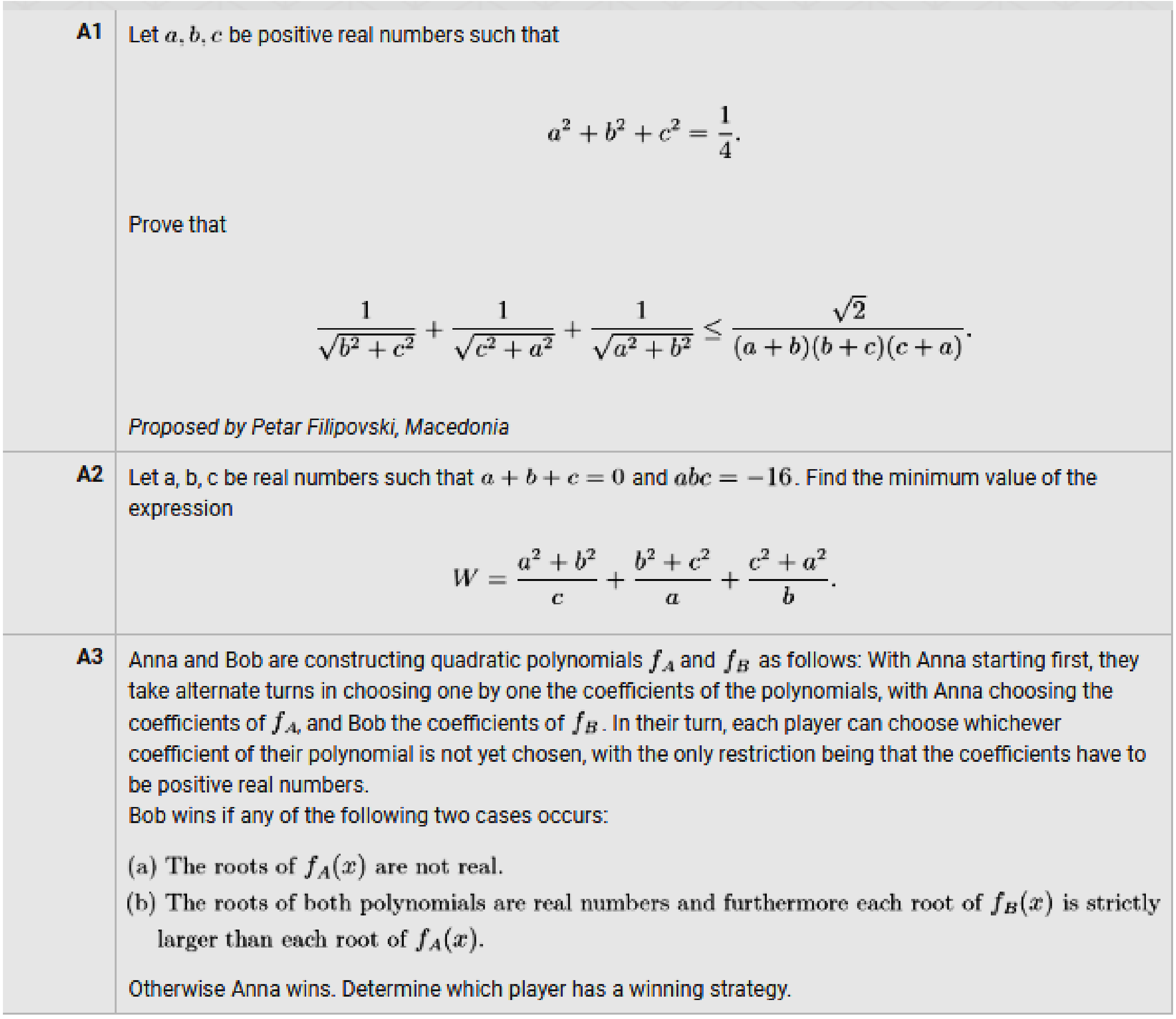


A1 Let a, b, c be positive real numbers such that

$$a^2+b^2+c^2=\frac{1}{4}.$$

Prove that

$$\frac{1}{\sqrt{b^2+c^2}}+\frac{1}{\sqrt{c^2+a^2}}+\frac{1}{\sqrt{a^2+b^2}}\leq\frac{\sqrt{2}}{(a+b)(b+c)(c+a)}.$$

Proposed by Petar Filipovski, Macedonia

A2 Let a, b, c be real numbers such that $a+b+c=0$ and $abc=-16$. Find the minimum value of the expression

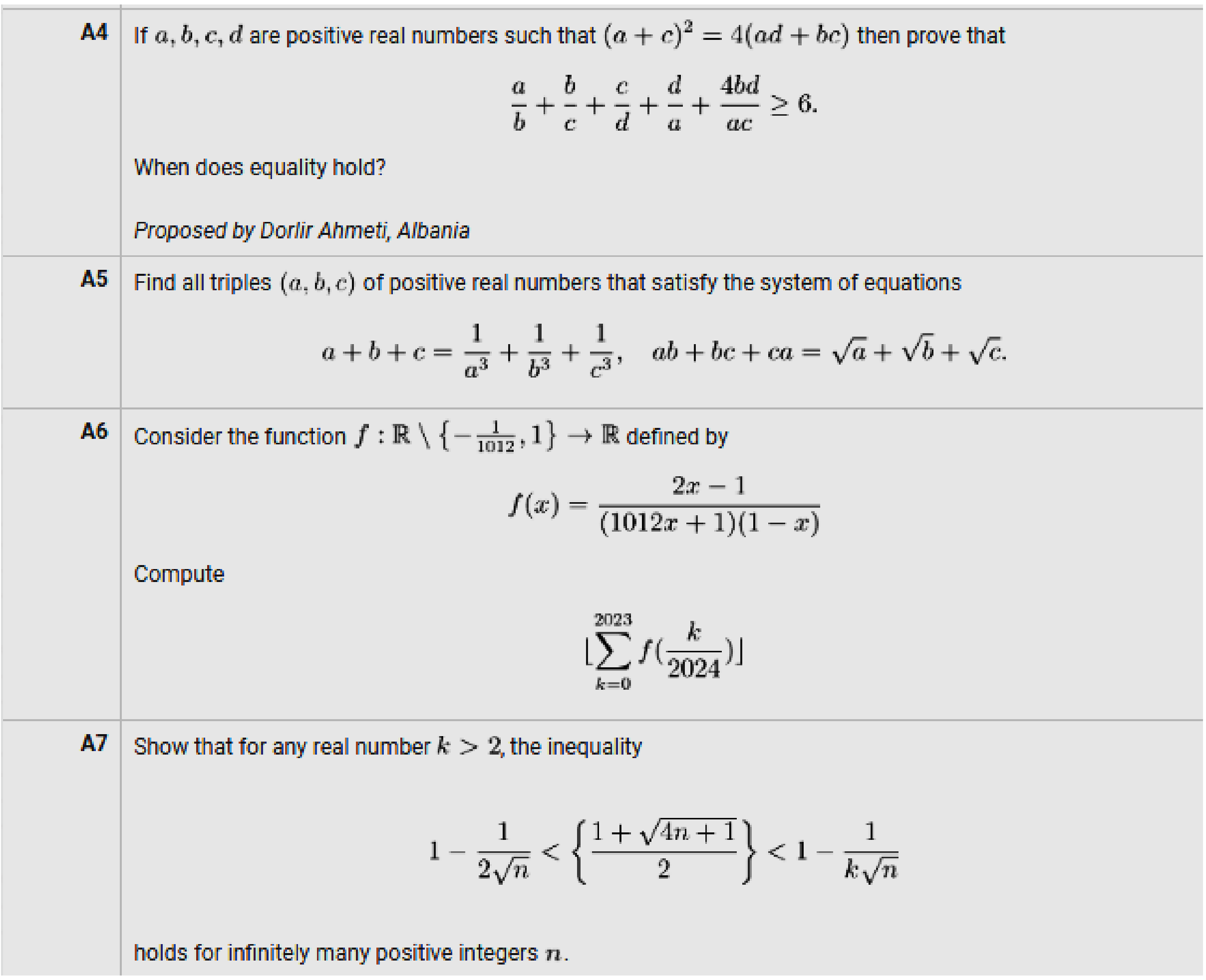
$$W=\frac{a^2+b^2}{c}+\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}.$$

A3 Anna and Bob are constructing quadratic polynomials f_A and f_B as follows: With Anna starting first, they take alternate turns in choosing one by one the coefficients of the polynomials, with Anna choosing the coefficients of f_A, and Bob the coefficients of f_B. In their turn, each player can choose whichever coefficient of their polynomial is not yet chosen, with the only restriction being that the coefficients have to be positive real numbers.
Bob wins if any of the following two cases occurs:

(a) The roots of $f_A(x)$ are not real.

(b) The roots of both polynomials are real numbers and furthermore each root of $f_B(x)$ is strictly larger than each root of $f_A(x)$.

Otherwise Anna wins. Determine which player has a winning strategy.



A4 If a, b, c, d are positive real numbers such that $(a+c)^2=4(ad+bc)$ then prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+\frac{4bd}{ac}\geq 6.$$

When does equality hold?

Proposed by Dorlir Ahmeti, Albania

A5 Find all triples (a, b, c) of positive real numbers that satisfy the system of equations

$$a+b+c=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3},\quad ab+bc+ca=\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

A6 Consider the function $f:\mathbb{R}\setminus\left\{-\frac{1}{1012},1\right\}\to\mathbb{R}$ defined by

$$f(x)=\frac{2x-1}{(1012x+1)(1-x)}$$

Compute

$$\lfloor\sum_{k=0}^{2023}f(\frac{k}{2024})\rfloor$$

A7 Show that for any real number $k>2$, the inequality

$$1-\frac{1}{2\sqrt{n}}<\left\{\frac{1+\sqrt{4n+1}}{2}\right\}<1-\frac{1}{k\sqrt{n}}$$

holds for infinitely many positive integers n.

C1	Determine the smallest positive integer k with the following property. For any subset $\mathbb{S}$ of the set $\{1,2,3,\cdot,2024\}$ with $\|S\| = k$, there are two distinct elements $a, b \in \mathbb{S}$ such that $ab+1$ is a perfect square
C2	Let n be a positive integer such that n^2-1 is divisible by 6. An $n \times n$ board is given. Prove that it is possible to place $\dfrac{n^2-1}{6}$ non-overlapping right triangles on the board with the lengths of $3, 4, 5$.
C3	A set of positive integers is called arithmetic if it contains three distinc elements which form an arithmetic sequence. Prove that at least 51% of the subsets of the set $\{1,2,3,\cdot,2024\}$ are arithmetic.
C4	All the unit cubes of a $5 \times 5 \times 3$ is colored white initially. We call two unit cubes are adjacent to each other if they share a common face. What is the largest number of unit cubes we can color in black such that each black unit cube is adjacent to at most one other black unit cube?
C5	Let n be a positive integer. Find the largest integer M such that for any distinct positive integers $a_1, \ldots, a_n$ with sum M there exist integers $b_1, \ldots, b_n \in \{-2,-1,0,1,2\}$, not all equal to 0, such that $\displaystyle\sum_{i=1}^{n} a_i b_i = 0$. Remark
C6	Three friends Archie, Billie, and Charlie play a game. At the beginning of the game, each of them has a pile of 2024 pebbles. Archie makes the first move, Billie makes the second, Charlie makes the third and they continue to make moves in the same order. In each move, the player making the move must choose a positive integer n greater than any previously chosen number by any player, take $2n$ pebbles from his pile and distribute them equally to the other two players. If a player cannot make a move, the game ends and that player loses the game. Determine all the players who have a strategy such that, regardless of how the other two players play, they will not lose the game. *Proposed by Ilija Jovčeski, Macedonia*

G1	Let $\triangle ABC$ be an acute-angled triangle with $AB = BC$. The perpendicular bisector of AB meets BC at D, and the circumcircle ω of $\triangle ADC$ again at the point E. Let F be diametrically opposite point of E in ω. Prove that $BD = DF$
G2	Let ABC be an acute-angled triangle, and A' and B' be the feet of the altitudes from A and B to BC, CA respectively. Let K and L be the reflections of A' with respect to AB and AC, respectively. Let M and N be the reflection of B' with respect to AB and BC, respectively. Prove that $KL = MN$.
G3	Let ABC be a triangle such that $AB < AC$. Let the excircle opposite to A be tangent to the lines AB, AC, and BC at points D, E, and F, respectively, and let J be its centre. Let P be a point on the side BC. The circumcircles of the triangles BDP and CEP intersect for the second time at Q. Let R be the foot of the perpendicular from A to the line FJ. Prove that the points P, Q, and R are collinear. (The *excircle* of a triangle ABC opposite to A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.) *Proposed by Bozhidar Dimitrov, Bulgaria*
G4	Let $ABCD$ be a circumscribed quadrilateral with circumcircle ω such that $AE = EC$, where E is the intersection point of the diagonals AC and BD. Point F is taken on ω such that $BF \parallel AC$. If G is the reflection of F with respect to A, prove that the circumcircle of $\triangle ADG$ is tangent to the line AC
G5	Let $ABCD$ be a rectangle, and H be the midpoint of the side AB. Point K is taken on DH such that $\angle BKD = 90^\circ$. Let F be a point on the diagonal AC. The perpendicular line to AB through F meets AB at G, and the parallel line to AB through F meets DH at L. If M is the midpoint of GB, prove that the angles $\angle AKF$ and $\angle LKM$ are equal.
G6	Let $ABCD$ be a trapezoid with $AB \parallel CD$. Let E and F be points on CD such that $AE \perp CD$ and $AF \perp AD$. Let G be a point on AE such that $BG \parallel AD$. Prove that the perpendicular line from A to BD bisects the segment FG.

G7	Let ABC be an acute-angled and scalene triangle, and D be a point on the side BC. Points E and F are taken on AD such that $EB \perp AB$ and $FC \perp AC$. Points S and T are taken on BC such that $SE \parallel AC$ and $TF \parallel AB$. The circumcircle of $\triangle BSE$ intersects AB for the second time at M, and the circumcircle of $\triangle CTF$ intersects AC for the second time at N. Prove that the lines MS, NT, and AD are concurrent
G8	Let ABC be a scalene triangle with smallest side BC, and D be a point on the side BC such that $\angle CAD = \angle DAB$. Let ω_1 and ω_2 be the circumcircles of $\triangle ABD$ and $\triangle ACD$ respectively. The line AC meets ω_1 again at F, and the line AB meets ω_2 again at E. The line DE meets ω_1 again at G, and the line DF meets ω_2 again at H. Prove that circumcircles of $\triangle ABC$, $\triangle AEF$ and $\triangle AGH$ have a common point other than A. *Proposed by Bruno Bajo, Albania*
N1	Find all pairs of positive integers (m, n) such that $\lvert 4^m - 7^n \rvert$ is a prime number. *Proposed by Dorlir Ahmeti, Albania*
N2	Find all the pairs of $(p;q)$ distinct prime numbers such that such that $q^p \mid p + p^q + p^{q^p}$
N3	Let $c \in \mathbb{Z}^+$ be a Turkish number if there is a positive integer m such that $m^3 - m = c!$ and $m^2 - 1$ has less than 12 positive divisors. Determine all Turkish numbers.
N4	For any positive integer n, let $s(n) = 1 + 2 + \cdots + n$. Define a strictly increasing sequence of positive integers $\{a_n\}_{n \geq 1}$ such that $a_1 = 1$ and $a_{n+1} = \min\{m \mid s(m) - s(a_n) \text{ is a perfect square}\}$ for all positive integers n. Find the value of a_{2024}.
N5	Does there exist a positive integer k such that the number of quadruples (a, b, c, n) of positive integers satisfying $a^5 + b^6 + c^{15} - n! = k$ is finite?
N6	Find all triples of positive integers (x, y, z) that satisfy the equation $2020^x + 2^y = 2024^z.$ *Proposed by Ognjen Tešić, Serbia*