41^{st} Balkan Mathematical Olympiad

27th April – 2nd May 2024
Varna, Bulgaria

Problem Shortlist with solutions

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These Shortlist Problems
have to be kept strictly confidential
until BMO 2025.

Contributing countries

The Organizing Committee and the Problem Selection Committee of BMO 2024 thank the following countries for submitting problems:

Albania, Cyprus, Greece, Kazakhstan, Republic of Moldova, Republic of North Macedonia, Romania, Serbia, Switzerland, United Kingdom, Uzbekistan.

Problem Selection Committee

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Algebra

A1.

Let u, v, w be positive reals. Prove that there is a cyclic permutation (x, y, z) of (u, v, w) such that the inequality:

$$\frac{a}{xa+yb+zc}+\frac{b}{xb+yc+za}+\frac{c}{xc+ya+zb}\geq\frac{3}{x+y+z}$$

holds for all positive real numbers a, b and c.

A2.

Prove that there is a positive integer number n such that the decimal representation of the number:

$$\sum_{k=1}^{\lfloor\frac{n}{3}\rfloor}\binom{n}{3k}8^k$$

ends in 2023 digits 8.

A3.

Find all triples (a, b, c) of positive real numbers that satisfy the system:

$$\left|\begin{array}{rcl} 11bc-36b-15c & = & abc \\ 12ca-10c-28a & = & abc \\ 13ab-21a-6b & = & abc. \end{array}\right.$$

A4.

Let $a \geq b \geq c \geq 0$ be real numbers such that $ab + bc + ca = 3$. Prove that:

$$3+(2-\sqrt{3})\cdot\frac{(b-c)^2}{b+(\sqrt{3}-1)c}\leq a+b+c$$

and determine all the cases when the equality occurs.

A5.

Let $\mathbb{R}^+ = (0, \infty)$ be the set of positive real numbers. Find all non-negative real numbers $c \geq 0$ such that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the property:

$$f(y^2 f(x) + y + c) = x f(x + y^2)$$

for all $x, y \in \mathbb{R}^+$.

A6.

Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $g(x)$ with non-negative coefficients and $g(0) = 0$ that satisfy the equality:

$$f(f(x) + g(y)) = f(x - y) + 2y$$

for all positive real numbers $x > y$.

Combinatorics

C1.

Let n, k be positive integers. Julia and Florian play a game on a $2n \times 2n$ board. Julia has secretly tiled the entire board with invisible dominos. Florian now chooses k cells. All dominos covering at least one of these cells then turn visible. Determine the minimal value of k such that Florian has a strategy to always deduce the entire tiling.

C2.

Let $n \geq 2$ and $S = \{1, 2, \ldots, n^2\}$. For any function $f : S \to S$ let $\mathrm{Fix}(f) = \{x \in S \mid f(x) = x\}$. Find the possible values of the expression

$$|\,\mathrm{Fix}(f)| + |\,\mathrm{Im}(f)| + \max_{k \in S} |f^{-1}(k)|$$

as f ranges over all functions $f : S \to S$.

C3.

Let $n \geqslant 3$. Alice and Bob play the following game: Alice chooses $k \in \{3, 4, \ldots, n\}$ and draws a $3 \times k$ table, then he fills the k cells of the first row with different numbers from $\{1, 2, \ldots, n\}$. Then, Bob fills on the second row some of the cells (eventually none) with distinct numbers from $\{1, 2, \ldots, n\}$, and the rest of them with 0. Finally, on each cell of the third row we write the sum of the two cells above. Show that regardless how Alice plays, Bob can guarantee that on the third row he can obtain, in some order, the terms of a non-constant arithmetical progression.

C4.

Prove that for every positive integer k there exists an integer n and distinct primes $p_1, p_2, \ldots, p_k$ such that, if $A(n)$ denotes the number of integers in $\{1, 2, \ldots, n\}$ which are relatively prime to $p_1 p_2 \cdots p_k$, then

$$\left| n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) - A(n) \right| > 2^{k-3}.$$

C5.

Let $n \geq 3$ be a natural number. Anna and Bob play the following game on the vertices of a regular n-gon: Anna places her token on a vertex of the n-gon. Afterwards Bob places his token on another vertex of the n-gon. Then, with Anna playing first, they move their tokens alternately as follows for $2n$ rounds: In Anna's turn on the k-th round, she moves her token k positions clockwise or anticlockwise. In Bob's turn on the k-th round, he moves his token 1 position clockwise or anticlockwise.
If at the end of any person's turn the two tokens are on the same vertex, then Anna wins the game. Otherwise Bob wins. Decide for each value of n which player has a winning strategy.

C6.

Let $\mathcal{D}$ be the set of lines in the plane and A a set of 17 points in the plane. For $d \in \mathcal{D}$, let $n_d(A)$ be the number of distinct points in which A projects on d. Find the maximum cardinality of

$$V_A = \{n_d(A) | d \in \mathcal{D}\}.$$

Geometry

G1.

Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E, F respectively. Let ℓ be a line through D meeting AC, AB at G, H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

G2.

Let ABC be an acute triangle and P be a point inside the triangle such that $\sphericalangle APB = \sphericalangle BPC = \sphericalangle CPA$. Denote with S the area and with α, β, γ the angles of $\triangle ABC$. Prove that

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} \geq \frac{PA^2 + PB^2 + PC^2}{2S} + \frac{4}{\sqrt{3}}.$$

When does the equality occur?

G3.

Let A, B, C, D be fixed points on this order on a line. Let ω be a variable circle through C and D and suppose that it meets the perpendicular bisector of CD at the points X and Y. Let Z and T be the other points of intersection of AX and BY with ω. Prove that XY passes through a fixed point which is independent of the circle ω.

G4.

Let ABC be an acute-angled triangle with $AB < AC$, orthocentre H, circumcircle Γ and circumcentre O. Let M be the midpoint of BC and let D be a point such that $ADOH$ is a parallelogram. Suppose that there exists a point X on Γ and on the opposite side of DH to A such that $\sphericalangle DXH + \sphericalangle DHA = 90°$. Let Y be the midpoint of OX. Prove that if $MY = OA$ then $OA = 2OH$.

G5.

Let ABC be a scalene acute triangle ABC, D be the orthogonal projection of A on BC, M and N are the midpoints of AB and AC respectively. Let P, Q are points on the minor arcs $\widehat{AB}$ and $\widehat{AC}$ of circumcircle of $\triangle ABC$ respectively such that $PQ \parallel BC$. Show that the circumcircles of $\triangle DPQ$ and $\triangle MND$ are tangent to each other if and only if PQ passes through M.

G6.

Let $\triangle ABC$ be a triangle and the points K and L on AB, M and N on BC and P and Q on CA are such that $AK = LB < \frac{1}{2}AB$, $BM = NC < \frac{1}{2}BC$ and $CP = QA < \frac{1}{2}CA$. The intersections of KN with MQ and LP are R and T respectively, and the intersections of NP with LM and KQ are D and E respectively. Prove that the lines DR, BE and CT pass through a common point.

G7.

Let $f : \pi \to \mathbb{R}$ be a function from the Euclidean plane to the real numbers such that

$$f(A) + f(B) + f(C) = f(O) + f(G) + f(H)$$

for any acute triangle ABC with circumcenter O, centroid G, and orthocenter H. Prove that f is constant.

Number Theory

N1.

Let n be a fixed natural number and

$$S_n = \{\overline{c_n c_{n-1} \dots c_1}_{(10)} \mid c_1, \dots, c_{n-1}, c_n \in \{1, 2, 3, 4\}\}.$$

Are there distinct numbers x and y, $x, y \in S_n$, such that $4^n \mid x - y$?

N2.

Prove that for every integer n, the number $n^4 - 12n^2 + 144$ is not a perfect cube of an integer.

N3.

Consider the sequence

$$a_n = 2^n + 3^{n+2} + 5^{n+1}$$

for $n \geqslant 1$. Prove that there are infinitely many prime numbers such that each one of them divides infinitely many terms of the sequence.

N4.

Let k be a positive integer. Find all sequences $(a_n)_{n \geq 1}$ of positive integers such that

$$a_{n+2}(a_{n+1} - k) = a_n(a_{n+1} + k)$$

for all $n \geq 1$.

N5.

Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

N6.

For a positive integer c, define a sequence by $a_1 = c$ and $a_n = a_{n-1}^3 + c$ for each $n \geq 2$. Call a prime number p *orange* if for every positive integer c there is some n such that $p \mid a_n$. Are there infinitely many orange primes?

N7.

Let a, b be positive integers such that $a + 1$, $b + 1$ and ab are all perfect squares. Prove that $\gcd(a, b) + 1$ is also a perfect square.

$p \mid a_n$. Are there infinitely many orange primes?

Algebra – Solutions

A1.

Let u, v, w be positive reals. Prove that there is a cyclic permutation (x, y, z) of (u, v, w) such that the inequality:

$$\frac{a}{xa+yb+zc}+\frac{b}{xb+yc+za}+\frac{c}{xc+ya+zb}\geq\frac{3}{x+y+z}$$

holds for all positive real numbers a, b and c.

Solution. Trivially, one of the inequalities $v+w\geq 2u, u+w\geq 2v,\ v+w\geq 2u$ must hold. Let (x, y, z) be such a cyclic permutation of (u, v, w) that $y+z\geq 2x$. Then the LHS of the desired inequality is equal to

$$\frac{a^2}{xa^2+yab+zac}+\frac{b^2}{xb^2+ybc+zab}+\frac{c^2}{xc^2+yac+zbc}.$$

Applying the Cauchy-Schwarz inequality (in the form of Arthur Engels) we get

$$\frac{a^2}{xa^2+yab+zac}+\frac{b^2}{xb^2+ybc+zab}+\frac{c^2}{xc^2+yac+zbc}\geq\frac{(a+b+c)^2}{x(a^2+b^2+c^2)+(y+z)(ab+ac+bc)}.$$

Denote $S=a^2+b^2+c^2, P=ab+ac+bc$. It suffices to prove $\frac{S+2P}{xS+(y+z)P}\geq\frac{3}{x+y+z}$. This inequality is equivalent to $(y+z-2x)(S-P)\geq 0$, which obviously holds since $S\geq P$ and $y+z\geq 2x$. □

Alternative Solution.

Lemma 1. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be convex, monotonically decreasing function. We prove that for any $x_1, x_2, x_3\in\mathbb{R}^+$ there is a cyclic permutations (x, y, z) of (x_1, x_2, x_3) such that the inequality:*

$$\alpha f(\alpha x+\beta y+\gamma z)+\beta f(\beta x+\gamma y+\alpha z)+\gamma f(\gamma x+\alpha y+\beta z)\geq f\left(\frac{x+y+z}{3}\right)$$

holds true for every choice of non-negative real numbers α, β, γ with sum $\alpha+\beta+\gamma=1$.

Proof. We choose the cyclic permutation (x, y, z) such that $x = \min(x_1, x_2, x_3)$. Let $\alpha+\beta+\gamma = 1$ and $\alpha, \beta, \gamma \geq 0$. Then by the convexity of f and Jensen inequality we have:

$$\alpha f(\alpha x+\beta y+\gamma z)+\beta f(\beta x+\gamma y+\alpha z)+\gamma f(\gamma x+\alpha y+\beta z) \geq f((\alpha^2+\beta^2+\gamma^2)x+(\alpha\beta+\beta\gamma+\gamma\alpha)(y+z)).$$

Setting $s = \alpha^2 + \beta^2 + \gamma^2$, we have that:

$$\alpha\beta + \beta\gamma + \gamma\alpha = \alpha\gamma + \beta\alpha + \gamma\beta = \frac{(\alpha+\beta+\gamma)^2 - s}{2} = \frac{1-s}{2}.$$

Therefore, in terms of s, we have:

$$\alpha f(\alpha x + \beta y + \gamma z) + \beta f(\beta x + \gamma y + \alpha z) + \gamma f(\gamma x + \alpha y + \beta z) \geq f\left(sx + \frac{1-s}{2}(y+z)\right).$$

Now note that $\alpha^2+\beta^2+\gamma^2 \geq \alpha\beta+\beta\gamma+\gamma\alpha$ and therefore $s \geq 1/2$. Finally, by the choice of $x \leq \min(y, z)$, a straightforward computation reveals that:

$$sx + \frac{1-s}{2}(y+z) \leq (x+y+z)/3, \quad \text{for this is equivalent to } (s-1/3)x \leq (-1/6+s/2)(y+z)$$

and the last inequality is obvious, since for $s \geq 1/2$, $(-1/6 + s/2)(y + z) \geq (-1/6 + s/2)2x = (-1/3 + s)x$.
Therefore, by the (decreasing) monotonicity of f, we get:

$$f(sx + \frac{1-s}{2}(y+z)) \geq f((x+y+z)/3)$$

and the desired inequality follows. □

Back to the original problem. Note that $f(t) = \frac{1}{t}$ is monotonically decreasing and convex function and therefore the assumptions of the Lemma are satisfied. Introducing $S = a + b + c$ for $a, b, c > 0$, and $\alpha = \frac{a}{S}$, $\beta = \frac{b}{S}$, $\gamma = \frac{c}{S}$, we have that:

$$\frac{a}{ax+by+cz}+\frac{b}{bx+cy+az}+\frac{c}{cx+ay+bz} = \alpha\frac{1}{\alpha x+\beta y+\gamma z}+\beta\frac{1}{\beta x+\gamma y+\alpha z}+\gamma\frac{1}{\gamma x+\alpha y+\beta z}.$$

With this remark, the desired inequality follows immediately from the conclusion of the Lemma. □

Comment. Note that the Lemma in the second solution suggests a natural generalisation of the proposed inequality to:

$$ag\left(\frac{ax+by+cz}{a+b+c}\right)+bg\left(\frac{bx+cy+az}{a+b+c}\right)+cg\left(\frac{cx+ay+bz}{a+b+c}\right) \geq (a+b+c)g\left(\frac{x+y+z}{3}\right),$$

where g is a monotone convex function.

A2.

Prove that there is a positive integer number n such that the decimal representation of the number:

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} 8^k$$

ends in 2023 digits 8.

Solution. Let $f(n) = \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} 8^k$ and $\omega \neq 1$ be a third root of the unity. Using the fact that for every integer $k \geq 0$:

$$1 + \omega^k + \omega^{2k} = \begin{cases} 3, & \text{if } 3 \mid k \\ 0, & \text{otherwise,} \end{cases}$$

we get that:

$$\begin{aligned} f(n) + 1 &= \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} 2^k = \frac{1}{3} \sum_{k=0}^{n} (1 + \omega^k + \omega^{2k}) \binom{n}{k} 2^k \\ &= \frac{1}{3} 3^n + \frac{1}{3}(1 + 2\omega)^n + \frac{1}{3}(1 + 2\omega^2)^n). \end{aligned}$$

Now note that 3, $1 + 2\omega$ and $1 + 2\omega^2$ are the roots of the polynomial:

$$P(x) = (x - 3)(x - 1 - 2\omega)(x - 1 - 2\omega^2) = (x - 1)^3 - 8 = x^3 - 3x^2 + 3x - 9$$

which, in turn, is the characteristic polynomial of the recursive sequence $(a_i)_{i \geq 0}$:

$$a_{i+3} = 3a_{i+2} - 3a_{i+1} + 9a_i \text{ for } i \geq 0.$$

Thus, if we set $a_i = f(i) + 1 = 1$ for $0 \leq i \leq 2$, then $f(n) + 1 = a_n$ for every $n \geq 0$.
Let $b_i = a_i \pmod{10^{2023}}$. Since $gcd(3, 10^{2023}) = 1$, any three consecutive terms of the sequence $(b_i)_{i \geq 0}$ uniquely the previous as well the next term of this sequence. Together with the fact that there are only finitely many residues modulo 10^{2023}, we conclude that the sequence $(b_i)_{i \geq 0}$ is periodic with some period $d > 3$ (since $b_3 = a_3 = 9$). Therefore:

$$9(f(d-1)+1) = 9a_{d-1} = a_{d+2} - 3a_{d+1} + 3a_d \equiv a_2 - 3a_1 + 3a_0 \pmod{10^{2023}} = 1 \pmod{10^{2023}}.$$

Finally, since $9 \mid 8.10^{2023} + 1$, we conclude that $9 \frac{8.10^{2023}+1}{9} \equiv 1 \pmod{10^{2023}}$ and consequently:

$$f(d-1) + 1 = a_{d-1} \equiv \frac{8.10^{2023} + 1}{9} = \underbrace{88 \ldots 8}_{2022}9 \pmod{10^{2023}}$$

and thus $f(d-1) \equiv \underbrace{88\ldots8}_{2022} \pmod{10^{2023}}$. Therefore $n = d-1$ has the desired property.

$\square$

Comment. We note that the idea of using roots of unity in order to study the properties of sums of binomial coefficients is broadly considered in Chapter 5 of the book of Titu Andreescu and Dorin Andrica *Complex Numbers from A to ...Z*, 2006, Birkhäuser. See for example Problem 4 on page 239.
Similar idea can be further found in the Romania TST 2004, Problem 10 by Catalin Popescu: https://artofproblemsolving.com/community/c6h5487p17788.

A3.

Find all triples (a, b, c) of positive real numbers that satisfy the system:

$$\left|\begin{array}{rcl} 11bc - 36b - 15c &=& abc \\ 12ca - 10c - 28a &=& abc \\ 13ab - 21a - 6b &=& abc. \end{array}\right.$$

Solution. Considering each of the equalities:

$$\begin{aligned} abc &= 11bc - 36b - 15c \\ abc &= 12ac - 10c - 28a \\ abc &= 13ab - 21a - 6b \end{aligned}$$

and dividing the first one by $bc > 0$, the second one by ac and third one by ab we obtain:

$$\begin{aligned} a &= 11 - \frac{36}{c} - \frac{15}{b} \\ b &= 12 - \frac{10}{a} - \frac{28}{c} \\ c &= 13 - \frac{21}{b} - \frac{6}{a}. \end{aligned}$$

Summing up all three equalities and rearranging, we conclude that:

$$a + \frac{16}{a} + b + \frac{36}{b} + c + \frac{64}{c} = 36.$$

Taking into account that a, b and c are positive and applying AM-GM, we get that $a + \frac{16}{a} \geq 8$, $b + \frac{36}{b} \geq 12$ and $c + \frac{64}{c} \geq 16$. Since $8 + 12 + 16 = 36$ we conclude that actually all three inequalities are satisfied with equality and this is possible only if:

$$a = 4, \quad b = 6, \quad c = 8.$$

For $(a, b, c) = (4, 6, 8)$, we have $\frac{36}{c} = \frac{9}{2}$ and $\frac{15}{b} = \frac{5}{2}$. Therefore:

$$a = 4 = 11 - 7 = 11 - \frac{9}{2} - \frac{5}{2} = 11 - \frac{36}{c} - \frac{15}{b}.$$

Similarly, $\frac{10}{a} = \frac{5}{2}$ and $\frac{28}{c} = \frac{7}{2}$ and therefore:

$$b = 6 = 12 - \frac{5}{2} - \frac{7}{2} = 12 - \frac{10}{a} - \frac{28}{c}.$$

Since two of the equalities are satisfied and the sum of the left hand sides of all three is equal to the sum of the right hand sides of all three equalities, we conclude that the third equality also holds. This shows that $(a, b, c) = (4, 6, 8)$ is indeed a solution of the given system. Hence the unique positive solution of the given system is $(a, b, c) = (4, 6, 8)$. $\square$

Comment. It seems that the dividing by abc and summing up is crucial for untangling the problem.
According to the author, one could solve directly via a 'brute-force approach'. However, note that there are actually *five* solutions for (a, b, c) without the positive assumption, namely:

$$(a, b, c) = \left(-11, \frac{21}{11}, \frac{28}{11}\right), (0, 0, 0), \left(\frac{15}{28}, -\frac{140}{9}, \frac{63}{20}\right), \left(\frac{10}{13}, \frac{15}{13}, -13\right), (4, 6, 8)$$

By considering the LCM of the denominators, this means that any brute force algebra approach to solve for one of the variables will require finding the roots of a non-trivial quartic polynomial and the leading coefficient will be at least 220. For example, solving directly for c gives the polynomial equation $220c^5 - 153c^4 - 27381c^3 + 139132c^2 - 183456c = 0$ and the ones for a, b are even more intractable.

A4.

Let $a \geq b \geq c \geq 0$ be real numbers such that $ab + bc + ca = 3$. Prove that:

$$3 + (2 - \sqrt{3}) \cdot \frac{(b-c)^2}{b + (\sqrt{3}-1)c} \leq a + b + c$$

and determine all the cases when the equality occurs.

Solution. We homogenize and prove a more general statement

$$\sqrt{3}\sqrt{ab + bc + ca} + (2 - \sqrt{3}) \cdot \frac{(b-c)^2}{b + (\sqrt{3}-1)c} \leq a + b + c \tag{1}$$

for all reals $a \geq b \geq c \geq 0$, with $b > 0$.

- Case 1. $c = 0$. Then (1) reduces to $\sqrt{3}\sqrt{ab} + (2 - \sqrt{3})b \leq a + b$. We set $\sqrt{ab} = p$ and $a + b = 2p$. We clearly have $b \leq p$, so we show $p\sqrt{3} + (2 - \sqrt{3})p \leq 2s \Leftrightarrow p \leq s$ which is true by applying AM-GM inequality, with equality when $a = b$.

- Case 2 $c > 0$. Since the inequality (1) is homogeneous, upon division of c and substituting $\frac{a}{c} \to a$ and $\frac{b}{c} \to b$, we may and we do assume that $c = 1$.

  Hence (1) can be rewritten as $\sqrt{3}\sqrt{ab + a + b} + (2 - \sqrt{3}) \cdot \frac{(b-1)^2}{b + \sqrt{3} - 1} \leq a + b + 1$.

  We introduce $2x := a + b \geq 2$. Then by applying AM-GM inequality we have $ab \leq x^2$ and so $\sqrt{3}\sqrt{ab + a + b} \leq \sqrt{3}\sqrt{x^2 + 2x}$.

  Also $b \leq x$. We will show that:

  $$\frac{(b-1)^2}{b + \sqrt{3} - 1} \leq \frac{(x-1)^2}{x + \sqrt{3} - 1}. \tag{2}$$

  We set $b - 1 = u$ and $x - 1 = y$.

  Then $0 \leq u \leq y$ and the inequality (2) takes the form:

  $$\frac{u^2}{u + \sqrt{3}} \leq \frac{y^2}{y + \sqrt{3}} \Leftrightarrow u^2 y + u^2\sqrt{3} \leq y^2 u + y^2\sqrt{3}.$$

  But this is true since $u^2 y \leq y^2 u$ and $u^2\sqrt{3} \leq y^2\sqrt{3}$. In summary:

  $$\sqrt{3}\sqrt{ab + a + b} + (2 - \sqrt{3}) \cdot \frac{(b-1)^2}{b + \sqrt{3} - 1} \leq \sqrt{3}\sqrt{x^2 + 2x} + (2 - \sqrt{3})\frac{(x-1)^2}{x + \sqrt{3} - 1}.$$

Hence, it suffices to show:

$$\sqrt{3}\sqrt{x^2+2x}+(2-\sqrt{3})\frac{(x-1)^2}{x+\sqrt{3}-1} \le 2x+1 \Leftrightarrow (2-\sqrt{3})\frac{(x-1)^2}{x+\sqrt{3}-1} \le \frac{x-1)^2}{\sqrt{3}\sqrt{x^2+2x}+2x+1}. \tag{3}$$

If $x = 1$ this is true, we have equality.

For $x > 1$ (3) is equivalent to:

$$\frac{1}{x+\sqrt{3}-1} \le \frac{2+\sqrt{3}}{\sqrt{3}\sqrt{x^2+2x}+2x+1} \Leftrightarrow \sqrt{3}\sqrt{x^2+2x} \le x\sqrt{3}+\sqrt{3} \Leftrightarrow \sqrt{x^2+2x} \le x+1,$$

which is obviously true, and the equality in the last line is strict, so no equality may occur in this case. Now the proof of the inequality complete.

As to the cases of equality, returning to the main inequality, from Case 1, it holds iff $a = b$ and $c = 0$, i.e. $(a, b, c) = (\sqrt{3}, \sqrt{3}, 0)$. As of the Case 2, it must be the case that $x = 1$, and thus $a + b = 2$. But $a \ge 1$ and $b \ge 1$, therefore $a = b = 1$. Consequently $(a, b, c) = (1, 1, 1)$. Obviously both triples $(\sqrt{3}, \sqrt{3}, 0)$ and $(1, 1, 1)$ satisfy the conditions of the problem and turn the inequality into an equality. □

Alternative Solution. Using the given condition $ab+bc+ca = 3$ we can easily express:

$$a = \frac{3-bc}{b+c}.$$

Thus the condition $a \ge b$ is equivalent to:

$$\frac{3-bc}{b+c} \ge b, \text{ or } b^2 + 2bc \le 3,$$

where we used that $b + c > 0$.
We apply the Sturm's method to show that under the given conditions the difference:

$$a+b+c-\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c}$$

attains minimum when $a = b$.
Let $\gamma \in (0, 1)$ and consider the triples:

$$\begin{aligned}(a, b, c) &= \left(\frac{3-bc}{b+c}, b, c\right) \text{ and}\\ (a', b', c') &= \left(\frac{3-\gamma^2 bc}{\gamma(b+c)}, \gamma b, \gamma c\right)\end{aligned}$$

with $b^2 + 2bc \le 3$. Then obviously $(b')^2 + 2b'c' = \gamma^2(b^2 + 2bc) \le 3$.
Note that:

$$\frac{(2-\sqrt{3})(b'-c')^2}{b'-c'+\sqrt{3}c'} = \gamma\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c} < \frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c},$$

because $\gamma \in (0,1)$. On the other hand:

$$\begin{aligned}
a'+b'+c'-(a+b+c) &= a'-a-(1-\gamma)(b+c) = \frac{3-\gamma^2 bc}{\gamma(b+c)} - \frac{3-bc}{b+c} - (1-\gamma)(b+c) \\
&= \frac{3(1-\gamma)+\gamma bc(1-\gamma)-\gamma(1-\gamma)(b+c)^2}{\gamma(b+c)} \\
&= (1-\gamma)\frac{3+\gamma bc-\gamma(b^2+2bc+c^2)}{\gamma(b+c)} \\
&= (1-\gamma)\frac{3-\gamma(b^2+2bc)+\gamma c(b-c)}{\gamma(b+c)} \\
&> 0,
\end{aligned}$$

where the last inequality follows by the condition that $b^2 + 2bc \le 3$, $\gamma \in (0,1)$ and $b \ge c$. This proves that $a'+b'+c' > a+b+c$ and since $\frac{(2-\sqrt{3})(b'-c')^2}{b'-c'+\sqrt{3}c'} < \frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c}$ we conclude that:

$$a+b+c-\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c} < a'+b'+c'-\frac{(2-\sqrt{3})(b'-c')^2}{b'-c'+\sqrt{3}c'}.$$

Thus if we have (a,b,c) satisfying the conditions of the problem and pick γ such that $b^2+2bc = 3\gamma^2$, then $\gamma \le 1$ and the triple (a',b',c') with $a' = b' = b/\gamma$, $c' = c/\gamma$ attains smaller value than (a,b,c).
Therefore the minimum of:

$$a+b+c-\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c}$$

under the conditions of the problem is attained for $a = b$ and $b^2 + 2bc = 3$ with $b \ge c$. Consequently $b \ge 1$ and $c = \frac{3}{2b} - \frac{b}{2}$. Now it is easy to compute that:

$$\begin{aligned}
b-c &= \frac{3}{2}(b-\frac{1}{b}) \\
b-c+\sqrt{3}c &= b+(\sqrt{3}-1)(\frac{3}{2b}-\frac{b}{2}) = \frac{3-\sqrt{3}}{2}b + \frac{3(\sqrt{3}-1)}{2b} = \frac{(\sqrt{3}-1)}{2}(\sqrt{3}b+\frac{3}{b}) \\
a+b+c &= 2b+c = 2b+\frac{3}{2b}-\frac{b}{2} = \frac{3}{2}(b+1/b).
\end{aligned}$$

Therefore:

$$\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c} = \frac{9(\sqrt{3}-1)^2(b-1/b)^2}{8}\frac{1}{\frac{(\sqrt{3}-1)}{2}(\sqrt{3}b+\frac{3}{b})} = \frac{9(\sqrt{3}-1)(b-1/b)^2}{4(\sqrt{3}b+\frac{3}{b})}$$

and thus by doing some technical but standard arithmetic we obtain:

$$\begin{aligned}
a+b+c-\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c} &= \frac{3}{2}(b+1/b)-\frac{9(\sqrt{3}-1)(b^2-2+1/b^2)}{4(\sqrt{3}b+3/b)} \\
&= \frac{(3-\sqrt{3})}{4}(\sqrt{3}b+3/b)+\frac{6\sqrt{3}+18}{4(\sqrt{3}b+3/b)} \\
&= \frac{(3-\sqrt{3})}{4}\left[(\sqrt{3}b+3/b)+\frac{(3+\sqrt{3})^2}{\sqrt{3}b+3/b}\right] \\
(AM \geq GM) \qquad &\geq \frac{(3-\sqrt{3})}{4}.2(3+\sqrt{3}) = 3
\end{aligned}$$

and the desired inequality is proven. In order to have equality as we have shown $a=b$ and in the last line it should be the case that $\sqrt{3}b+3/b = 3+\sqrt{3}$ with $b \geq 1$. This is equivalent to $\sqrt{3}(b-\sqrt{3})-\sqrt{3}/b(b-\sqrt{3}) = 0$. Therefore $b=\sqrt{3}$ or $b=1$. This shows that the only candidates for attaining the minimum are the triples $(\sqrt{3},\sqrt{3},0)$ and $(1,1,1)$. Substituting in the given expression we see that for $(a,b,c)=(\sqrt{3},\sqrt{3},0)$:

$$3+\frac{(2-\sqrt{3})(b-c)^2}{b-c+\sqrt{3}c} = 3\frac{(2-\sqrt{3}).3}{\sqrt{3}} = 3+2\sqrt{3}-3 = 2\sqrt{3} = a+b+0 = a+b+c$$

and for $(a,b,c)=(1,1,1)$ it is straightforward that the irregular term is 0 and that $3=1+1+1=a+b+c$.

□

Comment. The two solutions explore two different ideas. The first one uses a homogenisation and then trivialises the most inconvenient variable c. The second one does not change the problem but instead is guided by the basic idea to eliminate the least used variable a in the natural pursuit to determine the cases where the equality is attained. This is why the first solution might seem more trickier whereas the second one more technical.

A5.

Let $\mathbb{R}^+ = (0, \infty)$ be the set of positive real numbers. Find all non-negative real numbers $c \geq 0$ such that there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the property:

$$f(y^2 f(x) + y + c) = xf(x + y^2)$$

for all $x, y \in \mathbb{R}^+$.

Solution. We prove that no such real numbers exist.
For the sake of contradiction, assume that $c \geq 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the assumptions of the problem. We write $P(x, y)$ as an abbreviation for the property of $x, y \in \mathbb{R}^+$ that:

$$xf(x + y^2) = f(y^2 f(x) + y + c).$$

- Step 1. f is non-increasing. By $P(x, \sqrt{y})$ we first obtain:

  $$xf(x + y) = f(yf(x) + \sqrt{y} + c).$$

  Next, by $P(x + z, \sqrt{y})$ we further get:

  $$(x + z)f(x + y + z) = f(yf(x + z) + \sqrt{y} + c)$$

  and combining both equalities we conclude that:

  $$\begin{aligned}(x + z)f(x + y + z) &= xf(x + (y + z)) + zf((x + y) + z) \\ &= f((y + z)f(x) + \sqrt{y + z} + c) + f((x + y)f(z) + \sqrt{x + y} + c)\end{aligned}$$

  and therefore

  $$f(yf(x+z)+\sqrt{y}+c) = f((y+z)f(x)+\sqrt{y+z}+c)+f((x+y)f(z)+\sqrt{x+y}+c) \quad (1)$$

  Now assume that $x, z \in \mathbb{R}^+$ but, contrary to our endeavour, $f(x+z) > f(z)$. Fixing such a pair (x, z), we shall prove that there is $y \in \mathbb{R}^+$ with the property:

  $$yf(x + z) + \sqrt{y} + c = (x + y)f(z) + \sqrt{x + y} + c.$$

  We consider this as an equation with respect to y. All we need is to show that this equation admits a positive root. Substituting y with xy^2, we subsequently obtain:

  $$\begin{aligned}xy^2 f(x + z) + \sqrt{x}y + c &= (xy^2 + x)f(z) + \sqrt{x + xy^2} + c \\ \sqrt{x}y^2 f(x + z) + y &= \sqrt{x}y^2 f(z) + \sqrt{x}f(z) + \sqrt{1 + y^2} \\ \frac{y^2(\sqrt{x}f(x + z) - \sqrt{x}f(z)) + y - \sqrt{x}f(z)}{\sqrt{y^2 + 1}} &= 1.\end{aligned}$$

Note that letting y tend to 0 the left hand side tends to $-\sqrt{x}f(z) < 0 < 1$. On the other hand letting y tend to infinity, since $f(x+z) > f(z)$, the expression on the left hand side tends to ∞. Therefore the last equation has a positive root $y_0 \in (0, \infty)$ and thus:

$$f(y_0 f(x+z) + \sqrt{y_0} + c) = f((x+y_0)f(z) + \sqrt{x+y_0} + c).$$

But now by (1) with $y = y_0$ we obtain that:

$$f((y_0+z)f(x) + \sqrt{y_0+z} + c) = 0$$

which is a contradiction. Therefore no $x, z \in \mathbb{R}^+$ exist with the property $f(x+z) > f(x)$.

- Step 2. Now we prove that f is constant on some interval $[x_0, \infty)$. First note that if $a > b$ and $f(a) = f(b)$, then, since f is non-increasing, $f(x) = f(a)$ for every $x \in [b, a]$. Now by $P(1, \sqrt{y})$ we have:

$$f(y+1) = f(yf(1) + \sqrt{y} + c).$$

Next, we make case distinction w.r.t. $f(1)$:

  - $f(1) \geq 1$. Then for $y \geq 4$, we have that $yf(1)+\sqrt{y}+c \geq y+2$. Therefore by the above remark, we have that $f(y+1) = f(y+2)$ for any $y \geq 4$. Consequently, $f(y)$ is constant on $[5, \infty)$.
  - $f(1) < 1$. Thus for y large enough $yf(1) + \sqrt{y} < y$ and therefore, again by the remark above, $f(y) = f(y+1)$. Consequently there is x_0 such that on $[x_0, \infty)$, f is constant.

Now we arrive at a contradiction. Let $x > \max(x_0, 1)$ and $y > x_0$. Then $x + y^2 > x_0$ and $y^2 f(x) + y + c > x_0$. Consequently:

$$f(x+y^2) = f(y^2 f(x) + y + c).$$

On the other hand, by $P(x, y)$, we have:

$$xf(x+y^2) = f(y^2 f(x) + y + c).$$

For $x > 1$, by the positivity of f, this is impossible. □

Alternative Solution. This solution is based on considering the equation

$$(f(x)-1)y^2 + y + c - x = 0. \tag{2}$$

Indeed, if $f(x) > 1$ for some $x > c$ then (2) would have a positive solution, i.e., $y = \frac{-1+\sqrt{1-4(c-x)(f(x)-1)}}{2(f(x)-1)}$ and plugging that (x, y) into the original equation we find that $x = 1$. So, for all $x > c$ except possibly $x = 1$ we have $f(x) \leq 1$. On the other hand, choose $x \in (c, c+1/4)$ then if there is some x in the afore-mentioned interval such that $f(x) < 1$ then $1 - 4(c-x)(f(x)-1) > 1 + f(x) - 1 = f(x) > 0$ and the equation (2) has a positive solution, i.e., $y = \frac{-1-\sqrt{1-4(c-x)(f(x)-1)}}{2(f(x)-1)}$. By analogous reasoning, we find that $x = 1$. Hence, for all $x \in (c, c+1/4)$ except possibly 1 we have $f(x) = 1$. Now, choose $r, s \in (c, c+1/4), 1 \neq r \neq s \neq 1$ and choose y small enough to ensure that $r + y^2, s + y^2 \in (c, c+1/4)$ and $y^2 f(r) + y + c = y^2 + y + c \in (c, c+1/4)$. Then, plugging $(x, y) = (r, y), (s, y)$ in the original equality, it follows that $r = rf(r+y^2) = sf(s+y^2) = s$. Impossible. Therefore no function f with the desired properties exists. □

Alternative Solution. Assume that $c \geq 0$ is such that:

$$xf(x + y^2) = f(y^2 f(x) + y + c)$$

holds for any $x, y \in \mathbb{R}^+$.

1. Finding periods. Setting $x = 1$ and $u = y^2$, we get:

$$f(u + 1) = f(y^2 + 1) = f(f(1)y^2 + y + c) = f(f(1)u + \sqrt{u} + c).$$

   This shows that for any $u > 0$ there are x_1 and x_2 such that $f(x_1) = f(x_2)$ and:

$$x_2 - x_1 = |(f(1) - 1)u + \sqrt{u} + c - 1| =: T(u).$$

   Furthermore $x_1 = f(1)u + \sqrt{u} + c$ if $f(1)u + \sqrt{u} + c < u + 1$ and $x_1 = u + 1$, otherwise.

2. Some structure of the periods. On the other hand whenever $f(x_1) = f(x_2)$, $T = x_2 - x_1$ and $u > x_1$, we can write $u = x_1 + y^2$ for an appropriate $y > 0$. Therefore:

$$x_1 f(u) = x_1 f(x_1 + y^2) = f(y^2 f(x_1) + y + c) \text{ and}$$
$$x_2 f(u + T) = x_2 f(x_1 + T + y^2) = x_2 f(x_2 + y^2) = f(y^2 f(x_2) + y + c).$$

   Taking into account that $f(x_1) = f(x_2)$, we conclude that the right hand sides of both equalities are equal and consequently:

$$x_1 f(u) = x_2 f(u + T)$$

   holds for any $u > x_1$. Therefore for $u > x_1$ and recalling that $x_2 = x_1 + T$, we have:

$$\frac{x_1 + T}{x_1} = \frac{f(u+T)}{f(u)} \text{ and } \left(\frac{x_1 + T}{x_1}\right)^2 = \frac{f(u+2T)}{f(u+T)}\frac{f(u+T)}{f(u)} = \frac{f(u+2T)}{f(u)}.$$

3. Combining steps 1 and 2. To conclude the proof we combine steps 1 and 2. Let t be large enough. Then since $T(u) = |(f(1)-1)u + \sqrt{u} + c - 1|$ continuously tends to infinity when u tends to infinity, there is $u = U(t)$ such that $T(u) = t$. Let $x_1(t) = U(t) + 1$ if $U(t) + 1 < f(1)U(t) + \sqrt{U(t)} + c$ and $x_1(t) = U(t) + 1 - t$, otherwise. Then by step 2, we conclude that:

$$\left(1 + \frac{t}{x_1(t)}\right)^2 = \frac{f(y+2t)}{f(y)} = 1 + \frac{2t}{x_1(2t)},$$

where $y > \max(x_1(t) + t, x_1(2t) + 2t)$. Consequently, we get that:

$$\frac{2t}{x_1(t)} + \frac{t^2}{x_1^2(t)} = \frac{2t}{x_1(2t)}$$

or equivalently,

$$tx_1(2t) = 2x_1(t)(x_1(t) - x_1(2t)).$$

Yet, $x_1(t)$ tends to infinity as t does so. It follows that for t large enough $x_1(t) < x_1(2t)$. This is a contradiction since the left hand side is positive as is $2x_1(t)$.

Consequently, there are no non-negative real numbers c with the desired property.

□

Comment. Actually the original solution and the second alternative solution share the same ideas but they are organised in a different order. This suggests that the structure of the problem can be revealed from different angles and the pieces of the puzzle can be assembled differently to yield a complete solution.

The first alternative solution shows that the structure of the problem is much more regular and it can be revealed by systematically studying the behaviour of the quadratic function $f(x)y^2 + y + c = y^2 + x$ with respect to y.

A6.

Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $g(x)$ with non-negative coefficients and $g(0) = 0$ that satisfy the equality:

$$f(f(x) + g(y)) = f(x - y) + 2y$$

for all positive real numbers $x > y$.

Solution. Assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ and the polynomial g with non-negative coefficients and $g(0) = 0$ satisfy the conditions of the problem. For positive reals with $x > y$, we shall write $P(x, y)$ for the relation:

$$f(f(x) + g(y)) = f(x - y) + 2y.$$

1. Step 1. $f(x) \geq x$. Assume that this is not true. Since $g(0) = 0$, $g(x) + x$ is injective on positive reals. If $f(x) < x$ for some positive real x, then setting y such that $y + g(y) = x - f(x)$ (where obviously $y < x$), we shall get $f(x) + g(y) = x - y$ and by $P(x, y)$, $f(f(x) + g(y)) = f(x - y) + 2y$, we get $2y = 0$, a contradiction.

2. Step 2. $g(x) = cx$ for some non-negative real c. We will show $\deg g \leq 1$ and together with $g(0) = 0$ the result will follow. Assume the contrary. Hence there exists a positive l such that $g(x) \geq 2x$ for all $x \geq l$. By Step 1 we get

$$\forall x > y \geq l : f(x - y) + 2y = f(f(x) + g(y)) \geq f(x) + g(y) \geq f(x) + 2y$$

and therefore $f(x - y) \geq f(x)$. We get $f(y) \geq f(2y) \geq \cdots \geq f(ny) \geq ny$ for all positive integers n, which is a contradiction.

3. Step 3. If $c \neq 0$, then $f(f(x) + y + c^2 + 2) = f(x + 1) + y + 2c$. Indeed by $P(f(x + \frac{y}{2} + 1) + \frac{cy}{2} + c, c)$, we get

$$f(f(f(x + \frac{y}{2} + 1) + \frac{cy}{2} + c) + c^2) = f(f(x + \frac{y}{2} + 1) + \frac{cy}{2}) + 2c = f(x + 1) + y + 2c.$$

On the other hand by $P(x + \frac{y}{2} + 1, \frac{y}{2} + 1)$, we have:

$$f(x) + y + 2 = f\left(f\left(x + \frac{y}{2} + 1\right) + g\left(\frac{y}{2} + 1\right)\right) = f\left(f\left(x + \frac{y}{2} + 1\right) + \frac{cy}{2} + c\right).$$

Substituting in the LHS of $P(f(x + \frac{y}{2} + 1) + \frac{cy}{2} + c, c)$, we get $f(f(x) + y + 2 + c^2) = f(x + 1) + y + 2c$.

4. Step 4. There is x_0, such that $f(x)$ is linear on (x_0, ∞). If $c \neq 0$, then by Step 3, fixing $x = 1$, we get $f(y + f(1) + 2 + c^2) = y + f(2) + 2c$ which implies that f is linear for $y > f(1) + 2 + c^2$. As for the case $c = 0$, consider $y, z \in (0, \infty)$. Pick $x > \max(y, z)$, then by $P(x, x - y)$ and $P(x, x - z)$ we get:

$$f(y) + 2(x - y) = f(f(x)) = f(z) + 2(x - z)$$

which proves that $f(y) - 2y = f(z) - 2z$ and there fore f is linear on $(0, \infty)$.

5. Step 5. $g(y) = y$ and $f(x) = x$ on (x_0, ∞). By Step 4, let $f(x) = ax + b$ on (x_0, ∞). Since f takes only positive values, $a \geq 0$. If $a = 0$, then by $P(x + y, y)$ for $y > x_0$ we get:

$$2y + f(x) = f(f(x + y) + g(y)) = f(b + cy).$$

Since the LHS is not constant, we conclude $c \neq 0$, but then for $y > x_0/c$, we get that the RHS equals b which is a contradiction.

Hence $a > 0$. Now for $x > x_0$ and $x > (x_0 - b)/a$ large enough by $P(x + y, y)$ we get:

$$ax+b+2y = f(x)+2y = f(f(x+y)+g(y)) = f(ax+ay+b+cy) = a(ax+ay+b+cy)+b.$$

Comparing the coefficients before x, we see $a^2 = a$ and since $a \neq 0$, $a = 1$. Now $2b = b$ and thus $b = 0$. Finally, equalising the coefficients before y, we conclude $2 = 1 + c$ and therefore $c = 1$.

Now we know that $f(x) = x$ on (x_0, ∞) and $g(y) = y$. Let $y > x_0$. Then by $P(x + y, x)$ we conclude:

$$f(x) + 2y = f(f(x + y) + g(y)) = f(x + y + y) = x + 2y.$$

Therefore $f(x) = x$ for every x. Conversely, it is straightforward that $f(x) = x$ and $g(y) = y$ do indeed satisfy the conditions of the problem. $\square$

Alternative Solution. Assume that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and the polynomial with non-negative coefficients $g(y) = yg_1(y)$ satisfy the given equation. Fix $x = x_0 > 0$ and note that:

$$f(f(x_0 + y) + g(y)) = f(x_0 + y - y) + 2y = f(x_0) + 2y.$$

Assume that $g = 0$. Then $f(f(x + y)) = f(x) + 2y$ for $x, y > 0$. Let $x > 0$ and $z > 0$. Pick $y > 0$. Then:

$$2y + f(x + z) = f(f(x + y + z)) = f(f(x + z + y)) = f(x) + 2(z + y).$$

Therefore $f(x+z) = f(x) + 2z$ for any $x > 0$ and $z > 0$. Setting $c = f(1)$, we see that $f(z+1) = c + 2z$ for all positive z. Therefore if $x, y > 1$ we have that $f(x+y) = c + 2(x+y-1) > 1$. This shows that:

$$f(f(x+y)) = c + 2(f(x+y) - 1) = 3c + 4(x+y) - 4.$$

On the other hand $f(x)+2y = c+2x+2y$. Therefore the equality $f(f(x+y)) = f(x)+2y$ is not universally satisfied.
From now on, we assume that $g \neq 0$. Therefore g is strictly increasing with $g(0) = 0$, $\lim_{y\to\infty} g(y) = \infty$, i.e. g is bijective on $[0, \infty)$ and $g(0) = 0$.
Let $x > 0, y > 0$ and set $u = f(x+y)$, $v = g(y)$. From above, we have $u > 0$ and $v > 0$. Therefore:

$$f(f(u+v) + g(v)) = f(u) + 2v = f(f(x+y)) + 2g(y).$$

On the other hand $f(u+v) = f(f(x+y) + g(y)) = f(x) + 2y$. Therefore we obtain that:

$$f(f(x) + 2y + g(g(y))) = f(f(x+y)) + 2g(y).$$

Since g is bijective from $(0, \infty)$ to $(0, \infty)$ for any $z > 0$ there is t such that $g(t) = z$. Applying this observation to $z = g(g(y)) + 2y$ and setting $x' = x + t$, we obtain that:

$$f(f(x+t+y))+2g(y) = f(f(x'+y))+2g(y) = f(f(x')+g(g(y))+2y) = f(f(x+t)+g(t)) = f(x)+2t.$$

Thus if we denote $h(y) = g(g(y)) + 2y$, then $t = g^{-1}(h(y))$ and the above equality can be rewritten as:

$$f(f(x+g^{-1}(h(y))+y)) = f(x)+2g^{-1}(h(y))-2g(y) = f(x)+2g^{-1}(h(y))+2y-2y-2g(y).$$

Let $s(y) = g^{-1}(h(y)) + y$ and note that since h is continuous and monotone increasing, g is continuous and monotone increasing, then so are g^{-1} and consequently $g^{-1} \circ h$ and s. It is also clear, that $\lim_{y\to 0} s(y) = 0$ and $\lim_{y\to\infty} s(y) = \infty$. Therefore s is continuously bijective from $[0, \infty)$ to $[0, \infty)$ with $s(0) = 0$.
Thus we have:

$$f(f(x + s(y))) = f(x) + 2s(y) - 2y - 2g(y)$$

and using that s is invertible, we obtain:

$$f(f(x+y)) = f(x) + 2y - 2s^{-1}(y) - 2g(s^{-1}(y)).$$

Now fix x_0, then for any $x > x_0$ and any $y > 0$ we have:

$$\begin{aligned} f(x) + 2y - 2s^{-1}(y) - 2g(s^{-1}(y)) &= f(f(x+y)) = f(f(x_0 + x + y - x_0)) \\ &= f(x_0) + 2(x + y - x_0) - 2s^{-1}(x + y - x_0) - 2g(s^{-1}(x + y - x_0)). \end{aligned}$$

Setting $y = x_0$, we get:

$$f(x) + 2x_0 - 2s^{-1}(x_0) - 2g(s^{-1}(x_0)) = f(x_0) + 2x - 2s^{-1}(x) - 2g(s^{-1}(x)).$$

Since this equality is valid for any $x > x_0$ we actually have that:

$$f(x) - 2x + 2s^{-1}(x) + 2g(s^{-1}(x)) = c \text{ for some fixed constant } c \in \mathbb{R} \text{ and all } x \in \mathbb{R}^+.$$

Let $\phi(x) = -x + 2s^{-1}(x) + 2g(s^{-1}(x))$. Then:

$$f(f(x+y)+g(y)) = f(x+y+\phi(x+y)+c+g(y)) = x+y+g(y)+\phi(x+y)+2c+\phi(x+y+\phi(x+y)+c+g(y)).$$

On the other hand:

$$f(f(x+y)+g(y)) = f(x) + 2y = x + \phi(x) + 2y + c.$$

Therefore:

$$g(y) + \phi(x+y) + c + \phi(x + y + \phi(x+y) + c + g(y)) = \phi(x) + y + c.$$

Noting that ϕ is continuous on $[0, \infty)$, since it is sum of continuous functions, and letting y tend to 0, we obtain that:

$$\phi(x) + c + \phi(x + \phi(x) + c) = \phi(x).$$

Therefore $\phi(x + \phi(x) + c) + c = 0$ and substituting in the definition of $\phi(x) = -x + 2s^{-1}(x) + 2g(s^{-1}(x))$ we obtain:

$$-x - \phi(x) - c + c + 2s^{-1}(x + \phi(x) + c) + 2g(s^{-1}(x + \phi(x) + c)) = 0.$$

Consequently:

$$c + 2s^{-1}(x + \phi(x) + c) + 2g(s^{-1}(x + \phi(x) + c)) = x + \phi(x) + c.$$

Thus:

$$\phi(c + 2s^{-1}(x + \phi(x) + c) + 2g(s^{-1}(x + \phi(x) + c))) = \phi(x + \phi(x) + c) = -c.$$

Finally note that $x + \phi(x) + c = 2s^{-1}(x) + 2g(s^{-1}(x)) + c =: u(x)$ and since g and s^{-1} are monotone and bijective on $[0, \infty)$, $u(x)$ exhausts $[c, \infty)$ when x ranges on $[0, \infty)$. It follows that $\phi(x) = -c$ for $x \in [c, \infty)$. It follows that for $x > \max(c, 0)$:

$$f(x) = x + c - \phi(x) = x - 2c.$$

In particular, since $f(x) > 0$, $c \leq 0$. Now for $x > \max(c, 0)$ and $y > 0$ we have:

$$x + 2y - 2c = f(x) + 2y = f(f(x+y) + g(y)) = f(x + y - 2c + g(y)) = x + y + g(y) - 4c.$$

Since this is valid for any y, we conclude $g(y) = y$ and $c = 0$. Now it follows that $f(x) = x$ for $x \in (0, \infty)$.
It is also straightforward to check that $f(x) = x$ and $g(y) = y$ satisfy the equality:

$$f(f(x+y) + g(y)) = f(x + 2y) = x + 2y = f(x) + 2y.$$

$\square$

Comment. Considering the remark for the 1st solution; after finding $(f(z) + 2x + 2 + c^2) = f(z+1) + 2x + 2c$, we can fix z and conclude that f is linear for large enough x. Then, in the $(f(x+y)+cy) = f(x)+2y$, if we fix x and make y large enough, the problem would be broken.
The ideas in both solutions are similar. It is only that they are organised in a different way. Whereas the first solution first determines the structure of the solution (f, g), the approach in the second solution is more general and actually all it uses is that g (in the non-trivial case $g = 0$) can be extended to a monotone bijective (and thus continuous) function from $[0, \infty) \to [0, \infty)$ with $g(0) = 0$. Since the class of these functions is closed under addition and multiplication. The class of the polynomials with non-negative coefficients can be considered as the closure of the functions $x \to cx$ for $c > 0$ with respect to addition and multiplication. Even though the class of polynomials in $\mathbb{R}^+[x]$ does not exhaust the richness of the above functions, it is very natural and easy to describe – a property that is crucial for the wording of the problem.
Actually, one can further weaken the assumptions on g so that the functions s and h that appear in the second solution are bijective and continuously tend to 0 at 0, but it seems that these properties are not that simple to describe the class of these functions g in a natural way.

Combinatorics – Solutions

C1.

Let n, k be positive integers. Julia and Florian play a game on a $2n \times 2n$ board. Julia has secretly tiled the entire board with invisible dominos. Florian now chooses k cells. All dominos covering at least one of these cells then turn visible. Determine the minimal value of k such that Florian has a strategy to always deduce the entire tiling.

Solution. The minimal value of k with this property is n^2.

We first show that in order for Florian to be able to deduce the entire tiling, we must have $k \geq n^2$. If Julia picks an independent tiling on each of n^2 disjoint 2×2 regions, then Florian needs to reveal at least one domino from each region to deduce the entire tiling. Hence $k \geq n^2$.

Now colour the squares of the board with 4 different colours, such that the colouring is periodic with period 2 squares in both horizontal and vertical direction. We show that if Florian reveals the dominos covering the n^2 squares of one colour class, then there is at most one tiling of the board that contains this arrangement of revealed dominos.

Let red be one of the 4 colours and assume that we have two distinct tilings A and B of the board that agree on all the dominos covering a red square. We call a square of the board *augmented* if it is covered in a different way by A and B. Given an augmented square s, let $a(s)$ and $b(s)$ be the two squares covered by the same domino in the tiling A and B, respectively. By definition, we have $a(s) \neq b(s)$ and $a(s)$ and $b(s)$ must both be augmented as well. Repeating this argument, we find distinct augmented squares $s_1, s_2, s_3, \ldots, s_m$ with

$$(a(s_k), b(s_k)) = (s_{k+1}, s_{k-1}),$$

where indices are taken modulo m. Hence, there is a closed path P of orthogonally neighbouring squares that does not contain a red square and that is tiled by the restriction of both tilings A and B.

Note however, that any orthogonal, closed path that avoids red squares must enclose an orthogonally connected interior of odd size. This is easily shown by induction on the number of red squares enclosed: Consider the top-most, left-most red square x in the interior of the closed path. Of its four orthogonal neighbours either two, three or all four are in the path. In the last case the interior is a single square, in the others we can locally alter the path to exclude x from the interior while reducing the interior by an even number of squares.

Now, since the tilings A and B tile the path P, they must also tile the odd sized region enclosed by P. This is impossible since every domino covers exactly 2 squares, contradicting our initial assumption. We conclude that if two tilings agree on the red squares, they must in fact be the same tiling. Hence, if Florian knows how the red squares are tiled, there is a unique way to complete this tiling to the entire board, which he can deduce by searching though all the finitely many tilings of the board. $\square$

C2.

Let $n \geq 2$ and $S = \{1, 2, \dots, n^2\}$. For any function $f : S \to S$ let $\text{Fix}(f) = \{x \in S \mid f(x) = x\}$. Find the possible values of the expression

$$|\,\text{Fix}(f)| + |\,\text{Im}(f)| + \max_{k \in S} |f^{-1}(k)|$$

as f ranges over all functions $f : S \to S$.

Solution. We show that the answer is all values from $2n$ to $2n^2 + 1$. Assume f has $k \in \{0, 1, \dots, n^2\}$ fixed points. Then say $|\,\text{Im}(f)| = p$. Also let $s = \max_{k \in S} |f^{-1}(k)|$.
Upper Bound: From the definitions of s and p, we get $sp \geq n^2$. We also have the bound $s \leq n^2 - p + 1$ (as p values are in the image, each of which has at least 1 preimage). We deduce the upper bound:

$$k + p + s \leq 2k + i + (n^2 - p + 1) = n^2 + k + 1 \leq 2n^2 + 1.$$

Lower Bound: For the minimum value, using AM-GM we get $k + p + s \geq k + p + \frac{n^2}{p} \geq k + 2n \geq 2n$.

Now we show all those values are achievable by induction on n. A manual check solves the base case of $n = 2$ (the identity achieves the maximum value of 9, a 2-to-1 function can take the values $4, 5, 6$ depending on the number of fixed points, and a function that's 3-to-1 on three of the inputs can achieve 7 and 8). For the inductive hypothesis now, suppose we have a function $g : S \to S$ and let $T = \{1, 2, \dots, (n+1)^2\}$. We will build a function $f : T \to T$ from g by $f(x) = g(x)$ for $x \leq n^2$.
Values from $4n + 1$ to $2(n+1)^2 + 1$: For the values bigger than n^2, f can now be defined as any permutation of the numbers $\{n^2 + 1, \dots, (n+1)^2\}$. Obviously, this won't add to the maximum size of a preimeage, and it will add $2n+1$ to the size of the image. For the number of fixed points, this can be any number from $0, 1, \dots, 2n+1$. So using this, we can add any number from $2n+1$ to $4n+2$ to the value of the expression for g, which means, by the inductive hypothesis, we can hit all values $4n+1$ to $2n^2+1+(4n+2) = 2(n+1)^2+1$.
Values from $2n + 2$ to $4n$: If $s \geq n + 1$, then send n of the new points (from $n^2 + 1$ to $(n+1)^2$) to one of those new points and the other $n+1$ points to another one of the new points in such a way that we don't add new fixed points. This way, we don't increase the maximum preimage size or the number of fixed points, but we add 2 to the image size.
On the other hand, if $s \leq n$, then $|\,\text{Im}(f)| \geq n$. If $|\,\text{Im}(f)| \geq n + 1$, take $n + 1$ of the new points, assign each of them to exactly one of the elements in $\text{Im}(f)$. The other n new points will all get mapped to another new point (again, with no new fixed points). This

way we add no fixed points, we add 1 point in the image, and we increase the maximum preimage by 1, so again we add 2 overall. The remaining case is when $s = |\operatorname{Im}(f)| = n$. In this case we can only assign n of the new points to exactly one of the new elements in $\operatorname{Im}(f)$ and we map the other $n+1$ to a single new point.

In all cases, we add 2 to the expression overall, so we can get all values $2n+2, \ldots, 2n^2+3$, and that covers all values. $\square$

C3.

Let $n \geqslant 3$. Alice and Bob play the following game: Alice chooses $k \in \{3, 4, \ldots, n\}$ and draws a $3 \times k$ table, then he fills the k cells of the first row with different numbers from $\{1, 2, \ldots, n\}$. Then, Bob fills on the second row some of the cells (eventually none) with distinct numbers from $\{1, 2, \ldots, n\}$, and the rest of them with 0. Finally, on each cell of the third row we write the sum of the two cells above. Show that regardless how Alice plays, Bob can guarantee that on the third row he can obtain, in some order, the terms of a non-constant arithmetical progression.

Solution. Let $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant n$ be the numbers Alice chose. For a sequence $x_1 < x_2 < \cdots < x_k$ of positive integers, we call it's *deficit* the set $\mathbb{N} \cap [x_1, x_k] \setminus \{x_1, x_2, \ldots, x_k\}$.
Bob has the following strategy: he starts with $a_1 < a_2 < \cdots < a_k$. Let t be the maximum number of its deficit. If we denote $\delta = t - a_1 < n - 1$, Bob writes under a_1 the number $a_1' = \delta$. Then $a_1 + a_1' = t$. If $t < a_2$, then $a_2, a_3, \ldots, a_k$ are consecutive and $t = a_2 - 1$. So Bob writes under all the rest 0, and he gets on the third row a progression with unit ratio.
Otherwise, we have $t > a_2$ and Bob repeats the process, but for the sequence $t, a_2, \ldots, a_k$. The lowest term is a_2 and it's deficit does not have t, so has lower cardinal. Therefore, each step decreases the cardinality of the deficit. As long as the deficit is not empty, Bob can perform another step, so in the end the deficit will be empty and the numbers on the third row will form an arithmetic progression with unit ratio.
As the δ values are strictly decreasing, Bob fulfils the requirement of using numbers from 1 to n only once. □

Comment. The statement of the problem may be stated also as follows:
Show that for every integer sequence $a_1 < \cdots < a_k = n$ one can choose integers $0 \leq b_1, \ldots b_k \leq n$ such that sums $a_i + b_i$ form a permutation of a non-constant arithmetical progression and all the nonnegative b_i's are distinct.

C4.

Prove that for every positive integer k there exists an integer n and distinct primes $p_1, p_2, \ldots, p_k$ such that, if $A(n)$ denotes the number of integers in $\{1, 2, \ldots, n\}$ which are relatively prime to $p_1 p_2 \cdots p_k$, then

$$\left| n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) - A(n) \right| > 2^{k-3}.$$

Solution. If $k = 1$, choose $p_1 = 3$, $n = 2$. If $k = 2$, choose $p_1 = 3$, $p_2 = 7$, $n = 5$. Assume $k \geq 3$. Let $p_1, p_2, \ldots, p_k$ be primes congruent to 3 modulo 4. By the Chinese Remainder Theorem choose $n \equiv \frac{p_i+1}{4} \pmod{p_i}$ for every i, that is, choose an integer n such that $p_1 p_2 \cdots p_k \mid 4n - 1$.

Consider a fractional part $\theta = \left\{\frac{n}{q_1 q_2 \cdots q_r}\right\}$, where $q_1, q_2, \ldots, q_r$ are different primes among $p_1, p_2, \ldots, p_k$. Note that $\theta \approx \frac{1}{4}$ if r is odd and $\theta \approx \frac{3}{4}$ if r is even.

If r is odd, then working modulo 4, we get $4n - 1 = q_1 q_2 \cdots q_r \cdot (4m + 1)$, for some integer m. It follows that

$$\frac{n}{q_1 q_2 \cdots q_r} = m + \frac{1}{4} + \frac{1}{4q_1 q_2 \cdots q_r} \quad \text{and} \quad \left\{\frac{n}{q_1 q_2 \cdots q_r}\right\} = \frac{1}{4} + \frac{1}{4q_1 q_2 \cdots q_r}.$$

If r is even, then $4n - 1 = q_1 q_2 \cdots q_r \cdot (4m + 3)$, for some integer m. Hence

$$\frac{n}{q_1 q_2 \cdots q_r} = m + \frac{3}{4} + \frac{1}{4q_1 \cdots q_r} \quad \text{and} \quad \left\{\frac{n}{q_1 q_2 \cdots q_r}\right\} = \frac{3}{4} + \frac{1}{4q_1 q_2 \cdots q_r}.$$

The difference between $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$ and $A(n)$ equals to

$$\begin{aligned}
&\{n\} - \sum_{i=1}^{k}\left\{\frac{n}{p_i}\right\} + \sum_{1\leq i<j\leq k}\left\{\frac{n}{p_i p_j}\right\} - \cdots + (-1)^k\left\{\frac{n}{p_1 p_2 \cdots p_k}\right\} \\
&= -\sum_{i=1}^{k}\frac{1}{4} + \sum_{1\leq i<j\leq k}\frac{3}{4} - \cdots - \sum_{i=1}^{k}\frac{1}{4p_i} + \sum_{1\leq i<j\leq k}\frac{1}{4p_i p_j} - \cdots + (-1)^k\frac{1}{4p_1 p_2 \cdots p_k} \\
&= \frac{3}{4}\cdot 2^{k-1} - \frac{1}{4}\cdot 2^{k-1} + \frac{1}{4}\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_k}\right) - 1 \\
&= 2^{k-2} + \frac{1}{4}\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_k}\right) - 1 \\
&> 2^{k-3},
\end{aligned}$$

for $k \geq 3$, as desired. $\square$

Alternative Solution. Note that

$$A(n) = n - \sum_{i=1}^{k} \left\lfloor \frac{n}{p_i} \right\rfloor + \sum_{1 \leq i < j \leq k} \left\lfloor \frac{n}{p_i p_j} \right\rfloor - \cdots + (-1)^k \left\lfloor \frac{n}{p_1 p_2 \cdots p_k} \right\rfloor .$$

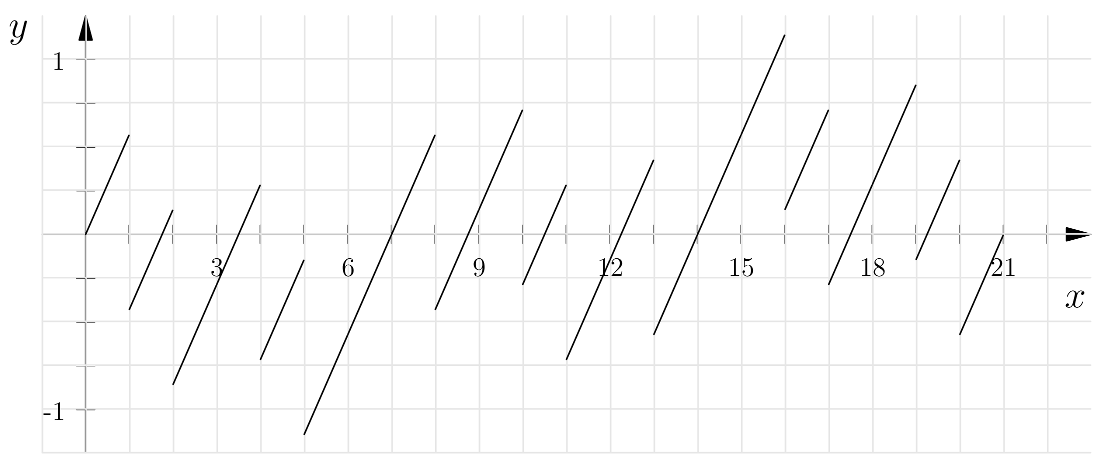
Denote by $\Pi_k = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$ and $f_k(n) = n\Pi_k - A(n)$, then

$$f_k(n) = \{n\} - \sum_{i=1}^{k} \left\{ \frac{n}{p_i} \right\} + \sum_{1 \leq i < j \leq k} \left\{ \frac{n}{p_i p_j} \right\} - \cdots + (-1)^k \left\{ \frac{n}{p_1 p_2 \cdots p_k} \right\} .$$

In particular, the function $f_k(n)$ satisfies

$$f(n) - f(n-1) = \begin{cases} \Pi_k - 1, \text{ if } \gcd(n, p_1 p_2 \cdots p_k) = 1; \\ \Pi_k, \text{ if } \gcd(n, p_1 p_2 \cdots p_k) > 1. \end{cases}$$

We consider the function $f_k(x)$ over the real numbers. Note that $f_k(x)$ is periodic: $f_k(x + p_1 p_2 \cdots p_k) = f_k(x)$. Also, it satisfies $f_k(x) = -f_k(p_1 p_2 \cdots p_k - x)$ for all x except integer points of discontinuity. For example, the graph of $f(x) = \{x\} - \left\{\frac{x}{3}\right\} - \left\{\frac{x}{7}\right\} + \left\{\frac{x}{21}\right\}$ looks as follows:



We prove by induction, for $k \geq 2$, that we can find primes $p_1, \ldots, p_k$ such that a slightly stronger inequality holds: $\max |f_k(n)| > 2^{k-3} + \Pi_k$. The base case: if $k = 2$ then choose $p_1 = 3$, $p_2 = 7$. We have $\max |f_2(n)| = |f_2(5)|$ and

$$|f_2(5)| = \left| \{5\} - \left\{ \frac{5}{3} \right\} - \left\{ \frac{5}{7} \right\} + \left\{ \frac{5}{21} \right\} \right| = \frac{8}{7} > \frac{1}{2} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right).$$

Given primes $p_1, \ldots, p_k$, suppose $-m_k \leq f_k(n) \leq M_k$, where $-m_k$ and M_k are the minimum and maximum values that $f_k(n)$, $n \in \mathbb{Z}$, can achieve. We can see that $\sup f_k(n) = m_k$ and $m_k = M_k + \Pi_k$ because of the discontinuity at the supremum integer point.

Suppose we can find primes $p_1, p_2, \ldots, p_k$ such that $\max |f_k(n)| > 2^{k-3} + \Pi_k$. We show how to find a prime p_{k+1} such that $\max |f_{k+1}(n)| > 2^{k-3} + \Pi_{k+1}$.

Let $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ be the two sets of residues modulo $p_1, p_2, \ldots, p_k$, respectively, which identify two integers for which minimum and maximum of $f_k(n)$ occurs. Note that $a_i \neq 0$ for all $1 \leq i \leq k$ because the minimum value occurs at a point of discontinuity, at an integer coprime with $p_1 p_2 \cdots p_k$.

Suppose we add a prime p_{k+1}, then $f_{k+1}(n) = f_k(n) - f_k\left(\frac{n}{p_{k+1}}\right)$. Set $n = mp_{k+1} + r$, where m is an integer and $0 \leq r \leq p_{k+1} - 1$. Since $\frac{r}{p_{k+1}} < 1$, we get

$$\begin{aligned} f_{k+1}(n) &= f_k(n) - f_k\left(\frac{n}{p_{k+1}}\right) \\ &= f_k(n) - f_k(m) - f_k\left(\frac{r}{p_{k+1}}\right) \\ &= f_k(n) - f_k(m) - \frac{r}{p_{k+1}} \cdot \Pi_k \\ &\geq f_k(n) - f_k(m) - \Pi_{k+1}. \end{aligned}$$

Pick an integer $r \not\equiv b_i \pmod{p_i}$, where $1 \leq i \leq k$. By Dirichlet's Theorem there exist infinitely many primes p_{k+1} such that $p_{k+1} \equiv (b_i - r) \cdot a_i^{-1} \pmod{p_i}$ for all $1 \leq i \leq k$, which satisfy $n = mp_{k+1} + r$, where $m \equiv a_i \pmod{p_i}$ and $n \equiv b_i \pmod{p_i}$. Therefore we can find a prime $p_{k+1} > r$ satisfying the given conditions and $M_{k+1} \geq M_k + m_k - \Pi_{k+1}$. Using induction hypothesis we conclude that

$$m_{k+1} = M_{k+1} + \Pi_{k+1} \geq M_k + m_k = 2m_k - \Pi_k > 2(2^{k-3} + \Pi_k) - \Pi_k > 2^{k-2} + \Pi_{k+1}.$$

□

Alternative Solution. The most obvious approach appears to be induction on k. Let's see if we can make that work.

Step 1. Say n and $p_1, p_2, \ldots, p_k$ work. We are going to keep these primes and add one new prime q to them. We'll set things up so that q is much larger than $p_1, p_2, \ldots, p_k$. We will select some new positive integer N to go with $p_1, p_2, ..., p_k, q$.

Let $P = p_1 p_2 \cdots p_k$. We decree right from the start that N is congruent to n modulo P. So $N = KP + n$ for some positive integer K. This way, we get to keep all fractional parts from the induction hypothesis.

Also, we want all of $p_1, p_2, \dots p_k$ to be odd, and we want none of $p_1, p_2, \dots p_k$ to divide n. Consider this part of the induction hypothesis.

Step 2. Now we'll satisfy some requirements on q, K, and N whose purpose will become clear later on.

Let C be the unique positive integer such that P divides $2n + C$ and $C < P$. By our assumptions in Step 1, none of $p_1, p_2, \dots, p_k$ divide C.

We require that Pq divides $nq + N + C$. Let's see how to ensure that.

Since P and q are relatively prime, this is the same as P divides $nq+n+C$ and q divides $N + C$.

The requirement that P divides $nq + n + C$ gives us q congruent to 1 modulo P. By Dirichlet, there are infinitely many primes q with this property.

(Note that this special case of Dirichlet has a known proof accessible to high-schoolers, unlike, to the best of my knowledge, the general case.)

So we set q to some crazy large prime in this arithmetic progression. We'll see how large later on.

Once we've fixed q, we moreover have to ensure that q divides $N + C = KP + n + C$. Since q and P are relatively prime, we can choose K such that this holds.

To recap, q is absolutely enormous and Pq divides $nq + N + C$.

Step 3. Let P' be any product of several (possibly zero) distinct primes out of $p_1, p_2, \dots, p_k$. So P' is any divisor of P.

Then $\{n/P'\}$ is a summand in our induction hypothesis and $\{N/P'q\}$ is a summand in what we hope will eventually amount to our induction step.

Consider $n/P' + N/P'q$. This is $(nq + N)/P'q$.

By Step 2, we have that $P'q$ divides $nq+N+C$. Also, remember that q is really insanely large. So $(nq + N)/P'q$ is $C/P'q$ below the nearest larger positive integer. Here, $C/P'q$ is at most C/q. So by making q absurdly large we can guarantee that $n/P' + N/P'q$ is arbitrarily close to a positive integer, from below , for all P'.

In particular, we can make it so close to a positive integer that its fractional part is larger than the fractional parts of all expressions of the form n/P'. Then it follows immediately that, for each expression of the form n/P', we have that $\{N/P'q\}$ is arbitrarily close to $1 - \{n/P'\}$.

Step 4. Let S be the sum of the fractional parts for $p_1, p_2, \dots, p_k$ and n; that is,

$$\{n\} - \sum\{n/p_i\} + \sum\{n/p_ip_j\} - \dots,$$

and so on. By the induction hypothesis, we know $|S| > 2^{k-3}$.

Let T be the analogous sum for $p_1, p_2, \dots, p_k$, q and N.

Since N is congruent to n modulo P, we have that T contains all terms from S.

The new terms are all thingies of the form $e\{N/P'q\}$, where $e = \pm 1$.

We know that each such thingie is very close to $e(1 - \{n/P'\}) = e + (-e)\{n/P'\}$.
Here, $(-e)\{n/P'\}$ is a term which appears in S as well, with this sign exactly. So T is really very close to $2S$ plus the sum of all the e's.
The sum of all the e's, on the other hand, is just the alternating sum of the binomial coefficients in row k of Pascal's triangle; so, zero. Therefore, T is very close to $2S$.
By making q sufficiently large, we can make "very close" amount to as close as we wish. Since $|S| > 2^{k-3}$ by the induction hypothesis, when T is sufficiently close to $2S$, we get $|T| > 2^{(k+1)-3}$, as needed.
(Also, q is odd because it is a very large prime and q does not divide N because q divides $N + C$ and q is much larger than C.)
The induction step is complete.
Step 5. We still have to take care of the base case. It looks like $k = 1$, $p_1 = 3$, and $n = 1$ works with $|S| = 1/3 > 1/4$. □

Comment. Note that

$$A(n) = n - \sum_{i=1}^{k} \left\lfloor \frac{n}{p_i} \right\rfloor + \sum_{1 \le i < j \le k} \left\lfloor \frac{n}{p_i p_j} \right\rfloor - \cdots + (-1)^k \left\lfloor \frac{n}{p_1 p_2 \cdots p_k} \right\rfloor$$

and the number $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$ represents an estimation of $A(n)$.
The positive difference $\left| n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) - A(n) \right|$ is equal to

$$\left| \{n\} - \sum_{i=1}^{k} \left\{ \frac{n}{p_i} \right\} + \sum_{1 \le i < j \le k} \left\{ \frac{n}{p_i p_j} \right\} - \cdots + (-1)^k \left\{ \frac{n}{p_1 p_2 \cdots p_k} \right\} \right|$$

and is clearly less than 2^{k-1}. The problem asks to show that this upper bound is relatively tight for some choice of primes $p_1, p_2, \ldots, p_k$ and integer n.

C5.

Let $n \geq 3$ be a natural number. Anna and Bob play the following game on the vertices of a regular n-gon: Anna places her token on a vertex of the n-gon. Afterwards Bob places his token on another vertex of the n-gon. Then, with Anna playing first, they move their tokens alternately as follows for $2n$ rounds: In Anna's turn on the k-th round, she moves her token k positions clockwise or anticlockwise. In Bob's turn on the k-th round, he moves his token 1 position clockwise or anticlockwise.
If at the end of any person's turn the two tokens are on the same vertex, then Anna wins the game. Otherwise Bob wins. Decide for each value of n which player has a winning strategy.

Solution. We will show that Bob wins if and only if $4|n$ and $n \neq 4$.
We will often say that Anna and Bob are at a distance d if we can move one token d positions clockwise or anticlockwise to reach the other token. Note that the value of this distance is not unique.
We first treat the case $4 \nmid n$. Given a positive integer r, we define

$$m_r = \frac{r^2+r+2}{2} \quad \text{and} \quad D_r = \{d \in \{1, 2, \ldots, m_r\} : d \equiv m_r \bmod 2\}$$

Lemma 1. *If it is Anna's turn on round $n-1-r \geqslant 1$ or round $2n-1-r \geqslant 1$, and she is at a distance d from Bob, for some $d \in D_r$, then she has a winning strategy.*

Before proving the Lemma, we show why this implies that Anna has a winning strategy in the case $4 \nmid n$.
Note that

$$m_{n-2} = \frac{n^2-3n+4}{2} \geqslant n$$

In particular, D_{n-2} consists of all odd or of all even numbers in $\{1, 2, \ldots, n-1\}$. If n is odd, the clockwise and the anti-clockwise distance of Anna from Bob have opposite parities so Anna is at a distance d from Bob for some $d \in D_{n-2}$. Applying the Lemma for $r = 1$ we see that Anna has a winning strategy.
If $n \equiv 2 \bmod 4$, then $n^2 - 3n + 4 \equiv 2 \bmod 4$, so m_{n-2} is odd. A same argument as above shows that Anna has a winning strategy if d is also odd. If d is even then we apply the Lemma in the same way with $r = 2n-2$ and Anna has a winning strategy since $m_{2n-2} = 2n^2 - 3n + 2$ is even (and $m_{2n-2} \geqslant n$).

Proof. (of Lemma 1) We proceed by induction on r. For $r = 1$ we have $m_1 = 2$ and $D_1 = \{2\}$ and since we are in round $n-2$ or $2n-2$ she has a winning strategy.

Assume the result is true for $r = k$. For the inductive step suppose it is now Anna's turn on round $n-1-(k+1) = n-(k+2)$ or round $2n-1-(k+1) = 2n-(k+2)$ and she is at a distance d from Bob, for some $d \in D_{k+1}$. By moving her token $n-(k+2)$, or $2n-(k+2)$ positions in the opposite direction, she is now at a distance of $|d-(k+2)|$ positions from Bob. After Bob's move they will have a distance of d' for some $d' \in \{d-k-3, d-k-1, k+3-d, k+1-d\}$. Note that all of these numbers have the same parity as $d-(k+1) \equiv m_{k+1}-(k+1) \equiv m_k \bmod 2$. Furthermore,

$$d-k-3 \leqslant d-k-1 \leqslant m_{k+1}-(k+1) = m_k$$

and

$$k+1-d \leqslant k+3-d \leqslant k+2 \leqslant m_k+1\,.$$

(Here we assumed that $d \geqslant 1$ as otherwise Anna already won.) Since in all cases $d' \leqslant m_k+1$ and $d' \equiv m_k \bmod 2$, then $d' \leqslant m_k$. Therefore Anna wins by the induction hypothesis. □

We now treat the case $4|n$, say $n = 4r$. If $r = 1$ it is easy to see that Anna wins in at most two rounds so assume $r > 1$.
Bob places his token so that $d = 3$. Note that Anna cannot win on her first move. Let d_{2k-1} denote the distance after Anna's move on the k-th round and d_{2k} the distance after Bob's move on the k-th round. Then modulo 2 the sequence is $0, 1, 1, 0, 1, 0, 0, 1, \ldots$ which then repeats periodically with period 8.
Bob's strategy consists of two parts. The first part is that he never places his token on Anna's token and also he never moves his token on a position where he will immediately lose on Anna's next step unless he is really forced to do this.
Before explaining the second part of Bob's strategy let us assume for contradiction that Anna has a winning strategy and look at Bob's last move. Due to the first part of his strategy he could perhaps lose only in the following two cases:

(a) Before his last move $d = 1$ so he is forced to make it $d = 2$ and then Anna wins.

(b) Before his last move $d = 2r$ so he is forced to make it $d = 2r-1$ ($d = 2r+1$ is the same) and then Anna wins.

In case (a) Anna wins on a round of the form $2 \bmod 4$ which is impossible as on those rounds d is odd after Anna's move
In case (b) Anna wins on rounds of the form $(2r-1) \bmod 4r$ or $(2r+1) \bmod 4r$. Actually rounds of the form $(2r+1) \bmod 4r$ are rejected since in that case we would have $d = 2r$ when Bob was playing on round $2r \bmod 4k$ but that could only be possible if $d = 0$ when Anna was playing on round $2r \bmod 4r$. This is rejected as it means that Anna won on an earlier round.

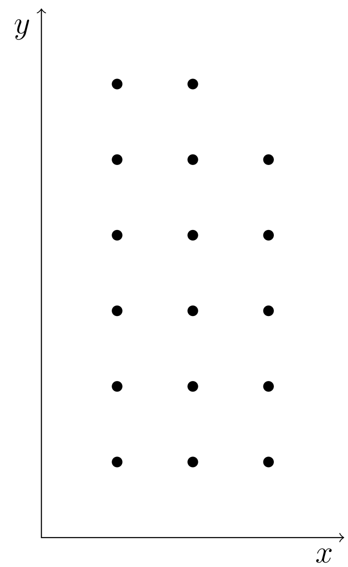
So in case (b) Anna wins on rounds of the form $(2r-1) \bmod 4r$. If r is even, say $r = 2s$, this is impossible as on round $(2r-1) \equiv 3 \bmod 4$ we have that d is odd after Anna's move.
So we need to show how Bob can avoid case (b) if r is odd, say $r = 2s+1$. He needs to avoid $d = 2r$ when it's his turn to play on rounds of the form $(2r-2) \bmod 4r$. This can only occur if $d = 2$ when it's Anna's turn to play on rounds of the form $(2r-2) \bmod 4r$. Bob can avoid this unless $d = 1$ when it's his turn to play on rounds of the form $(2r-3) \bmod 4r$. This can only occur if $d = 2r-2$ or $d = 2r-4$ when it's Anna's turn to play on rounds of the form $(2r-3) \bmod 4r$. Bob can avoid both of these cases unless $d = (2r-3)$ when it's his turn to play on rounds of the form $(2r-4) \bmod 4r$. This can only occur if $d = 1$ or $d = 7$ when it's Anna's turn to play on rounds of the form $(2r-4) \bmod 4r$. But Bob can avoid both of these on his move (on rounds of the form $(2r-5) \bmod 4r$). The only potential issue would be if $n = 10$ which is not the case here. □

C6.

Let $\mathcal{D}$ be the set of lines in the plane and A a set of 17 points in the plane. For $d \in \mathcal{D}$, let $n_d(A)$ be the number of distinct points in which A projects on d. Find the maximum cardinality of

$$V_A = \{n_d(A) | d \in \mathcal{D}\}.$$

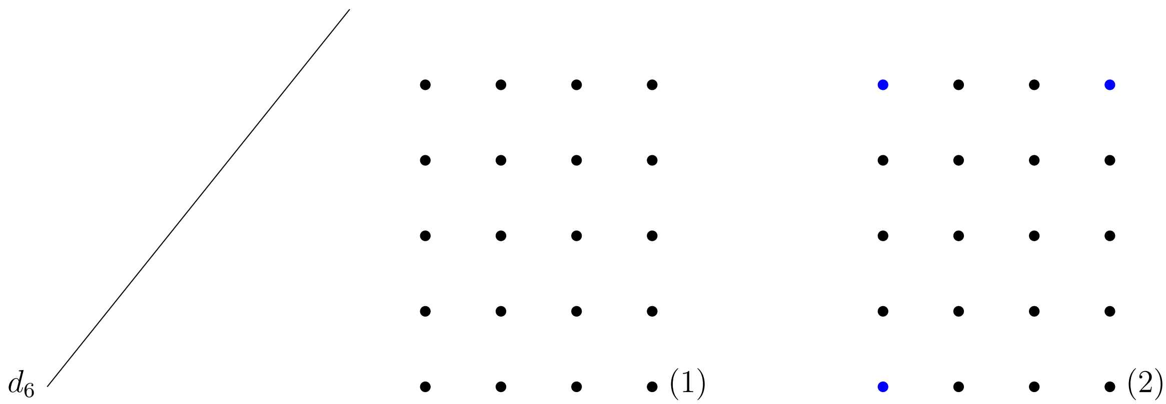
Solution. Let m be the required maximum. We see that $m \geqslant 13$, analyzing this example: $A = \{(x, y) | x \in \{1, 2, 3\}, y \in \{1, 2, 3, 4, 5, 6\}\} \setminus \{(3, 6)\}$.



For this A we have $V_A = \{3, 6, 7, 8, \ldots, 17\}$, so $|V_A| = 13$.

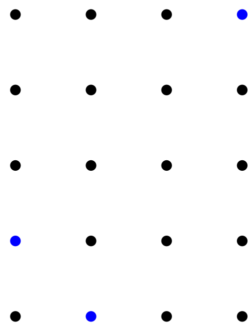
To show that $m = 13$ is the required maximum, we consider for the sake of contradiction a set A of 17 points for which $V_A = n \geqslant 14$.
We see that if $a, b \in V_A$, we can consider $d_a, d_b \in \mathcal{D}$ such that $n_{d_a}(A) = a$ and $n_{d_b}(A) = b$. Considering the a lines which are perpendicular on d_a in the a points where A projects and similarly the b lines perpendicular on d_b where the points of A projects on d_b, then the points in A are on this grid. Thus $|A| \leq ab$.
Let $V_A = \{a_1 < a_2 < \cdots < a_n\}$. Then, from the observation above, we have $a_1 a_2 \geqslant 17$. If $a_1 \leqslant 3$, then $3a_2 \geqslant 17$, so $a_2 \geqslant 6$. Then, $a_3 \geqslant 7, a_4 \geqslant 8, \ldots, a_n \geqslant n + 4 \geqslant 18$, impossible. So $a_1 \geqslant 4$, then $a_2 \geqslant 5, \ldots, a_n \geqslant n + 3 \geqslant 17$. As A has 17 points, all these inequalities become equalities, so $V_A = \{4, 5, \ldots, 17\}$. Then, for each $k \in \{4, 5, \ldots, 17\}$ there is a set of k parallel lines (we call these the k-lines support) whose union contains A. For each

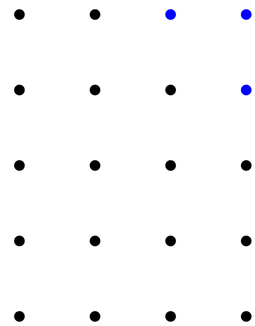
$k \in \{4, 5, \ldots, 17\}$ we choose a line d_k perpendicular to the k-lines support, on which A projects in k points.
In particular the points of A are part of a 4×5 grid G formed at the intersection of the 4-lines support with the 5-lines support. So A is constituted of 17 out of these $4.5 = 20$ points. In the following schematical representation of G we will assume $d_4 \perp d_5$. Clearly, d_6 is has direction different from the one of d_4 and d_5.
W.l.o.g. assume that G belongs to one of the two half planes determined by d_6. Consider the 8 grid points on the 2 closest adjacent sides of the convex hull of G plus the 6 points next to them $(A_{11}, A_{12}, A_{13}, A_{14}, A_{21}, A_{22}, A_{23}, A_{24}, A_{31}, A_{32}, A_{41}, A_{42}, A_{51}, A_{52})$ (the labelling of the points is matrix inspired).



Out of these 14 points, apart from A_{14} and A_{51}, we can have at most 2 points projecting in the same point of d_6 (otherwise, out of 3, at least two belong to the 8-points outer border, or to the 6-points inner border, impossible). So we have at least $1 + 1 + (14 - 2)/2 = 8$ points of projection. To achieve 6 points of projection, the set A must eliminate the "cause" of at least 2 of this projections. Since A can eliminate at most 3 points, we have two cases:
1) A eliminates A_{14} and A_{51}. Then the points in A lie on the lines $(L_1 : (A_{11}, A_{22}, A_{33}, A_{44}); L_2 : (A_{21}, A_{32}, A_{43}, A_{54}); L_3 : (A_{12}, A_{23}, A_{34}); L_4 : (A_{13}, A_{24}); L_5 : (A_{31}, A_{42}, A_{53}); L_6 : (A_{41}, A_{52}))$.
The collinearity on these lines implies that the grid consists of congruent parallelograms. Apart from this direction, the minimum number of parallel lines we can put the 18 points of $G \setminus \{A_{14}, A_{51}\}$ is 8 (achieved for the perpendicular direction), and is achieved when A_{11} and A_{54} are each the only one point on one of these lines. This means that, in order to achieve $7 \in V_A$, one of A_{11} and A_{54} has to be out of A. Then A will be like the representation in (2). In this case we cannot have anymore $8 \in V_A$.
2) A eliminates one of A_{14} and A_{51}, say A_{14}, and a pair, but the line determined by its points has no other grid point. Amongst $L_1, L_2, , L_3$ we have 2 which not include the missing pair from A, so their points collinearity can assure that the grid consists of

congruent parallelograms. Then, the only possible position of the missing pair is next to a corner, as shown in (3) or (4). But in none of these cases we have $7 \in V_A$.


(3)


(4)

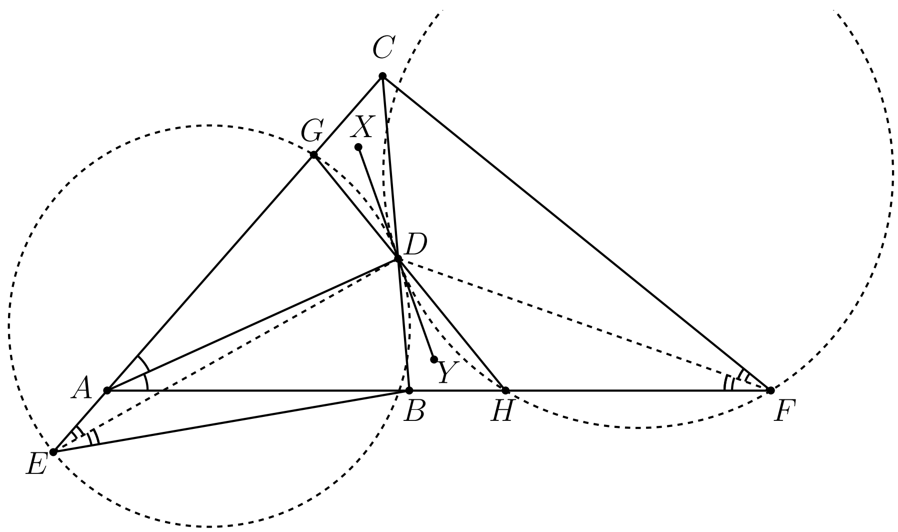
$\square$

Geometry – Solutions

G1.

Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E, F respectively. Let ℓ be a line through D meeting AC, AB at G, H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

Solution. Let X and Y lie on the tangent to the circumcircle of $\triangle EDG$ on the opposite side to D as shown in the figure below. Regarding diagram dependency, the acute condition with $AC > AB$ ensures E lies on extension of CA beyond A, and F lies on extension of AB beyond B. The condition on ℓ means the points lie in the orders E, A, G, C and A, B, H, F.



Using two applications of the alternate segment theorem, the condition that $\odot EDG$ and $\odot FDH$ are tangent at D can be rewritten as

$$\sphericalangle XDG = \sphericalangle YDH \Leftrightarrow \sphericalangle DEG = \sphericalangle DFH.$$

So we can remove G, H from the figure, and we will prove that $\sphericalangle DEA = \sphericalangle DFB$.

The reflection property means that AD and BD are external angle bisectors in $\triangle EAB$ and hence D is the E-*excentre* of this triangle. Thus DE (internally) bisects $\sphericalangle BEA$, giving

$$\sphericalangle DEA = \frac{1}{2}\sphericalangle BEA = \frac{1}{2}\sphericalangle BEC.$$

Similarly, in $\triangle FAC$, AD and CD are both internal angle bisectors, and so D is the *incentre* of this triangle and FD is also the internal angle bisector of $\sphericalangle CFH$, giving

$$\sphericalangle DFB = \sphericalangle DFA = \frac{1}{2}\sphericalangle CFA = \frac{1}{2}\sphericalangle CFB.$$

Now observe that the pairs of lines (BE, CE) and (BF, CF) are reflections in BC thus E, F are reflections in BC. Hence $\sphericalangle BEC = \sphericalangle CFB$ and so

$$\sphericalangle DEA = \frac{1}{2}BEC = \frac{1}{2}CFB = \sphericalangle DFB,$$

as required. □

G2.

Let ABC be an acute triangle and P be a point inside the triangle such that $\sphericalangle APB = \sphericalangle BPC = \sphericalangle CPA$. Denote with S the area and with α, β, γ the angles of $\triangle ABC$. Prove that

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} \geq \frac{PA^2 + PB^2 + PC^2}{2S} + \frac{4}{\sqrt{3}}.$$

When does the equality occur?

Solution. The inequality can be rewritten as

$$2S\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} - \frac{4}{\sqrt{3}}\right) \geq PA^2 + PB^2 + PC^2.$$

Note that $AB \cdot AC \cdot \sin\alpha = AB \cdot BC \cdot \sin\beta = AC \cdot BC \cdot \sin\gamma = 2S$ and

$$2S = 2(S_{PAB} + S_{PBC} + S_{PCA}) = (PA \cdot PB + PB \cdot PC + PC \cdot PA)\frac{\sqrt{3}}{2}.$$

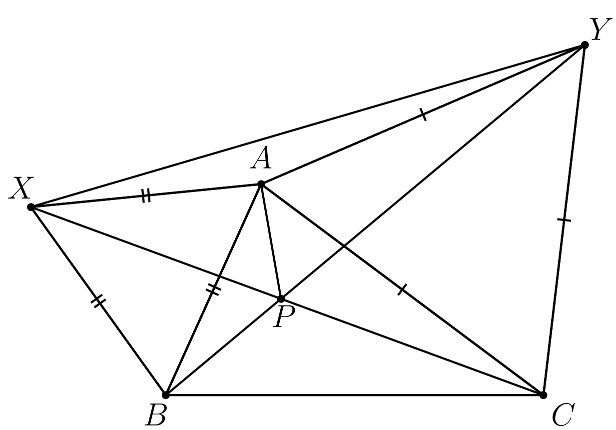
The inequality can be further rewritten as

$$AB \cdot AC + AC \cdot BC + BC \cdot AB - 2(PA \cdot PB + PB \cdot PC + PC \cdot PA) \geq PA^2 + PB^2 + PC^2$$

that is equivalent to

$$AB \cdot AC + AC \cdot BC + BC \cdot AB \geq (PA + PB + PC)^2$$

Consider the points X and Y such that $\triangle XAB$ and $\triangle YAC$ are equilateral (X and C lie on different halfplanes with respect to AB, similarly Y and B with respect to AC).



$\sphericalangle AXB + \sphericalangle APB = 180^\circ = \sphericalangle AYC + \sphericalangle APC \Rightarrow XAPB$ and $YAPC$ are cyclic quadrilaterals. Also note that $\sphericalangle BPA + \sphericalangle APY = 120^\circ + \sphericalangle ACY = 120^\circ + 60^\circ = 180^\circ$ hence B, P and Y are collinear. Similarly, points C, P and X are collinear.
From Ptolemy's Theorem in cyclic quadrilateral $XAPB$ we have that $PA \cdot XB + PB \cdot XA = PX \cdot AB$, but since $\triangle XAB$ is equilateral, then $XA = XB = AB$ and we have that $PA + PB = PX$. From here, $CX = PX + PC = PA + PB + PC$. Similarly, $BY = PA + PB + PC$.
Now, we apply Ptolemy's Inequality in quadrilateral $XBCY$ and get that $XB \cdot YC + XY \cdot BC \geq CX \cdot BY$. From Triangle Inequality we have that $AX + AY \geq XY$ so $XB \cdot YC + (AX + AY)BC \geq CX \cdot BY$. Rewriting the inequality based on the above relations we have that $AB \cdot AC + AB \cdot BC + AC \cdot BC \geq (PA + PB + PC)^2$.
The equality occurs if and only if both equality cases of Ptolemy's Inequality and Triangle's Inequality occur. The equality case of Triangle's Inequality occurs when X, A and Y are collinear $\iff 180^\circ = \sphericalangle XAB + \sphericalangle BAC + \sphericalangle CAY = 60^\circ + \sphericalangle BAC + 60^\circ \Rightarrow \sphericalangle BAC = 60^\circ$.
The Ptolemy's Inequality equality case occurs if and only if $XBCY$ is cyclic $\iff 180^\circ = \sphericalangle BXY + \sphericalangle BCY = 60^\circ + \sphericalangle BCA + \sphericalangle ACY = 60^\circ + \sphericalangle BCA + 60^\circ \Rightarrow \sphericalangle BCA = 60^\circ$. So the equality case happens if and only if $\triangle ABC$ is equilateral. □

Alternative Solution. Applying cotangent rule for $\triangle PAB$, $\triangle PBC$ and $\triangle PAC$ we have:

$$AB^2 = PA^2 + PB^2 + \frac{4}{\sqrt{3}} S_{PAB},$$

$$BC^2 = PB^2 + PC^2 + \frac{4}{\sqrt{3}} S_{PBC},$$

$$CA^2 = PC^2 + PA^2 + + \frac{4}{\sqrt{3}} S_{PCA}$$

$$\Rightarrow PA^2 + PB^2 + PC^2 = \frac{AB^2 + BC^2 + CA^2}{2} - \frac{2}{\sqrt{3}} S$$

and the inequality is equivalent to

$$AB \cdot AC + AC \cdot BC + BC \cdot AB \geqq \frac{AB^2 + BC^2 + CA^2}{2} + 2\sqrt{3}S$$

Using standard notations for a triangle we have

$$a + b + c = 2p,\ ab + bc + ac = p^2 + r^2 + 4rR,\ S = pr.$$

Therefore $a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8rR$ and we need to prove that

$$\sqrt{3}p \leqq r + 4R.$$

From Leibniz's inequality $a^2 + b^2 + c^2 \leqq 9R^2$ and Euler's inequality $r \leqq \dfrac{R}{2}$ we have

$$3p^2 \leqq 3r^2 + 12Rr + \frac{27}{2}R^2 = (r + 4R)^2 + 2r^2 + 4Rr - \frac{5}{2}R^2 \leqq (r + 4R)^2,$$

where the equation occurs if and only if $\triangle ABC$ is equilateral. $\square$

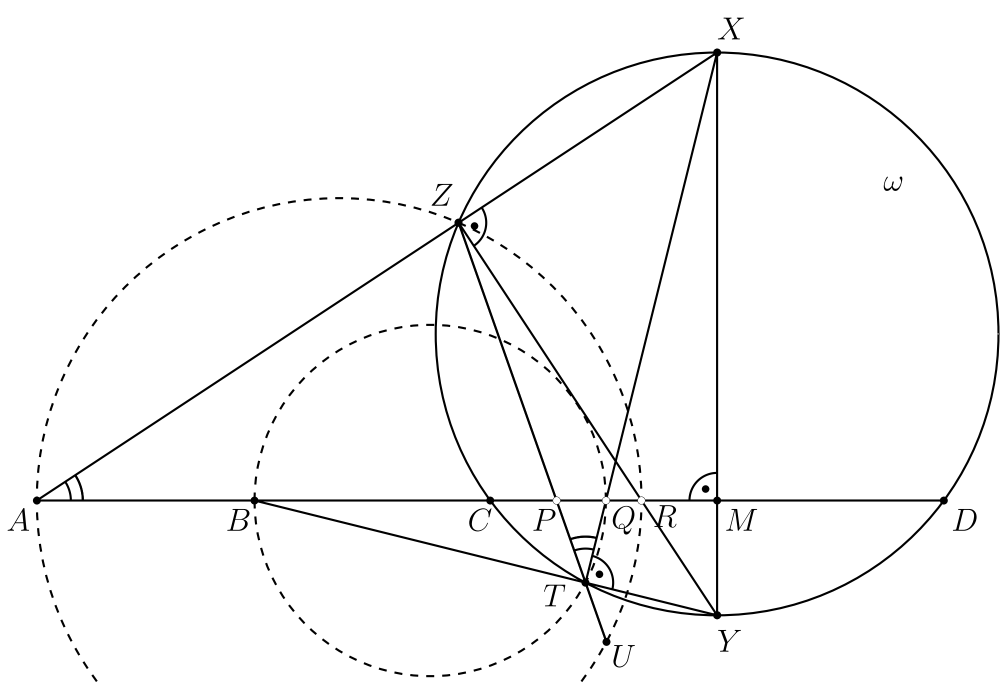
G3.

Let A, B, C, D be fixed points on this order on a line. Let ω be a variable circle through C and D and suppose that it meets the perpendicular bisector of CD at the points X and Y. Let Z and T be the other points of intersection of AX and BY with ω. Prove that XY passes through a fixed point which is independent of the circle ω.

Solution. Let M be the midpoint of CD and let Q and R be the points of intersection of XT and YZ with CD respectively. Since

$$\sphericalangle RZX = \sphericalangle YZX = 90^\circ = \sphericalangle RMX \quad \text{and} \quad \sphericalangle QTY = \sphericalangle XTY = 90^\circ = \sphericalangle QMY$$

the quadrilaterals $XZRM$ and $YTQM$ are cyclic.



By the power of the point A with respect to the circumcircle of $XZRM$ and with respect to ω we have

$$AR.AM = AZ.AX = AC.AD\,.$$

It follows that

$$AR = \frac{AC.AD}{AM}$$

which is independent of the circle ω. So Z is a point on the fixed circle of diameter AR. Similarly, Q is independent of the circle ω and T is a point on the fixed circle of diameter BQ. Let P be the point of intersection of ZT with CD. We will show that P is a fixed point independent of ω.
Since

$$\sphericalangle QTZ = \sphericalangle XTZ = \sphericalangle XYZ = 90^\circ - \sphericalangle YXZ = \sphericalangle ZAQ$$

then $ATQZ$ is cyclic, thus

$$PT.PZ = PA.PQ\,.$$

Letting U be the point of intersection of ZT with the circumcircle of $\triangle AZR$ we also have

$$PU.PZ = PA.PR\,.$$

We deduce that

$$\frac{PT}{PU} = \frac{PQ}{PR}$$

from which it follows that P is the centre of homothety of the two fixed circles with diameters AR and BQ. Thus P is indeed a fixed point. □

Alternative Solution. Applying the properties of cross (double) ratio we get:

$$(A, C; P, D) = Z(X, C; T, D) \overset{\omega}{=} Y(X, C; T, D) = (M, C; B, D)$$

It follows that

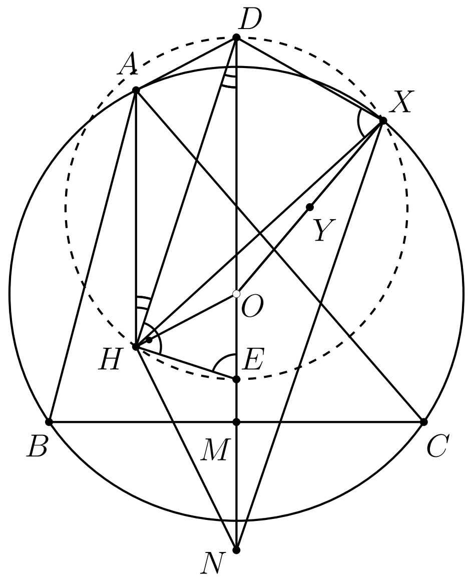
$$\frac{AP}{CP} : \frac{AD}{CD} = \frac{MB}{BC} : \frac{MD}{CD} \Rightarrow \frac{AP}{CP} = \frac{MB}{BC} . \frac{AD}{MD} = const$$

Therefore P is a fixed point. □

G4.

Let ABC be an acute-angled triangle with $AB < AC$, orthocentre H, circumcircle Γ and circumcentre O. Let M be the midpoint of BC and let D be a point such that $ADOH$ is a parallelogram. Suppose that there exists a point X on Γ and on the opposite side of DH to A such that $\measuredangle DXH + \measuredangle DHA = 90°$. Let Y be the midpoint of OX. Prove that if $MY = OA$ then $OA = 2OH$.

Solution. Let E lie on OM such that $\measuredangle EHD = 90°$ and let N be the reflection of O in BC.



Since $AH \parallel DE$ we have:

$$\measuredangle DXH = 90° - \measuredangle DHA = 90° - \measuredangle HDE = \measuredangle DEH$$

So $DXEH$ is cyclic - call this circle ω.
It's well-known that $AH = 2OM = ON$ so $NH = OA$ (which we'll use later) and $ON = AH = DO$ (as $ADOH$ is a parallelogram). This means $\frac{OD}{ND} = \frac{1}{2}$. Also, by considering homothety factor 2 at O:

$$NX = 2MY = 2OA = 2OX \implies \frac{OX}{NX} = \frac{1}{2}$$

so, as the diameter of ω lies on line ON, ω is in fact a circle of Apollonius with focii at O, N. Using the above and that $NH = OA$, which we noted before:

$$\frac{1}{2} = \frac{OH}{NH} = \frac{OH}{OA} \Rightarrow OA = 2OH$$

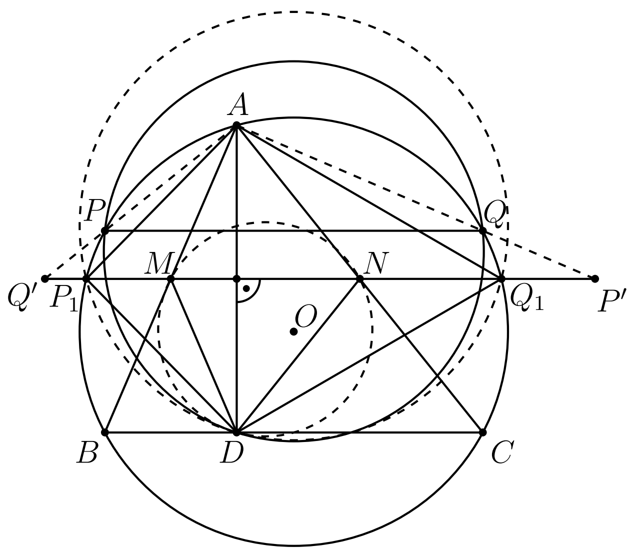
which is what we wanted to prove. $\square$

Comment. The converse of the condition is also true and can be proven in a similar way.

G5.

Let ABC be a scalene acute triangle ABC, D be the orthogonal projection of A on BC, M and N are the midpoints of AB and AC respectively. Let P, Q are points on the minor arcs $\widehat{AB}$ and $\widehat{AC}$ of circumcircle of $\triangle ABC$ respectively such that $PQ \parallel BC$. Show that the circumcircles of $\triangle DPQ$ and $\triangle MND$ are tangent to each other if and only if PQ passes through M.

Solution. Assume PQ does not pass through the midpoint of AB but (DPQ) is tangent to circle (MND).



Consider $i =$ the inversion of pole A and $k = \frac{AB \cdot AC}{2}$ followed by the reflection with respect to angle bisector of $\sphericalangle BAC$. Denote $X' = i(X)$ for any X in the plane.
Notice that $B' = N$, $C' = M$, the midpoints of AC and AB respectively and that Euler's circle of the triangle ABC is (DMN), so it's 'inverse' is the circle $(D'M'N')$. But now, $M' = C$, $N' = B$ and D' is the circumcenter of ABC, denoted by O: Indeed, the line $B - D - C$ is sent to the circle $(AND'C)$, which is the circle with diameter AO. And since AO and AD are isogonals, it follows that $D' = O$.Hence the circle (DMN) is sent to (OBC). At the same time circle ABC is sent to the line MN.
As $PQ \parallel BC$, it follows that arcs PB and QC are equal, so AP and AQ are isogonals, $P' = AQ \cap MN$, $Q' = AP \cap MN$ and the circle PDQ is sent to $P'OQ'$.
Let P_1, Q_1 are the points where the line MN cuts the minor arcs AB and AC. We will prove that (DMN) is tangent to (DP_1Q_1). By the same argument as before, we get

$P_1' = Q_1$ and $Q_1' = Q_1$ and then the circles (OP_1Q_1) and (OBC) are tangent as they are isosceles with $OB = OC$ and $OP = OQ$. Hence (P_1DQ_1) and (MDN) are tangent at D and $M \in P_1Q_1$.
Now (DPQ) and (DMN) are tangent if and only if $(OP'Q')$ and (OBC) are tangent, so if and only if the tangent at O to (OBC) is the tangent at O to $(OP'Q')$ which happens if and only if the triangle $OP'Q'$ is isosceles with base $P'Q'$ which is equivalent to $OP' = OQ'$. So we have $OP' = OQ'$ and also $OP_1 = OQ_1$. It follows that

$$Q'P_1 = Q_1P' \Leftrightarrow Q'Q_1' = P'P_1' \Leftrightarrow QQ_1 \cdot \frac{k}{AQ \cdot AQ_1} = PP_1 \cdot \frac{k}{AP \cdot AP_1} \Leftrightarrow$$

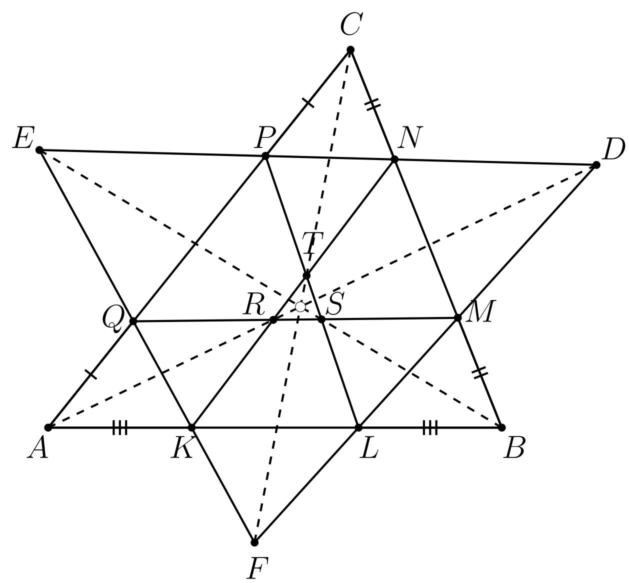
$$\Leftrightarrow AQ \cdot AQ_1 = AP \cdot AP_1 \Leftrightarrow S_{AQQ_1} = S_{APP_1} \Leftrightarrow dist(A, QQ_1) = dist(A, PP_1).$$

But this means that A lies on the segment bisector of P_1Q_1 and PQ respectively. So the minor arcs AB and AC are equal and the triangle is isosceles, contradiction. The conclusion follows now. □

G6.

Let $\triangle ABC$ be a triangle and the points K and L on AB, M and N on BC and P and Q on CA are such that $AK = LB < \frac{1}{2}AB$, $BM = NC < \frac{1}{2}BC$ and $CP = QA < \frac{1}{2}CA$. The intersections of KN with MQ and LP are R and T respectively, and the intersections of NP with LM and KQ are D and E respectively. Prove that the lines DR, BE and CT pass through a common point.

Solution. Let U, V and W be the intersections of AB, BC and CA with MQ, KN and PL respectively. Here it is allowed for the points to be infinite points on the respective lines.



From Menelaus theorem for the triangle $\triangle ABC$ and the lines MQ, KN and PL we get

$$\frac{\overline{AU}}{\overline{UB}} = -\frac{\overline{AQ}}{\overline{QC}} \cdot \frac{\overline{CM}}{\overline{MB}},$$

$$\frac{\overline{BV}}{\overline{VC}} = -\frac{\overline{BL}}{\overline{LA}} \cdot \frac{\overline{AP}}{\overline{PC}},$$

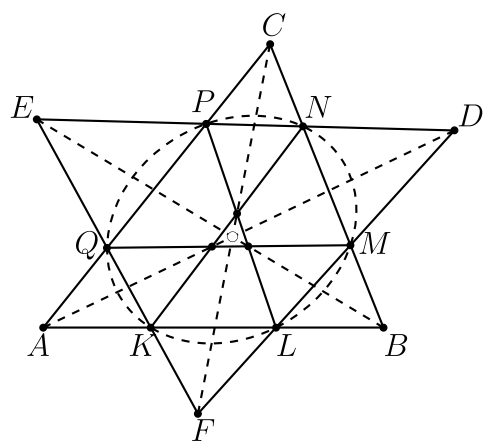
$$\frac{\overline{CW}}{\overline{WA}} = -\frac{\overline{CN}}{\overline{NB}} \cdot \frac{\overline{BK}}{\overline{KA}}.$$

After multiplying them we get:

$$\frac{\overline{AU}}{\overline{UB}} \cdot \frac{\overline{BV}}{\overline{VC}} \cdot \frac{\overline{CW}}{\overline{WA}} = -\frac{\overline{AQ}}{\overline{QC}} \cdot \frac{\overline{CM}}{\overline{MB}} \cdot \frac{\overline{BL}}{\overline{LA}} \cdot \frac{\overline{AP}}{\overline{PC}} \cdot \frac{\overline{CN}}{\overline{NB}} \cdot \frac{\overline{BK}}{\overline{KA}} = -1.$$

Hence by the converse Menelaus theorem the points U, V and W are collinear, implying $\triangle ALP$ and $\triangle RMN$ are coaxial. Now by Desargues theorem, $\triangle ALP$ and $\triangle RMN$ are copolar, hence A, R and D are collinear.
Let S be the intersection of LP and MQ. Similarly $\triangle BKN$ and $\triangle SQP$ are coaxial, implying they are copolar, hence B, S and E are collinear.
Now since U, V and W are collinear, $\triangle ABC$ and $\triangle RST$ are coaxial and by Desargues theorem they are copolar, hence $AR \equiv DR$, $BS \equiv BE$ and CT are concurrent. The lines AR, BS and CT cannot be parallel as R, S and T are inside $\triangle ABC$. □

Comment. The colinearity of points U, V and W lead us to the conclusion that the points K, L, M, N, P and Q lie on an ellipse (Braikenridge Maclaurin Theorem - The Converse of Pascal's Theorem). A lot of properties for this configuaration including the given one are shown in "'Geometry in Figures" - Arseniy Akopyan, Chapter 11.2 - Conics intersecting a triangle.



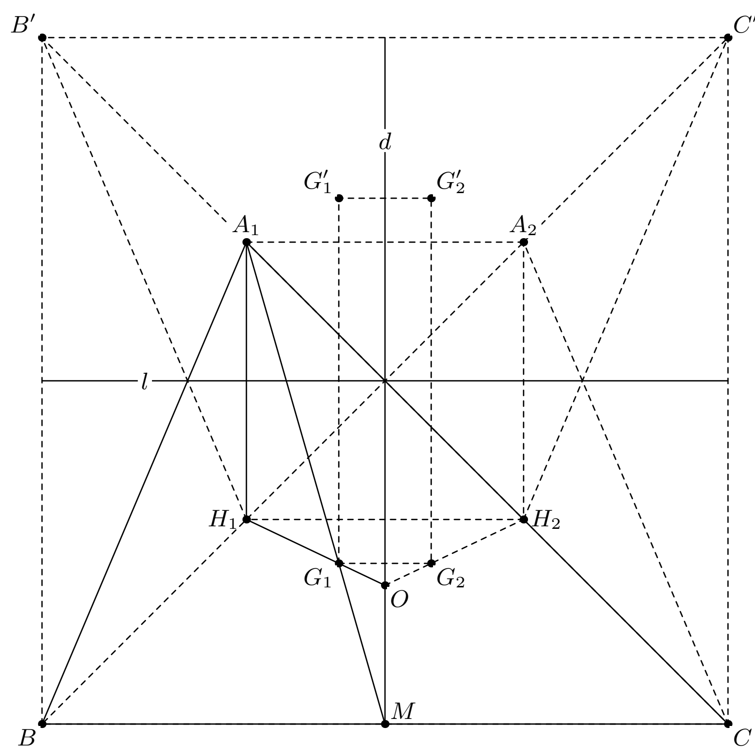
G7.

Let $f : \pi \to \mathbb{R}$ be a function from the Euclidean plane to the real numbers such that

$$f(A) + f(B) + f(C) = f(O) + f(G) + f(H)$$

for any acute triangle ABC with circumcenter O, centroid G, and orthocenter H. Prove that f is constant.

Solution.



Let $G_1 \neq G_2$ be arbitrary points and let d be the perpendicular bisector of G_1G_2. We shall construct two congruent triangles, symmetric with respect to d and with centroids at G_1 and G_2. Choose an acute triangle A_1BC with centroid G_1, and rotate it around G_1 if necessary to ensure that $BC \parallel G_1G_2$. By applying a homotethy centered at G_1 if necessary, we may ensure that the midpoint M of BC lies on d. Let H_1, O be the orthocenter and circumcenter of $\triangle ABC$. By construction, we have $O \in d$ and $A_1H_1 \parallel OM \perp BC$. If A_2, H_2 are the reflections of points A_1, H_1 across d, then one sees that $\triangle A_1BC \equiv \triangle A_2CB$, and H_2, G_2, O are the orthocenter, centroid and cirumcenter of $\triangle A_2BC$. Therefore, applying the property of f to $\triangle A_1BC$ and $\triangle A_2BC$, we have

$$\begin{aligned} f(A_1) + f(B) + f(C) &= f(O) + f(G_1) + f(H_1) \\ f(A_2) + f(B) + f(C) &= f(O) + f(G_2) + f(H_2) \end{aligned}$$

Subtracting, we obtain

$$f(A_1) - f(A_2) = f(G_1) - f(G_2) + f(H_1) - f(H_2) \quad (1)$$

Now let's reflect the picture across line l, the perpendicular bisector of A_1H_1 (l is a symmetry axis of rectangle $A_1A_2H_2H_1$), denoting by X' the image of point X. It follows that $A_1' = H_1, A_2' = H_2$ and triangles $H_1B'C'$,$H_2B'C'$ are symmetric about d. Using this symmetry, relation (1) becomes

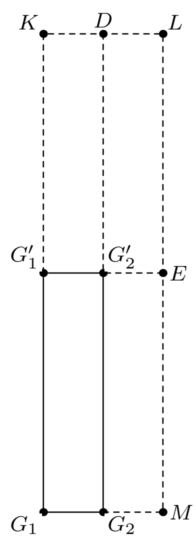
$$f(H_1) - f(H_2) = f(G_1') - f(G_2') + f(A_1) - f(A_2) \quad (2)$$

From (2) and (3) we obtain

$$f(G_1) - f(G_2) + f(G_1') - f(G_2') = 0 \quad (3)$$

The argument up to now holds if we *scale* the picture. It follows that for any rectangle $XYZT$ which is similar to the rectangle $G_1G_1'G_2'G_2$, we have

$$f(X) - f(T) + f(Y) - f(Z) = 0 \quad (4)$$



Let us double up the rectangle $G_1G_1'G_2'G_2$ to a rectangle G_1KLM (see the picture), with D and E the midpoints of KL and LM. Applying (4) to rectangle $G_2G_2'EM$, we have $f(G_2) - f(M) + f(G_2') - f(E) = 0$ which added to (3) yields

$$f(G_1) - f(M) + f(G_1') - f(E) = 0$$

Similarly, one obtains
$$f(G_1') - f(E) + f(K) - f(L) = 0$$
Subtracting the last relations we get
$$f(G_1) - f(M) - f(K) + f(L) = 0$$
On the other hand, applying (4) to the rectangle G_1KLM we have
$$f(G_1) - f(M) + f(K) - f(L) = 0$$
The last 2 relations imply $f(G_1) - f(M) = 0$, and scaling back we finally obtain $f(G_1) - f(G_2) = 0$. The choice of $G_1 \neq G_2$ being arbitrary, it follows that the function f must be constant. □

Alternative Solution. Assume that $f : \pi \to \mathbb{R}$ satisfies the condition of the problem.

1. Step 1. (Rhombus 60°) Consider a rhombus $ABCD$ with $\angle BAC = 60°$. Let M and N lie on the diagonal AC such that $AM = MN = NC$. Then since $\triangle ABD$ and $\triangle CBD$ are equilateral we have that M and N are their centroids and therefore:
$$\begin{aligned} f(A) + f(B) + f(D) &= 3f(M) \\ f(C) + f(B) + f(D) &= 3f(N). \end{aligned}$$
This proves that $f(A) - f(C) = 3(f(M) - f(N))$. As a consequence we have that whenever A, M, N, C are collinear and $AM = MN = NC$ then:
$$f(A) - f(C) = 3(f(M) - f(N)).$$

2. Step 2. Now consider five collinear points A, B, C, D, E with $AB = BC = CD = DE$. Then by Step 1 we have:
$$\begin{aligned} f(A) - f(D) &= 3(f(B) - f(C)) \\ f(B) - f(E) &= 3(f(C) - f(D)). \end{aligned}$$
Summing up both equalities we obtain:
$$f(A) - f(E) = 2(f(B) - f(D)).$$
Thus whenever A, B, D, E are collinear (in this order) with $AB = DE = \frac{1}{2}BD$ it holds:
$$f(A) - f(E) = 2(f(B) - f(D)).$$

3. Step 3. Consider a trapezoid $ABCD$ with bases $AB \parallel CD$ and $AB = 2CD$. Let O be the intersection point of the diagonals AC and BD and M and N be the midpoints of AO and BO. Finally let $P = DM \cap AB$ and $Q = CN \cap AB$. To start with note that by construction $AM = MO = OC$ and $BN = NO = OD$ therefore by Step 1:

$$\begin{aligned} f(A) - f(C) &= 3(f(M) - f(O)) \\ f(B) - f(D) &= 3(f(N) - f(O)). \end{aligned}$$

Subtracting the second equality from the first one we obtain:

$$f(A) - f(B) = f(C) - f(D) + 3(f(M) - f(N)).$$

Now note that $AP : CD = AM : MC = 1 : 2 = BN : ND = BQ : CD$. Therefore $AP = BQ$ and $PQ = AB - 2AP = AB - CD = CD = 2AP$. Therefore A, P, Q, B are collinear (in this order) with $AP = QB = 1/2PQ$ and by Step 2 we have:

$$f(A) - f(B) = 2(f(P) - f(Q)).$$

Substituting in the above equality we obtain:

$$\begin{aligned} 2(f(P) - f(Q)) &= f(C) - f(D) + 3(f(M) - f(N)) \text{ or after rearrangement} \\ 2f(P) + f(D) - 3f(M) &= 2f(Q) + f(C) - 3f(N). \end{aligned}$$

Finally note that $PQCD$ is a parallelogram ($PQ \overset{\parallel}{=} DC$) and $PM = QN = 1/3PD$. Clearly we can reconstruct the points $A = CM \cap PQ$ and $B = DN \cap PQ$ and by Thales's Theorem they will satisfy $AP = 1/2CD = BQ$. To summarise for every parallelogram $PQCD$ and points $M \in PD$ and $N \in QC$ with $PM = 1/3PD = QN$ we have:

$$2f(P) + f(D) - 3f(M) = 2f(Q) + f(C) - 3f(N).$$

4. Step 4. Let $PQCD$ be parallelogram and $M, M' \in PD$ are $N, N' \in QC$ be such that $PM = MM' = M'D$ and $QN = NN' = N'D$. Applying Step 3 to the parallelogram $CDPQ$ and points N' and M' we have:

$$2f(C) + f(Q) - 3f(N') = 2f(D) + f(P) - 3f(M').$$

Summing up we the result from Step 3 for the original parallelogram $PQCD$ and the points M and N we get:

$$3(f(P) + f(D) - f(M) - f(M')) = 3(f(C) + f(Q) - f(N) - f(N'))$$

or $f(P)+f(D)-f(M)-f(M')=f(C)+f(Q)-f(N)-f(N')$ whenever the line segments $PD \overset{\parallel}{=} QC$ and the points M, M' and N, N' split PD and QC in three equal parts, respectively. We can express this result as follows. There is a function g such that whenever A_1, A_2, A_3, A_4 are collinear and $\overrightarrow{v} = \overrightarrow{A_iA_{i+1}}$ for $i = 1, 2, 3$, then:

$$g(\overrightarrow{v}) = f(A_1) - f(A_2) - f(A_3) + f(A_4).$$

5. Step 5. Let $\overrightarrow{v} \neq 0$ be a vector in the plane and let $\overrightarrow{A_iA_{i+1}} = \overrightarrow{v}$ for $i = 1, 2, 3, 4$. Then by Step 4 we have:

$$f(A_1) - f(A_2) - f(A_3) + f(A_4) = g(\overrightarrow{v})$$
$$f(A_2) - f(A_3) - f(A_4) + f(A_5) = g(\overrightarrow{v})$$

Summing up both equalities we obtain:

$$f(A_1) - 2f(A_3) + f(A_5) = 2g(\overrightarrow{v}).$$

Therefore by scaling with $1/2$ we also have $f(A_1) - 2f(A_2) + f(A_3) = 2g(\overrightarrow{v}/2)$. Consequently:

$$\begin{aligned} g(\overrightarrow{v}) &= f(A_1) - f(A_2) - f(A_3) + f(A_4) \\ &= (f(A_1) - 2f(A_2) + f(A_3)) + (f(A_2) - 2f(A_3) + f(A_4)) = 4g(\overrightarrow{v}/2). \end{aligned}$$

Therefore $g(\overrightarrow{v}/2) = g(v)/4$ or, after rescaling, $4g(\overrightarrow{v}) = g(2\overrightarrow{v})$.

6. Step 6. Let H, G and O be the orthocenter, centroid and the centre of the circumcircle, respectively, of a triangle. Then, by Euler's Theorem, they are collinear with G belonging to the line segment OH and $OH = 2OG$. Conversely, if O, G and H satisfy the above condition, then there is an acute (isosceles) triangle ABC such that H, G and O are the orthocenter, centroid and the centre of the circumcircle of $\triangle ABC$. To see this one can take an arbitrary acute triangle $\triangle A'B'C'$ with orthocenter, centroid and the centre of the circumcircle, H', G' and O', respectively. Then scale $\triangle A'B'C'$ so that $O'H' = OH$ and finally translate and rotate the diagram so that $O'H'$ match OH. During this transformation, clearly, the triangle $A'B'C'$ is transformed to a similar triangle ABC.

7. Step 7. Let O, G, H be collinear with $\overrightarrow{HG} = 2\overrightarrow{GO}$. Let $\triangle ABC$ be arbitrary acute triangle such that O, G, H are its centre of the circumcircle, centroid and orthocenter, respectively. Let A_1, B_1, C_1 be the midpoints BC, AC and AB, respectively. Finally, let A_2, B_2 and C_2 be the midpoints of B_1C_1, C_1A_1 and B_1A_1, respectively.

We denote with O_i, G_i, H_i the centre of the circumcircle, centroid and orthocenter, respectively, of triangle $A_iB_iC_i$ for $i = 1, 2$.

Note that by Step 5:

$$f(A) - 2f(C_1) + f(B) = 2g(\overrightarrow{AC_1}/2) = 8g(\overrightarrow{A_1C_2}/2) = 4(f(A_1) - 2f(C_2) + f(B_1)).$$

Similarly we have:

$$f(B) - 2f(A_1) + f(C) = 4(f(B_1) - f(A_2) + f(C_1)$$
$$f(C) - 2f(B_1) + f(A) = 4(f(C_1) - f(B_2) + f(A_1)).$$

Summing up all three equalities and taking into account that $f(A)+f(B)+f(C) = f(O)+f(G)+f(H)$ and $f(A_i)+f(B_i)+f(C_i) = f(O_i)+f(G_i)+f(H_i)$ we conclude that:

$$f(O)+f(G)+f(H)-f(O_1)-f(G_1)-f(H_1) = 4(f(O_1)+f(G_1)+f(H_1)-f(O_2)-f(G_2)-f(H_2)).$$

However, obviously $G = G_1 = G_2$ and a simple homothetic argument at G shows that $O = H_1$ and $O_1 = H_2$. Therefore:

$$f(H) - f(O_1) = 4(f(H_1) - f(O_2)).$$

Now note that O_1 is the midpoint of OH and similarly O_2 is the midpoint of $O_1H_1 = O_1O$. Since O, G, H were arbitrary with $\overrightarrow{HG} = 2\overrightarrow{GO}$, we conclude that whenever $\overrightarrow{HO_1} = 2\overrightarrow{O_1O_2} = 2\overrightarrow{O_2H_1}$ we have:

$$f(H) - f(O_1) = 4(f(H_1) - f(O_2)).$$

Introducing M to be the midpoint of HO_1 and replacing (H, H_1) with (H_1, H) we get:

$$f(H_1) - f(O_1) = 4(f(H) - f(M)).$$

Subtracting from the first equality the second one, we arrive at:

$$f(H)-f(H_1) = 4(f(H_1)-f(H)-f(O_1)+f(M)) \text{ or } 5(f(H)-f(H_1)) = 4(f(M)-f(O_1)).$$

Now, since $\overrightarrow{MO_1} = 2\overrightarrow{HM} = 2\overrightarrow{O_1H_1}$ by Step 2 we also have:

$$f(H) - f(H_1) = 2(f(M) - f(O_1)).$$

This already shows that $10(f(M)-f(O_1)) = 4(f(M)-f(O_1))$ and therefore $f(M) = f(O_1)$, implying that $f(H) = f(H_1) = f(O)$. Since O and H can be considered arbitrary, as G is uniquely determined by the choice of O and H, we conclude that f is constant

□

Comment. 1. An earlier variant of the problem assumed $f(A)+f(B)+f(C) = 3f(H)$, where H is the orthocenter of ABC. This version is easier (but with similar ideas) than the above. Note that if we allow using obtuse-angled triangles, there is an even shorter solution, by just applying the relation to the 4 triangles ABC, BCH, CAH, ABH with orthocenters H, A, B, C, respectively. Replacing H with O also simplifies the problem, as one can fix 2 points on a circle, and allow the third point to move along an arc of the circle.

2. A known problem (USAMO 2001) had instead the relation $f(A)+f(B)+f(C) = 3f(I)$, where I is the incenter of ABC.

Number Theory – Solutions

N1.

Let n be a fixed natural number and

$$S_n = \{\overline{c_n c_{n-1} \ldots c_1}_{(10)} \mid c_1, \ldots, c_{n-1}, c_n \in \{1,2,3,4\}\}.$$

Are there distinct numbers x and y, $x, y \in S_n$, such that $4^n \mid x - y$?

Solution. For $n = 1$, the answer is negative. For $n > 1$ we will show that answer is positive. In contrary, since $|S_n| = 4^n$, we see that the set S_n is complete system of remainders modulo 4^n. Thus,

$$\sum_{x \in S_n} x^3 \equiv \sum_{i=1}^{4^n} i^3 \pmod{4^n} \equiv \left(\frac{4^n(4^n+1)}{2}\right)^2 \pmod{4^n} \equiv 0 \pmod{4^n}. \qquad (\heartsuit)$$

Let us calculate $\sum_{x \in S_n} x^3$. Denote by $A_n = \sum_{x \in S_n} x,\ B_n = \sum_{x \in S_n} x^2$ and $C_n = \sum_{x \in S_n} x^3$, for all $n \in \mathbb{N}$.
Then:

$$\begin{aligned} A_n &= \sum_{k=1}^{n} 10^{k-1} 4^{n-1}(1+2+3+4) \equiv 2 \cdot 4^{n-1} \pmod{4^n}, & (1) \\ B_{n+1} &= \sum_{(c,x_n) \in \{1,2,3,4\} \times S_n} (10^n \cdot c + x_n)^2 = \sum_{(c,x_n) \in \{1,2,3,4\} \times S_n} (10^{2n} \cdot c^2 + 2 \cdot 10^n \cdot c \cdot x_n + x_n^2) \\ &= 4^n \cdot 10^{2n} \cdot \sum_{c=1}^{4} c^2 + 2 \cdot 10^n \cdot \sum_{c=1}^{4} c \cdot A_n + 4 \cdot B_n \equiv 4B_n \pmod{4^{n+1}}. \end{aligned}$$

Since $B_1 = 1^2 + 2^2 + 3^2 + 4^2 = 30 \equiv 2 \pmod 4$, by induction, it follows that

$$B_n \equiv 2 \cdot 4^{n-1} \pmod{4^n} \qquad (2)$$

holds for all $n \in \mathbb{N}$.

So, by using (1) and (2), we have:

$$\begin{aligned}
C_{n+1} &= \sum_{(c,x_n)\in\{1,2,3,4\}\times S_n} (10^n\cdot c + x_n)^3 \\
&= \sum_{(c,x_n)\in\{1,2,3,4\}\times S_n} (10^{3n}\cdot c^3 + 3\cdot 10^{2n}\cdot c^2\cdot x_n + 3\cdot 10^n\cdot c\cdot x_n^2 + x_n^3) \\
&= 4^n\cdot 10^{3n}\cdot\sum_{c=1}^{4} c^3 + 3\cdot 10^{2n}\cdot\sum_{c=1}^{4} c^2\cdot A_n + 3\cdot 10^n\cdot\sum_{c=1}^{4} c\cdot B_n + 4\cdot C_n \\
&\overset{(1),(2)}{\equiv} 0 + 0 + 3\cdot 10^{n+1}\cdot B_n + 4C_n \pmod{4^{n+1}} \\
&\equiv 3\cdot 10^{n+1}\cdot(k\cdot 4^n + 2\cdot 4^{n-1}) + 4C_n \pmod{4^{n+1}} \qquad (3) \\
&\equiv 3\cdot 5^{n+1}\cdot 2^{n+2}\cdot 4^{n-1} + 4C_n \pmod{4^{n+1}}. \qquad (4)
\end{aligned}$$

Since $C_1 = 100$, from (3), we obtain $C_2 \equiv 1000 \pmod{16} \equiv 8 \pmod{16}$ and for $n \geq 2$, since $4^2 \mid 2^{n+2}$, we get

$$C_{n+1} \equiv 4C_n \equiv 4^{n+1}. \qquad (5)$$

Finally, from (5), by induction, we see that for all $n \in \mathbb{N}\setminus\{1\}$

$$C_n \equiv_{4^n} 2\cdot 4^{n-1} \qquad (\diamondsuit)$$

holds and by ($\heartsuit$) and ($\diamondsuit$) we arrive at a contradiction. $\square$

Alternative Solution. Assume such x and y do not exist and let $n > 1$. Since $|S_n| = 4^n$, we see that the set S_n is complete system of remainders modulo 4^n.
Note that the numbers

$$\overline{c_nc_{n-1}...c_21}, \overline{c_nc_{n-1}...c_22}, \overline{c_nc_{n-1}...c_23}, \overline{c_nc_{n-1}...c_24}$$

give four consecutive remainders modulo 4^n. Call such four consecutive remainders *a block*.
Therefore all 4^n remainders modulo 4^n can be split into blocks.
Thus, the difference of any two of the 4^{n-1} numbers $\overline{c_nc_{n-1}...c_20}$ is congruent to $4t$ modulo 4^n. But

$$\overline{c_nc_{n-1}...c_320} - \overline{c_nc_{n-1}...c_310} = 10,$$

implying $4t \equiv 10 \pmod{4^n}$, a contradiction. $\square$

N2.

Prove that for every integer n, the number $n^4 - 12n^2 + 144$ is not a perfect cube of an integer.

Solution. Suppose otherwise and let $m \in \mathbb{Z}$ be such that $n^4 - 12n^2 + 144 = m^3$. Firstly, m is clearly positive and we can assume that n is a positive integer (since $n = 0$ clearly doesn't work). Note that the polynomial $x^4 - 12x^2 + 144$ may be factored as

$$x^2 - 12x^2 + 144 = (x^2 + 12)^2 - (6x)^2 = (x^2 - 6x + 12)(x^2 + 6x + 12).$$

By repeatedly applying Euclid's algorithm, we get

$$\gcd(n^2 - 6n + 12, n^2 + 6n + 12) = \gcd(n^2 - 6n + 12, 12n), \tag{1}$$

and then

$$\gcd(n^2 - 6n + 12, n) = \gcd(n^2 - 6n + 12 - (n - 6)n, n) = \gcd(12, n). \tag{2}$$

We now distinguish three cases.

Case I. n is even. Write $n = 2k$ for some $k \in \mathbb{N}$ and we have $16(k^4 - 3k^2 + 9) = m^3$. Hence m is divisible by 4 which means that m^3 is divisible by 64. But $k^4 - 3k^2 + 9$ is always odd, so $16(k^4 - 3k^2 + 9)$ cannot be divisible by 64, a contradiction.

Case II. n is divisible by 3. Write $n = 3l$ for some $l \in \mathbb{N}$ and we have $9(9l^4 - 3l^2 + 16) = m^3$. Hence m is divisible by 3 which means m^3 is divisible by 27. But $9l^4 - 3l^2 + 16$ is not divisible by 3, so $9(9l^4 - 3l^2 + 16)$ cannot be divisible by 27, a contradiction.

Case III. Suppose that $\gcd(n, 6) = 1$. Then $\gcd(n^2 - 6n + 12, n^2 + 6n + 12) = 1$ by (1) and (2) (since $\gcd(n^2 - 6n + 12, 12) = \gcd(n(n - 6), 12) = 1$, because $\gcd(n, 6) = 1$), thus $n^2 - 6n + 12 = (n - 3)^2 + 3$ and $n^2 + 6n + 12 = (n + 3)^2 + 3$ are both perfect cubes of integers. However, we get a contradiction with the following lemma.

Lemma 1. *For every even integer x, the number $x^2 + 3$ is not a perfect cube of an integer.*

Proof. Suppose otherwise, namely that there exists a positive integer y such that $x^2 + 3 = y^3$. The last relation modulo 4 gives $y \equiv -1 \mod 4$. The equation then turns to

$$x^2 + 2^2 = y^3 + 1 = (y + 1)(y^2 - y + 1).$$

But we have $y^2 - y + 1 \equiv (-1)^2 - (-1) + 1 \equiv -1 \pmod 4$, hence there exists a prime number $p \equiv -1 \pmod 4$ such that $p \mid y^2 - y + 1 \mid x^2 + 2^2$. But it is well known that from here we should have $p \mid x$ and $p \mid 2$. This means that $p = 2$, which contradicts $p \equiv -1 \mod 4$. $\square$

We conclude that no such integer m can exist, thus $n^4 - 12n^2 + 144$ is never a perfect cube of an integer. $\square$

Comment. In the Shortlisted Problems of JBMO 2019 one can find the following similar problem: https://artofproblemsolving.com/community/c6h2610073p22542084.
More generally, Diofantine equations of the form $x^2+k=y^3$ are called Mordell's Equation and can be solved by factoring both sides and looking at the prime divisors of k. In this particular case we have $(n^2-6)^2+108=y^3$ and $108=2^2.3^3$.

N3.

Consider the sequence

$$a_n = 2^n + 3^{n+2} + 5^{n+1}$$

for $n \geqslant 1$. Prove that there are infinitely many prime numbers such that each one of them divides infinitely many terms of the sequence.

Solution. Let S be the set of all primes $p \neq 2,3,5$ for which 2 and 5 are quadratic residues modulo p but 3 is not.
If $p \in S$, then by Euler's criterion we have

$$2^{\frac{p-1}{2}} \equiv 5^{\frac{p-1}{2}} \equiv 1 \bmod p \quad \text{and} \quad 3^{\frac{p-1}{2}} \equiv -1 \bmod p\,.$$

So for $n = k(p-1) + \frac{p-1}{2} + 1$, where $k \in \mathbb{N}$, by Fermat's Little Theorem we have

$$\begin{aligned} 2^n &\equiv 2^{k(p-1)+\frac{p-1}{2}+1} \equiv \left(2^{p-1}\right)^k \cdot 2^{\frac{p-1}{2}} \cdot 2 \equiv 2 \bmod p\,, \\ 3^{n+2} &\equiv 3^{k(p-1)+\frac{p-1}{2}+3} \equiv \left(3^{p-1}\right)^k \cdot 3^{\frac{p-1}{2}} \cdot 3^3 \equiv -27 \bmod p\,, \\ 5^{n+1} &\equiv 3^{k(p-1)+\frac{p-1}{2}+2} \equiv \left(5^{p-1}\right)^k \cdot 5^{\frac{p-1}{2}} \cdot 5^2 \equiv 25 \bmod p\,. \end{aligned}$$

Therefore p divides a_n for infinitely many values of n.
It remains to show that S contains infinitely many terms. By Dirichlet's Theorem about primes in arithmetic progressions, there are infinitely many primes p with $p \equiv 41 \bmod 120$. We claim that every such p belongs to S. (Other congruence classes also work.) Indeed this follows since

- $p \equiv 41 \bmod 120 \implies p \equiv 1 \bmod 8 \implies \left(\dfrac{2}{p}\right) = +1$
- $p \equiv 41 \bmod 120 \implies \left\{\begin{array}{l} p \equiv 1 \bmod 4 \\ p \equiv 2 \bmod 3 \end{array}\right\} \implies \left(\dfrac{3}{p}\right) = \left(\dfrac{p}{3}\right) = -1$
- $p \equiv 41 \bmod 120 \implies p \equiv 1 \bmod 5 \implies \left(\dfrac{5}{p}\right) = \left(\dfrac{p}{5}\right) = +1$

□

Comment. Note that the following problem appeared at Italia Olympiad 2010:https://artofproblemsolving.com/community/c6h466737p2613923
We also note that the following result has been proven by George Polya:

Theorem 1. *Let* $0 < b_1 < \cdots < b_r$ *and* $P_1(x),\ \ldots, P_r(x)$ *are all polynomials with integer coefficients. For a set* $I \subseteq \{1, 2, \ldots, n\}$*, set*

$$x_{I,n} = \sum_{i \in I} b_i^n P_i(n).$$

Assume that $r \geq 2, \gcd(b_1,\ \cdots, b_r)\ = 1$ *and that for any non-empty set* $I \subseteq \{1, 2, \ldots, n\}$*,* $|x_{I,n}|$ *tends to infinity. Then the set of primes dividing some* x_n *is infinite.*

N4.

Let k be a positive integer. Find all sequences $(a_n)_{n\geq 1}$ of positive integers such that

$$a_{n+2}(a_{n+1}-k) = a_n(a_{n+1}+k)$$

for all $n \geq 1$.

Solution. Denote the given equation by (1). Note that if $a_n \leq k$, for $n \geq 2$, we get a contradiction, so $a_n > k$ for all $n \geq 2$. Then (1) becomes

$$a_{n+2} = \frac{a_n(a_{n+1}+k)}{a_{n+1}-k}.$$

We deduce that $a_{n+2} \geq a_n + 1$.

Now $a_{n+2} = \frac{a_n(a_{n+1}+k)}{a_{n+1}-k}$ so $a_{n+1} - k \mid a_n(a_{n+1}+k)$. However, $a_{n+1} - k \mid a_n(a_{n+1}-k)$ and subtracting gives $a_{n+1} - k \mid 2ka_n$. Let $r \geq 1$ such that $r(a_{n+1}-k) = 2ka_n$. Applying (1), $a_{n+2} = a_n + r$. We apply (1) again too get

$$\begin{aligned} a_{n+3} &= \frac{a_{n+1}(a_{n+2}+k)}{a_{n+2}-k} = \frac{a_{n+1}(a_n+r+k)}{a_n+r-k} = a_{n+1} + \frac{2ka_{n+1}}{a_n+r-k} \\ &= a_{n+1} + \frac{2ka_{n+1}}{\frac{r(a_{n+1}-k)}{2k}+r-k} = a_{n+1} + \frac{4k^2a_{n+1}}{ra_{n+1}+rk-2k^2}. \end{aligned}$$

Thus, $ra_{n+1}+rk-2k^2 \mid 4k^2a_{n+1}$ so $ra_{n+1}+rk-2k^2 \leq 4k^2a_{n+1}$. If $r \geq 4k^2$ we get a contradiction, so $r < 4k^2$. Now we use the fact that if $a, b, c \in \mathbb{Z}$, $a, b, c \neq 0$ then there is a finite number of natural numbers x for which $ax + b \mid c$. So if $rk \neq 2k^2$ (whence $r \neq 2k$), we get a finite number of possibilities for a_{n+1} (which depends on k but not on r!). Let S be this finite set. On the other hand, if $r = 2k$, it leads us to $a_{n+1} = a_n + k$.

In short, for all $n \geq 2$ we either have $a_n \in S$ or $a_n = a_{n-1} + k$. But $a_{n+2} \geq a_n + 1$ for all $n \geq 2$ so for n sufficiently large, we can't have $a_n \in S$. So there exists $N \geq 2$ such that $a_n = a_{n-1} + k$ for all $n \geq N$. Using $a_{N+1} = a_N + k$ we use (backward) induction to get $a_n = a_{n-1} + k$ for all n so $(a_n)_{n\geq 1}$ is an arithmetic progression of common difference k and, conversely, these satisfy (1). $\square$

N5.

Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Solution. Obviously we have $a > b$. Let $a = bq + r$, where $0 \leq r < b$. Then

$$3^a \equiv 3^{bq+r} \equiv (-2)^q.3^r \equiv -2 \pmod{3^b + 2}$$

So $3^b + 2$ divides $A = (-2)^q.3^r + 2$ and it follows that

$$|(-2)^q.3^r + 2| \geq 3^b + 2 \text{ or } (-2)^q.3^r + 2 = 0.$$

We make case distinction:

1. $(-2)^q.3^r + 2 = 0$. Then $q = 1$ and $r = 0$ or $a = b$, a contradiction.

2. q is even. Then
$$A = 2^q.3^r + 2 = (3^b + 2).k.$$
Consider both sides of the last equation modulo 3^r. Since $b > r$:
$$2 \equiv 2^q.3^r + 2 = (3^b + 2)k \equiv 2k \pmod{3^r},$$
so it follows that $3^r | k - 1$. If $k = 1$ then $2^q.3^r = 3^b$, a contradiction. So $k \geq 3^r + 1$, and therefore:
$$A = 2^q.3^r + 2 = (3^b + 2)k \geq (3^b + 2)(3^r + 1) > 3^b.3^r + 2$$
It follows that
$$2^q.3^r > 3^b.3^r, \text{ i.e. } 2^q > 3^b, \text{ which implies } 3^{b^2} < 2^{bq} < 3^{bq} \leq 3^{bq+r} = 3^a.$$
Consequently $a > b^2$.

3. If q is odd. Then
$$2^q.3^r - 2 = (3^b + 2)k.$$
Considering both sides of the last equation modulo 3^r, and since $b > r$, we get: $k+1$ is divisible by 3^r and therefore $k \geq 3^r - 1$. Thus $r > 0$ because $k > 0$, and:
$$2^q.3^r - 2 = (3^b + 2)k \geq (3^b + 2)(3^r - 1), \text{ and therefore}$$
$$2^q.3^r > (3^b + 2)(3^r - 1) > 3^b(3^r - 1) > 3^b\frac{3^r}{2}, \text{ which shows}$$
$$2^q + 1 > 3^b.$$
But for $q > 1$ we have $2^q + 1 < 3^q$, which combined with the above inequality, implies that $3^{b^2} < (2^q + 1)^b < 3^{qb} \leq 3^a$, q.e.d. Finally, If $q = 1$ then $2^q.3^r - 2 = (3^b + 2)k$ and consequently $2.3^r - 2 \geq 3^b + 2 \geq 3^{r+1} + 2 > 2.3^r - 2$, a contradiction.

□

Alternative Solution. $D = a - b$, and we shall show $D > b^2 - b$. We have $3^b + 2 | 3^a + 2$, so $3^b + 2 | 3^D - 1$. Let $D = bq + r$ where $r < b$. First suppose that $r \neq 0$. We have

$$1 \equiv 3^D \equiv 3^{bq+r} \equiv (-2)^{q+1} 3^{r-b} \pmod{3^b + 2} \Longrightarrow 3^{b-r} \equiv (-2)^{q+1} \pmod{3^b + 2}$$

Therefore

$$3^b + 2 \leq |(-2)^{q+1} - 3^{b-r}| \leq 2^{q+1} + 3^{b-r} \leq 2^{q+1} + 3^{b-1}$$

Hence

$$2 \times 3^{b-1} + 2 \leq 2^{q+1} \Longrightarrow 3^{b-1} < 2^q \Longrightarrow \frac{\log 3}{\log 2}(b-1) < q$$

Which yields $D = bq + r > bq > \frac{\log 3}{\log 2} b(b-1) \geq b^2 - b$ as desired. Now for the case $r = 0$, $(-2)^q \equiv 1 \pmod{3^b + 2}$ and so

$$3^b + 2 \leq |(-2)^q - 1| \leq 2^q + 1 \Longrightarrow 3^{b-1} < 3^b < 2^q \Longrightarrow \frac{\log 3}{\log 2}(b-1) < q$$

and analogous to the previous case, $D = bq + r = bq > \frac{\log 3}{\log 2} b(b-1) \geq b^2 - b$. □

N6.

For a positive integer c, define a sequence by $a_1 = c$ and $a_n = a_{n-1}^3 + c$ for each $n \geq 2$. Call a prime number p *orange* if for every positive integer c there is some n such that $p \mid a_n$. Are there infinitely many orange primes?

Solution. We claim that every prime $p \equiv 2 \pmod 3$ is orange. There are infinitely many such primes either i) by quoting Dirichlet's theorem; ii) or noting that if there were a finite list $p_1, \ldots, p_k$ then $3p_1 \cdots p_k - 1$ has a prime divsor that is 2 modulo 3 but can't be any of the primes already on our list giving a contradiction.

Claim 1. *Let* $p \equiv 2 \pmod 3$ *and* $x, y \in \mathbb{Z}$. *If* $p \mid x^3 - y^3$ *then* $p \mid x - y$.

Proof. If $p \mid x$ or $p \mid y$ then we're done so we can assume $p \nmid x, y$. Write $p = 3k + 2$ where $k \in \mathbb{N}$. By Fermat's little theorem we have $x^{p-1} = x^{3k+1} \equiv 1 \pmod p$ and similarly for y so:

$$x^3 \equiv y^3 \pmod p \Longrightarrow x \equiv x \cdot \left(x^{3k+1}\right)^2 \equiv \left(x^3\right)^{2k+1} \equiv \left(y^3\right)^{2k+1} \equiv y \pmod p$$

which gives $p \mid x - y$. □

Now take p as in the claim and assume for contradiction that p is not orange. By the pigeonhole principle, there are two terms in the sequence which are from the same residue class modulo p. Consider the pair $i < j$ with the smallest possible value of $i + j$ such that $p \mid a_i - a_j$. If $i > 1$ then note that:

$$p \mid a_{i-1}^3 - a_{j-1}^3 \Longrightarrow p \mid a_{i-1} - a_{j-1}$$

by our claim which is a contradiction. Otherwise $i = 1$ and:

$$p \mid a_{j-1}^3 + c - c = a_{j-1}^3 \Longrightarrow p \mid a_{j-1}$$

so p is orange as desired. □

Comment. We can classify all orange primes. By a direct check, 3 is an orange prime and by the above $p \equiv 2 \pmod 3$ is orange. If $p \equiv 1 \pmod 3$ then we can choose $-c$ to not be a cube modulo p (as $x \mapsto x^3$ is not injective) and for this value of c, p won't divide any terms in the sequence so p is not orange.
This is of course an alternative possible statement for the problem, requiring at least one extra idea for a solution.

Comment. This problem is a simple implication of the conclusion of the following China TST problem:https://artofproblemsolving.com/community/c6h1212540p6016827

N7.

Let a, b be positive integers such that $a+1$, $b+1$ and ab are all perfect squares. Prove that $\gcd(a,b)+1$ is also a perfect square.

Solution. Firstly observe that when $a=b$, $\gcd(a,b)+1=a+1$ is a perfect square. So we now assume WLOG $a<b$.
Let $a+1=A^2$, $b+1=B^2$ where $1<A<B$ are positive integers. We prove the result by induction on $\max\{a,b\}=b$.
Define $C:=AB-\sqrt{ab}=AB-\sqrt{(A^2-1)(B^2-1)}$ which is an integer as ab is a perfect square. Define $c:=C^2-1$ We will now prove the following:

- $1<c<b$
- ac is a perfect square (noting that it follows immediately from the construction that $a+1$, $c+1$ are perfect squares.)
- $\gcd(a,c)=\gcd(a,b)$

We will then be done by induction as by repeatedly applying this process, we preserve the gcd and must eventually reach a case with $a=b$ which we have proved above.
For the first part,

$$C=AB-\sqrt{A^2B^2-A^2-B^2+1}>AB-\sqrt{A^2B^2-2AB+1}=1$$

where the inequality is strict since $A\neq B$. Furthermore,

$$2(A-1)B^2\geq 2B^2>A^2-1 \implies A^2+B^2-1<2AB^2-B^2$$

leading to

$$C=AB-\sqrt{A^2B^2-A^2-B^2+1}<AB-\sqrt{A^2B^2-2AB^2+B^2}=B.$$

So $1<C<B$ and therefore $1<c<b$.
For the second part, expanding the definition of C we get:

$$(AB-C)^2=\left(A^2-1\right)\left(B^2-1\right) \implies A^2+B^2+C^2=2ABC+1 \quad (1)$$

We can rearrange this to:

$$ac=\left(A^2-1\right)\left(C^2-1\right)=(B-AC)^2 \quad (2)$$

which shows ac is a perfect square.
For the final part, note that if $d \mid A^2 - 1$ and $d \mid C^2 - 1$ then from (2), $d \mid B - AC$. Then from (1) we have:

$$B^2 - 1 = 2ABC - A^2 - C^2 \equiv 2A^2C^2 - A^2 - C^2 \equiv 2 - 1 - 1 \equiv 0 \pmod{d}$$

Thus $d \mid B^2 - 1$. Setting $d = \gcd(a, c)$, we see that $\gcd(a, c) \mid \gcd(a, b)$. But the condition we're using in (1) is symmetric in A, B, C and so in a similar way we get $\gcd(a, b) \mid \gcd(a, c)$.
Combining, we must have $\gcd(a, b) = \gcd(a, c)$. □

Alternative Solution. Using the notation from the solution above, define $g := \gcd(a, b)$ then, as ab is a perfect square, there exists positive integers x, y with $\gcd(x, y) = 1$ such that:

$$a = A^2 - 1 = gx^2 \quad \text{and} \quad b = B^2 - 1 = gy^2$$

Then the pairs (A, x) and (B, y) satisfy the Pell equation:

$$p^2 - gq^2 = 1$$

If (p_1, q_1) is the fundamental solution to this equation, then all other solutions are generated from the recurrences:

$$\begin{aligned} p_{n+1} &= p_1 p_n + g q_1 q_n \\ q_{n+1} &= p_1 q_n + q_1 p_n \end{aligned}$$

and in particular, $q_1 | q_n$ for all n. Thus $q_1 | x, y$ and hence $q_1 | \gcd(x, y) = 1$ so $q_1 = 1$. But then, considering the fundamental solution:

$$1 = p_1^2 - gq_1^2 = p_1^2 - g \implies g + 1 = p_1^2$$

as required. □

Comment. Letting $p_1 = \cosh\theta$, an easy induction on the recurrence shows that

$$p_n = \cosh(n\theta), \quad q_n = \frac{\sinh(n\theta)}{\sinh(\theta)},$$

hence $p_n = T_n(p_1)$ where T_n is the n^{th} Chebyshev polynomial of the first kind. Writing $p_1 := m \in \mathbb{Z}_{\geq 1}$, we have $g = m^2 - 1$ and we see a general solution is given by

$$(A, B) = (T_i(m), T_j(m)) \quad \text{or} \quad (a, b) = \left(T_i(m)^2 - 1, T_j(m)^2 - 1\right),$$

for any integers $i, j, m \in \mathbb{Z}_{\geq 1}$.

Comment. This problem appeared as a part of a problem whose solution can be found in Journal Kvant, 2000, Number 4, Problem 1740. The problem was proposed by V. Senderov.