Balkan MO Shortlist 2023

–	Algebra

A1 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(x + f(y)) = (y - x)f(f(x)).$$

A2 Let a, b, c, d be non-negative reals such that $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} = 1$. Show that there exists a permutation (x_1, x_2, x_3, x_4) of (a, b, c, d), such that

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \geq 4.$$

A3 Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(xy + f(x^2)) = xf(x + y)$$

for all reals x, y.

A4 Prove that there exists a real $c < \frac{3}{4}$, such that for each sequence $x_1, x_2, \ldots$ satisfying $0 \leq x_i \leq 1$ for all i, there exist infinitely many (m, n) with $m > n$, such that

$$|x_m - x_n| \leq \frac{c}{m}.$$

A5 Are there polynomials P, Q with real coefficients, such that $P(P(x)) \cdot Q(Q(x))$ has exactly 2023 distinct real roots and $P(Q(x)) \cdot Q(P(x))$ has exactly 2024 distinct real roots?

A6 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$f(x^{2023} + f(x)f(y)) = x^{2023} + yf(x)$$

for all $x, y > 0$.

C1   Joe and Penny play a game. Initially there are 5000 stones in a pile, and the two players remove stones from the pile by making a sequence of moves. On the k-th move, any number of stones between 1 and k inclusive may be removed. Joe makes the odd-numbered moves and Penny makes the even-numbered moves. The player who removes the very last stone is the winner. Who wins if both players play perfectly?

C2   For an integer $n > 2$, the tuple $(1, 2, \ldots, n)$ is written on a blackboard. On each turn, one can choose two numbers from the tuple such that their sum is a perfect square and swap them to obtain a new tuple. Find all integers $n > 2$ for which all permutations of $\{1, 2, \ldots, n\}$ can appear on the blackboard in this way.

C3   In a given community of people, each person has at least two friends within the community. Whenever some people from this community sit on a round table such that each adjacent pair of people are friends, it happens that no non-adjacent pair of people are friends. Prove that there exist two people in this community such that each has exactly two friends and they have at least one common friend.

C4   Once upon a time there are n pairs of princes and princesses who are in love with each other. One day a witch comes along and turns all the princes into frogs; the frogs can be distinguished by sight but the princesses cannot tell which frog corresponds to which prince. The witch tells the princesses that if any of them kisses the frog that corresponds to the prince very that she loves then that frog will immediately transform back into a prince. If each princess can stand kissing at most k frogs, what is the maximum number of princes they can be sure to save?

(The princesses may take turns kissing in any order, communicate with each other and vary their strategy for future kisses depending on information gained from past kisses.)

C5   Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly k numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

G1   Let $ABCD$ be a circumscribed quadrilateral and let X be the intersection point of its diagonals AC and BD. Let I_1, I_2, I_3, I_4 be the incenters of $\triangle DXC$, $\triangle BXC$, $\triangle AXB$, and $\triangle DXA$, respectively. The circumcircle of $\triangle CI_1I_2$ intersects the sides CB and CD at points P and Q, respectively. The circumcircle of $\triangle AI_3I_4$ intersects the sides AB and AD at points M and N, respectively. Prove that $AM + CQ = AN + CP$

G2   Let $ABCD$ be a cyclic quadrilateral with circumcenter O lying in the interior. Let E and F be the midpoints of the segments BC and AD, respectively. Let X be the point lying on the same side

of the line EF as the vertex C such that $\triangle EXF$ and $\triangle BOA$ are similar. Prove that $XC = XD$.

G3 In triangle ABC, the incircle touches sides BC, CA, AB at D, E, F respectively. Assume there exists a point X on the line EF such that

$$\angle XBC = \angle XCB = 45°.$$

Let M be the midpoint of the arc BC on the circumcircle of ABC not containing A. Prove that the line MD passes through E or F.

G4 Let O and H be the circumcenter and orthocenter of a scalene triangle ABC, respectively. Let D be the intersection point of the lines AH and BC. Suppose the line OH meets the side BC at X. Let P and Q be the second intersection points of the circumcircles of $\triangle BDH$ and $\triangle CDH$ with the circumcircle of $\triangle ABC$, respectively. Show that the four points P, D, Q and X lie on a circle.

G5 Let ABC be a triangle with circumcenter O. Point X is the intersection of the parallel line from O to AB with the perpendicular line to AC from C. Let Y be the point where the external bisector of $\angle BXC$ intersects with AC. Let K be the projection of X onto BY. Prove that the lines AK, XO, BC have a common point.

G6 Let ABC be an acute triangle ($AB < BC < AC$) with circumcircle Γ. Assume there exists $X \in AC$ satisfying $AB = BX$ and $AX = BC$. Points $D, E \in \Gamma$ are taken such that $\angle ADB < 90°$, $DA = DB$ and $BC = CE$. Let P be the intersection point of AE with the tangent line to Γ at B, and let Q be the intersection point of AB with tangent line to Γ at C. Show that the projection of D onto PQ lies on the circumcircle of $\triangle PAB$.

– Number Theory

N1 For positive integers a, b, c (not necessarily distinct), suppose that $a + bc, b + ac, c + ab$ are all perfect squares. Show that

$$a^2(b + c) + b^2(a + c) + c^2(a + b) + 2abc$$

can be written as sum of two squares.

N2 Find all positive integers, such that there exist positive integers a, b, c, satisfying $\gcd(a, b, c) = 1$ and $n = \gcd(ab + c, ac - b) = a + b + c$.

N3 For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$