Balkan MO Shortlist 2014

– Algebra

A1 A1 Let a, b, c be positive reals numbers such that $a + b + c = 1$.Prove that $2(a^2 + b^2 + c^2) \geq \frac{1}{9} + 15abc$

A2 Let x, y and z be positive real numbers such that $xy + yz + xz = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geq 2(x + y + z) - 3$$

and determine when equality holds.

UK - David Monk

A3 $A3$ The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = a_2 = 1, a_{2n+1} = 2a_{2n} - a_n$ and $a_{2n+2} = 2a_{2n+1}$ for $n \in N$.Prove that if $n > 3$ and $n - 3$ is divisible by 8 then a_n is divisible by 5

A4 $A4$ Let m_1, m_2, m_3, n_1, n_2 and n_3 be positive real numbers such that

$$(m_1 - n_1)(m_2 - n_2)(m_3 - n_3) = m_1m_2m_3 - n_1n_2n_3$$

Prove that

$$(m_1 + n_1)(m_2 + n_2)(m_3 + n_3) \geq 8m_1m_2m_3$$

A5 $A5$ Let $n \in N, n > 2$,and suppose $a_1, a_2, ..., a_{2n}$ is a permutation of the numbers $1, 2, ..., 2n$ such that $a_1 < a_3 < ... < a_{2n-1}$ and $a_2 > a_4 > ... > a_{2n}$.Prove that

$$(a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2 > n^3$$

A6 $A6$ The sequence $a_0, a_1, ...$ is defined by the initial conditions $a_0 = 1, a_1 = 6$ and the recursion $a_{n+1} = 4a_n - a_{n-1} + 2$ for $n > 1$.Prove that a_{2^k-1} has at least three prime factors for every positive integer $k > 3$.

A7 $A7$ Prove that for all $x, y, z > 0$ with $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ and $0 \leq a, b, c < 1$ the following inequality holds

$$\frac{x^2 + y^2}{1 - a^z} + \frac{y^2 + z^2}{1 - b^x} + \frac{z^2 + x^2}{1 - c^y} \geq \frac{6(x + y + z)}{1 - abc}$$

–	Combinatorics
C1	The International Mathematical Olympiad is being organized in Japan, where a folklore belief is that the number 4 brings bad luck. The opening ceremony takes place at the Grand Theatre where each row has the capacity of 55 seats. What is the maximum number of contestants that can be seated in a single row with the restriction that no two of them are 4 seats apart (so that bad luck during the competition is avoided)?
C2	Let $M = \{1, 2, ..., 2013\}$ and let Γ be a circle. For every nonempty subset B of the set M, denote by $S(B)$ sum of elements of the set B, and define $S(\varnothing) = 0$ ($\varnothing$ is the empty set). Is it possible to join every subset B of M with some point A on the circle Γ so that following conditions are fulfilled: $1.$ Different subsets are joined with different points; $2.$ All joined points are vertices of a regular polygon; $3.$ If $A_1, A_2, ..., A_k$ are some of the joined points, $k > 2$, such that $A_1A_2...A_k$ is a regular $k-gon$, then 2014 divides $S(B_1) + S(B_2) + ... + S(B_k)$?
C3	Let n be a positive integer. A regular hexagon with side length n is divided into equilateral triangles with side length 1 by lines parallel to its sides. Find the number of regular hexagons all of whose vertices are among the vertices of those equilateral triangles. *UK - Sahl Khan*

–	Geometry
G1	Let ABC be an isosceles triangle, in which $AB = AC$, and let M and N be two points on the sides BC and AC, respectively such that $\angle BAM = \angle MNC$. Suppose that the lines MN and AB intersects at P. Prove that the bisectors of the angles $\angle BAM$ and $\angle BPM$ intersects at a point lying on the line BC
G2	Triangle ABC is said to be perpendicular to triangle DEF if the perpendiculars from A to EF,from B to FD and from C to DE are concurrent.Prove that if ABC is perpendicular to DEF,then DEF is perpendicular to ABC
G3	Let $\triangle ABC$ be an isosceles.$(AB = AC)$.Let D and E be two points on the side BC such that $D \in BE$,$E \in DC$ and $2\angle DAE = \angle BAC$.Prove that we can construct a triangle XYZ such that $XY = BD$,$YZ = DE$ and $ZX = EC$.Find $\angle BAC + \angle YXZ$.
G4	Let $A_0B_0C_0$ be a triangle with area equal to $\sqrt{2}$. We consider the excenters A_1,B_1 and C_1 then we consider the excenters ,say A_2, B_2 and C_2,of the triangle $A_1B_1C_1$. By continuing this procedure ,examine if it is possible to arrive to a triangle $A_nB_nC_n$ with all coordinates rational.
G5	Let $ABCD$ be a trapezium inscribed in a circle k with diameter AB. A circle with center B and radius BE,where E is the intersection point of the diagonals AC and BD meets k at points K and L. If the line ,perpendicular to BD at E,intersects CD at M,prove that $KM \perp DL$.
G6	In $\triangle ABC$ with $AB = AC$,M is the midpoint of BC,H is the projection of M onto AB and D is arbitrary point on the side AC.Let E be the intersection point of the parallel line through B to HD with the parallel line through C to AB.Prove that DM is the bisector of $\angle ADE$.
G7	Let I be the incenter of $\triangle ABC$ and let H_a, H_b, and H_c be the orthocenters of $\triangle BIC$, $\triangle CIA$, and $\triangle AIB$, respectively. The lines H_aH_b meets AB at X and the line H_aH_c meets AC at Y. If the midpoint T of the median AM of $\triangle ABC$ lies on XY, prove that the line H_aT is perpendicular to BC

– Number Theory

N1 $\boxed{N1}$Let n be a positive integer,$g(n)$ be the number of positive divisors of n of the form $6k+1$ and $h(n)$ be the number of positive divisors of n of the form $6k-1$,where k is a nonnegative integer.Find all positive integers n such that $g(n)$ and $h(n)$ have different parity.

N2 $\boxed{N2}$ Let p be a prime numbers and $x_1, x_2, ..., x_n$ be integers.Show that if

$$x_1^n + x_2^n + ... + x_p^n \equiv 0 \pmod{p}$$

for all positive integers n then $x_1 \equiv x_2 \equiv ... \equiv x_p \pmod{p}$.

N3 $\boxed{N3}$Prove that there exist infinitely many non isosceles triangles with rational side lengths,rational lentghs of altitudes and, perimeter equal to 3.

N4 A *special number* is a positive integer n for which there exists positive integers a, b, c, and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that

i) there are infinitely many special numbers;
ii) 2014 is not a special number.

Romania

N5 $\boxed{N5}$Let a, b, c, p, q, r be positive integers such that $a^p + b^q + c^r = a^q + b^r + c^p = a^r + b^p + c^q$. Prove that $a = b = c$ or $p = q = r$.

N6 Let $f : \mathbb{N} \to \mathbb{N}$ be a function from the positive integers to the positive integers for which $f(1) = 1, f(2n) = f(n)$ and $f(2n+1) = f(n) + f(n+1)$ for all $n \in \mathbb{N}$. Prove that for any natural number n, the number of odd natural numbers m such that $f(m) = n$ is equal to the number of positive integers not greater than n having no common prime factors with n.