Macedonian

17 јуни 2026

Задача 1. Нека a, b, c се позитивни реални броеви такви што

$$a^2 + b^2 + c^2 \geqslant 3.$$

Докажи дека

$$\frac{a^4}{a^2 + 2b + 2c} + \frac{b^4}{b^2 + 2c + 2a} + \frac{c^4}{c^2 + 2a + 2b} \geqslant \frac{3}{5}.$$

Задача 2. Одреди ги сите парови (a, b) од позитивни цели броеви такви што

$$(a + 1) \text{ е делител на } (b + 2) \quad \text{и} \quad b \text{ е делител на } 2a^2.$$

Задача 3. Нека $n \geqslant 3$ е природен број. Во круг се распоредени n ламби во боја. Со едно притискање на дугмето на ламбата се менува нејзината боја на следниов начин:

зелено $\to$ црвено, црвено $\to$ плаво, плаво $\to$ зелено.

На почеток, сите ламби се обоени во црвена боја. Аце прави потег на овие ламби. Секој потег се состои од следниве три чекори:

- избира ламба L, без да го притиска нејзиното дугме;
- притиска еднаш на дугмето на ламбата која е соседна на ламбата L во насока на стрелките на часовникот;
- два пати го притиска дугмето на соседната ламба на L, која е во спротивна насока на стрелките на часовникот.

За секој n, одреди го максималниот можен број на ламби кои истовремено се обоени зелено, по конечно многу потези.

Задача 4. Нека ABC е триаголник со $AB \neq AC$ и нека I е центарот на впишаната кружница во $\triangle ABC$. Нека P и Q се точки од внатрешноста на $\triangle ABC$ такви што $PB = PC > QC = QB$. Правите BP и CQ се сечат во точката X. Да претпоставиме дека AI е тангента на опишаната кружница околу $\triangle IPQ$. Докажи дека опишаните кружници околу триаголниците ABQ, ACP и PQX имаат заедничка точка.

Време за работа: 4 часа и 30 минути
Секоја задача се вреднува со 10 поени

30th JBMO Buzău România 2026

English

17 June 2026

Problem 1. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 \geqslant 3.$$

Prove that

$$\frac{a^4}{a^2 + 2b + 2c} + \frac{b^4}{b^2 + 2c + 2a} + \frac{c^4}{c^2 + 2a + 2b} \geqslant \frac{3}{5}.$$

Problem 2. Determine all pairs (a, b) of positive integers such that

$$(a + 1) \text{ divides } (b + 2) \quad \text{and} \quad b \text{ divides } 2a^2.$$

Problem 3. Let $n \geqslant 3$ be an integer. There are n colored lamps arranged in a circle. Pressing the button of a lamp once changes its color as follows:

green → red, red → blue, blue → green.

Initially, all lamps are colored red. Aladdin makes moves on these lamps. Each move consists of the following three steps:

- he chooses a lamp L, without pressing its button;
- he presses the button of the clockwise neighbor of L once;
- he presses the button of the counterclockwise neighbor of L twice.

For each n, determine the maximum possible number of lamps that are colored green simultaneously, after finitely many moves.

Problem 4. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Let P and Q be points inside triangle ABC such that $PB = PC > QC = QB$. Lines BP and CQ meet at X. Suppose that line AI is tangent to the circumcircle of triangle IPQ. Prove that the circumcircles of triangles ABQ, ACP and PQX have a common point.

Time: 4 hours and 30 minutes
Every problem is worth 10 points

Problem 1. Let a, b, c be positive real numbers such that $a^2+b^2+c^2 \geqslant 3$. Prove that

$$\frac{a^4}{a^2+2b+2c}+\frac{b^4}{b^2+2c+2a}+\frac{c^4}{c^2+2a+2b} \geqslant \frac{3}{5}.$$

Serbia

Let a, b, c be positive real numbers such that $a^2+b^2+c^2 \geqslant 3$. Prove that

$$\frac{a^4}{a^2+2b+2c}+\frac{b^4}{b^2+2c+2a}+\frac{c^4}{c^2+2a+2b} \geqslant \frac{3}{5}.$$

Solution 1. Using $x^2+1 \geqslant 2x$ and QM-AM, we have

$$\text{LHS} \geqslant \frac{a^4+b^4+c^4}{a^2+b^2+c^2+2} \geqslant \frac{\left(a^2+b^2+c^2\right)^2}{3\left(a^2+b^2+c^2+2\right)}.$$

Denote $a^2+b^2+c^2=t$. Then the given condition yields $t \geqslant 3$, and for the conclusion, it suffices to prove that

$$\frac{t^2}{3(t+2)} \geqslant \frac{3}{5} \iff 5t^2-9t-18 \geqslant 0.$$

V1. One way to finish is to factor the above into

$$(t-3)(5t+6) \geqslant 0,$$

which is true since $5t+6>0$ and $t \geqslant 3$.

V2. Another way to see this is by noting that since $t \geqslant 3$, we have $3t^2 \geqslant 9t$ and $2t^2 \geqslant 18$. Adding these two inequalities up yields the desired result.

Solution 2. By Cauchy-Schwarz (in Titu form) and by the given condition, we have

$$\text{LHS} \geqslant \frac{\left(a^2+b^2+c^2\right)^2}{a^2+b^2+c^2+4(a+b+c)} \geqslant \frac{3\left(a^2+b^2+c^2\right)}{a^2+b^2+c^2+4(a+b+c)},$$

so the desired inequality reduces to showing that

$$a^2+b^2+c^2 \geqslant a+b+c.$$

By QM-AM and the given condition

$$a+b+c \leqslant \sqrt{3\left(a^2+b^2+c^2\right)} \leqslant a^2+b^2+c^2,$$

which is precisely what we wanted.

Solution 3. Denote $a^2+b^2+c^2=t$. By QM-AM, we have $a+b+c \leqslant \sqrt{3t}$.

By Cauchy-Schwarz (in Titu form), it follows that

$$\text{LHS} \geqslant \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+12(a+b+c)} \geqslant \frac{t^2}{t+4\sqrt{3t}} = \frac{t\sqrt{t}}{\sqrt{t}+4\sqrt{3}}.$$

Therefore, it suffices to show that

$$f(t) := \frac{t\sqrt{t}}{\sqrt{t}+4\sqrt{3}} = \frac{t}{1+4\sqrt{3/t}} \geqslant \frac{3}{5}.$$

Then f has increasing positive numerator and decreasing positive denominator. This shows that f is increasing, so $f(t) \geqslant f(3) = 3/5$, as desired.

Problem 2. Determine all pairs (a, b) of positive integers such that

$$(a+1) \text{ divides } (b+2) \quad \text{and} \quad b \text{ divides } 2a^2.$$

Kazakhstan

Answer: $(2,1)$, $(3,2)$, $(3,18)$, and $(a, 2a)$, where a is a positive integer.

In all of the solutions, a pair (a, b) satisfying the given properties will be called *good*.

Solution 1. If $b < a$, then $b+2 \leqslant a+1$ which, combined with the condition $a+1 \mid b+2$, implies $b+2 = a+1$, hence $b = a-1$. Substituting this into the second condition gives $a-1 \mid 2a^2$. Since $2a^2 = 2(a^2-1)+2 = 2(a-1)(a+1)+2$, this reduces to $a-1 \mid 2 \implies a-1 \in \{1,2\} \implies a \in \{2,3\}$. This yields the pairs $(2,1)$ and $(3,2)$, both of which are good pairs.

If $b \geqslant a$, let $k \in \mathbb{N}^*$ such that $2a^2 = kb$. From $a+1 \mid b+2$, it follows that $a+1 \mid k(b+2) = kb+2k = 2a^2+2k$. Since $2a^2 = 2(a^2-1)+2 = 2(a-1)(a+1)+2$, we have $2a^2 \equiv 2 \pmod{a+1}$. Therefore $a+1 \mid 2k+2$. Let $p \in \mathbb{N}^*$ such that $2k+2 = p(a+1)$. Since $b \geqslant a$, we have

$$k = \frac{2a^2}{b} = 2a \cdot \frac{a}{b} \leqslant 2a \implies p = \frac{2k+2}{a+1} \leqslant \frac{4a+2}{a+1} < 4$$

Thus, $p \in \{1,2,3\}$. We analyze each subcase:

I. If $p = 1$, then $2k+2 = a+1$, so $2k = a-1$. This gives $4a^2 = 2kb = (a-1)b$. Since $\gcd(a-1, a^2) = \gcd(a-1, 1) = 1$, $a-1$ must be a divisor of 4. On the other hand $a-1 = 2k$ is an even number, so $a \in \{3,5\}$.

- For $a = 3$, we get $b = 18$, giving the good pair $(3,18)$.
- For $a = 5$, we get $b = 25$, but the pair $(5,25)$ is not good because $5+1 \nmid 25+2$.

II. If $p = 2$, then $2k+2 = 2(a+1) \implies k = a$. Substituting this into $2a^2 = kb$ gives $2a^2 = ab$, which implies $b = 2a$.

This yields the family of good pairs $(a, 2a)$ for any $a \in \mathbb{N}^*$.

III. If $p = 3$, then $2k+2 = 3(a+1) \implies 2k = 3a+1$.

This yields $4a^2 = 2kb = (3a+1)b$. Since $\gcd(3a+1, a^2) = \gcd(1, a^2) = 1$, $3a+1$ must divide 4. Since $a \in \mathbb{N}^*$, we have $3a+1 \geqslant 4$, which leaves only $3a+1 = 4 \implies a = 1$.

If $a = 1$, then $b = 1$, but the pair $(1,1)$ is not a good because $1+1 \nmid 1+2$.

Solution 2. Write $b+2 = t(a+1)$ and $2a^2 = kb$, with t, k positive integers. Then $2a^2 - kta - kt + 2k = 0$. The discriminant of this quadratic equation in $a \in \mathbb{Z}$ is

$$\Delta = k^2t^2 + 8kt - 16k = (kt+4)^2 - 16k - 16$$

and must be a perfect square. Note that $\Delta < (kt+4)^2$, and Δ and $kt+4$ have the same parity, so the only possible cases are the following.

I. $\Delta = (kt+2)^2$. This gives $k(t-4) = 1$, whence $k = 1$, $t = 5$, yielding the good pair (3,18).

II. $\Delta = (kt)^2$. This gives $k(t-2) = 0$, whence $t = 2$, yielding the good pairs $(a, 2a)$, with a any positive integer.

III. $\Delta \leqslant (kt-2)^2$. This gives $k(3t-4) \leqslant 1$, which is possible only for $t = 1$. In this case $\Delta = k^2 - 8k = s^2 \iff (k-4+s)(k-4-s) = 16$.

Since the two factors have the same parity, this leads to $k-4+s = 8$, $k-4-s = 2$ or $k-4+s = k-4-s = 4$. The first subcase yields $k = 9$, leading to the good pair $(3,2)$ and the second subcase yields leads $k = 8$, leading to the good pair $(2,1)$.

Solution 3. Let $b+2 = c\,(a+1)$, where c is a positive integer. We observe that

$$2ac \equiv \left(2a^2+2a\right) c = 2a\,(b+2) \equiv 4a \pmod{b},$$

so $b \mid 2a\,(c-2)$.

- If $c = 1$, then we find the good pairs $(2,1)$ and $(3,2)$.
- If $c = 2$, then we find the infinite family $(a, 2a)$, where a is a positive integer.

So we can assume that $c \geqslant 3$ and now proceed in one of the following two ways.

V1. Let $2a\,(c-2) = bd$, where d is a positive integer.

i) If $d = 1$, then we find that $b = 2a\,(c-2)$ and that $b+2 = (a+1)\,c$. Combining these two relations gives that

$$ac - 4a - c + 2 = 0 \iff (a-1)\,(c-4) = 2,$$

yielding only the good pair $(a,b) = (3,18)$.

ii) If $d \geqslant 2$, then we have $2a\,(c-2) \geqslant 2b$. However,

$$b = ac + c - 2 \leqslant a\,(c-2)\,.$$

Combining these inequalities gives that $c + 2a \leqslant 2$, a contradiction.

V2. Use that $2b = 2c(a+1)-4$ and so notice that $b \mid 2a(c-2)-2b$, which means that $b \mid 4-2a-2c$.

As $2a+2c-4 \geq 2c-4 > 0$, this implies that $c(a+1)-2 \leq 2a+2c-4$.

This can be rearranged into $(a-1)(c-4) \leq 2$ and now we finish by analysing the remaining cases.

Problem 3. Let $n \geqslant 3$ be an integer. There are n colored lamps arranged in a circle. Pressing the button of a lamp once changes its color as follows:

$$\text{green} \to \text{red}, \qquad \text{red} \to \text{blue}, \qquad \text{blue} \to \text{green}.$$

Initially, all lamps are colored red. Aladdin makes moves on these lamps. Each move consists of the following three steps:

- he chooses a lamp L, without pressing its button;
- he presses the button of the clockwise neighbor of L once;
- he presses the button of the counterclockwise neighbor of L twice.

For each n, determine the maximum possible number of lamps that are colored green simultaneously, after finitely many moves.

Albania

Answer:

$$M(n) = \begin{cases} n, & 3 \mid n, \\ n-1, & 3 \nmid n \text{ and } n \text{ is odd}, \\ n-2, & 3 \nmid n \text{ and } n \text{ is even}. \end{cases}$$

Solution 1. Assign to each color a remainder modulo 3 as follows: red $= 0$, blue $= 1$, green $= 2$. Then pressing the button once means adding 1 modulo 3, while pressing it twice means adding 2, which is the same as subtracting 1 modulo 3.

When n is odd, the sequence $0, 2, 4, \ldots, 2(n-1) \pmod n$ visits all the lamps exactly once. Thus, we may label the lamps $w_1, w_2, \ldots, w_n$ in this order. A move then changes two neighboring lamps modulo 3 by $w_k \mapsto w_k - 1,\ w_{k+1} \mapsto w_{k+1} + 1$, hence the sum $S := w_1 + w_2 + \cdots + w_n$ is invariant mod 3. Since all lamps are initially red, we have $S \equiv 0 \pmod 3$, which holds for all configurations. This condition is also sufficient since using moves of the form $(w_k, w_{k+1}) \mapsto (w_k - 1, w_{k+1} + 1)$, we can transfer 1's around the circle. Therefore, every configuration with $S \equiv 0 \pmod 3$ is reachable. It remains to maximize the number of green lamps. If all n lamps are green, then the total sum is $2n$, which is 0 mod 3 when $3 \mid n$. Hence, if $3 \mid n$, then all lamps can be green, otherwise we can only achieve $n-1$ green lamps by picking the n^{th} one to be red or blue, according to n mod 3. It follows for odd n that $M(n) = n$ when $3 \mid n$ and $M(n) = n - 1$ if $3 \nmid n$.

When n is even, further mark the lamps alternately with black and white. Observe that if Aladdin chooses a black lamp, then its affected neighbors are white and vice-versa. This shows that the black and white lamps form two groups evolving independently, each containing $m := n/2$ lamps. In each of these groups the total sum is invariant mod 3 and 0 initially, and the same argument as above applies. As a result, in each group of m lamps we get m green lamps when $3 \mid n$ and $m - 1$ otherwise. It follows for even n that $M(n) = n$ when $3 \mid n$ and $M(n) = n - 2$ if $3 \nmid n$.

Solution 2. If $n \geqslant 6$, Aladdin picks a group of 6 consecutive lamps – label them from 1 to 6. Then he performs moves, in order, on lamps 2, 3, 4, 4, 5, 5. As a result, all 6 lamps will increase their status by 1 modulo 3, and the others' status will remain unchanged. Call such a sequence of moves a *smart move*.

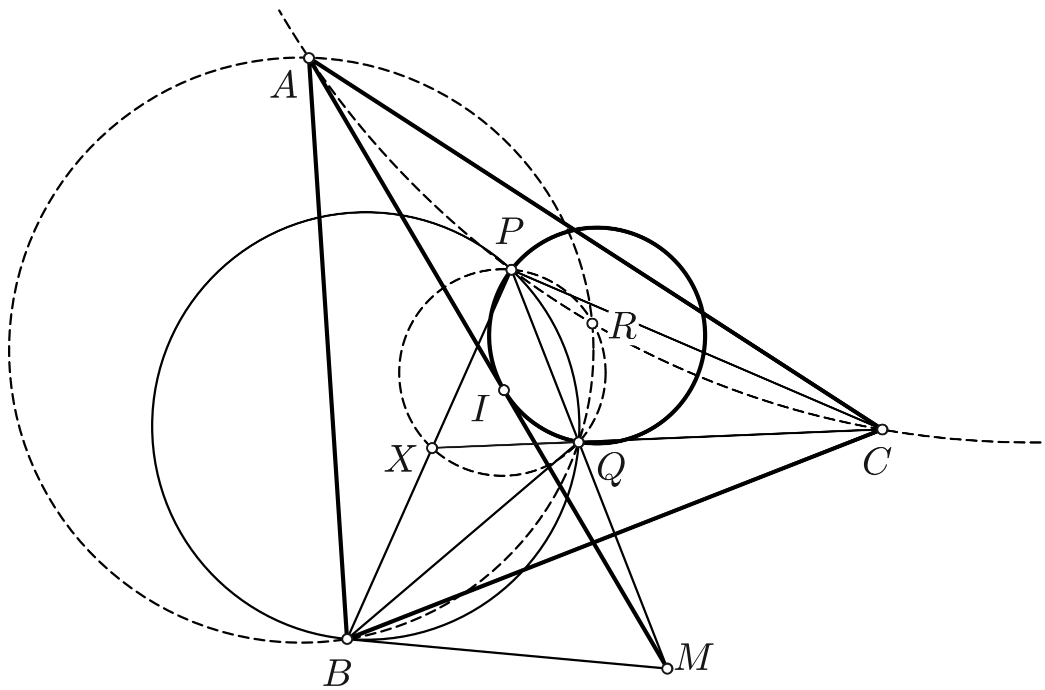
Let $n = 6k + r$, with integers $k \geq 0$ and $0 \leq r \leq 5$. Then applying appropriate smart moves $2k$ times brings the first $6k$ lamps to status 2 and the last $2r \pmod 6$ lamps to status 1. If $r = 0$ then all lamps are green, while if $r = 3$ we are left with 6 lamps, on which we perform a final smart move. Therefore, in this case Aladdin can turn green all the lamps, so the answer is n.

This also shows that if $r = 1$ or $r = 5$, Aladdin can turn green all the lamps but one (if $r = 5$, Aladdin must make one more smart move). The argument that the sum of the statuses of the lamps is constant $\pmod 3$ proves that in the case $r = 1$ or $r = 5$ the maximum is $n - 1$.

Finally, when $r = 2$ or $r = 4$, Aladdin can turn green $n - 2$ lamps and the final argument in Solution 1 proves that $n - 2$ is indeed the maximum in this case.

Problem 4. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Let P and Q be points inside triangle ABC such that $PB = PC > QC = QB$. Lines BP and CQ meet at X. Suppose that line AI is tangent to the circumcircle of triangle IPQ. Prove that the circumcircles of triangles ABQ, ACP and PQX have a common point.

BULGARIA



Solution. Without loss of generality, assume that $AB < AC$.

Step 1. First, we translate AI being tangent to (IPQ) into an angle condition. Let M be the second intersection of AI and (ABC). Since $MB = MC$, it follows that M lies on PQ. Using the incenter-excenter lemma and the given tangency, we have

$$MB^2 = MI^2 = MP \cdot MQ,$$

so MB is tangent to (BPQ) (or equivalently, the triangles $\triangle PMB$ and $\triangle BMQ$ are similar). This gives that

$$\begin{aligned} \angle MBQ = \angle MPB &\iff \angle MBC + \angle QBC = 90^\circ - \angle PBC \\ &\iff \angle QBC + \angle PBC = 90^\circ - \frac{1}{2}\angle BAC. \end{aligned} \tag{*}$$

Step 2. We now show that the circle concurrency reduces to the same angle equality. Let (AQB) and (APC) meet a second time at $R \neq A$.

Observe that

$$\angle PXQ = \angle PBC + \angle QCB = \angle PBC + \angle QBC.$$

On the other hand,

$$\begin{aligned}\angle PRQ &= \angle ARQ - \angle ARP \\ &= (180^\circ - \angle ABQ) - \angle ACP \\ &= 180^\circ - (\angle ABC - \angle QBC) - (\angle ACB - \angle PCB) \\ &= \angle BAC + \angle QBC + \angle PBC.\end{aligned}$$

Finally, X lies on (PQR) if and only if

$$\angle PRQ = 180^\circ - \angle PXQ \iff \angle PBC + \angle QBC = 90^\circ - \frac{1}{2}\angle BAC,$$

which is the same as condition (*) found at **Step 1**.

Our proof is complete.