Problems and solutions of the 14th European Mathematical Cup

13th December 2025 - 21st December 2025

Junior Category

Problem 1. Let ω_1 and ω_2 be two circles intersecting at A and B. The common tangent, closer to A, of ω_1 and ω_2 touches ω_1 at P and ω_2 at Q. The tangent of ω_1 at A meets ω_2 at C, which is different from A, and the extension of AP meets QC at D. Let E be the centre of the circumcircle of triangle ABD. The lines AD and QE intersect at F. Prove that F lies on the circle with diameter PQ.

(Steve Vo Dinh)

Solution. It suffices to show that $\angle PFQ = 90^\circ$. Since $AE = DE$ (since E is the centre of the circumcircle of ABD), this is equivalent to QE being the perpendicular bisector of $\overline{AD}$, since we know that all points on the perpendicular bisector are equidistant from the endpoints of $\overline{AD}$. So, it is sufficient to prove $QA = QD$, and with this, we completely remove E from the diagram.

3 points.

For convenience, denote $\angle AQP = \alpha$, $\angle APQ = \beta$. Then:

$$\angle QAD = \alpha + \beta$$

from triangle APQ (since the exterior angle is equal to the sum of the two remaining interior angles).

1 point.

It remains to compute $\angle QDA = \angle ACD + \angle DAC$. We have

$$\angle ACD = \angle ACQ = \angle AQP = \alpha$$

from the tangent QP to ω_2 (by the Chord Tangent Lemma, also known as the Alternate Segment Theorem)

1 point.

On the other hand, $\angle DAC = 180^\circ - \angle PAC$ since P, A and D lie on the same line. However, since AC is tangent to ω_1, we can find $\angle CAB = \angle APB$ (again by the Chord Tangent Lemma).

Now, we can break the $180^\circ = \angle PAD = \angle PAB + \angle BAC + \angle CAD$ around point A, but we can also find the sum of internal angles in APB to be $\angle PAB + \angle BPA + \angle ABP = 180^\circ$, so by previous conclusion, we get that $\angle ABP = \angle CAD$.

3 points.

Lastly, $\angle ABP = \angle APQ = \beta$ since PQ is tangent to ω_1 (using the Chord Tangent Lemma one last time).

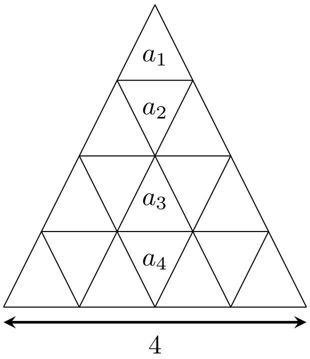
By looking at triangle CDA, we conclude that $\angle QDA = \angle ACD + \angle DAC$, so we have

$$\angle QDA = \angle ACD + \angle DAC = \alpha + \beta = \angle AQP + \angle APQ = \angle QAD$$

concluding $QA = QD$, as desired.

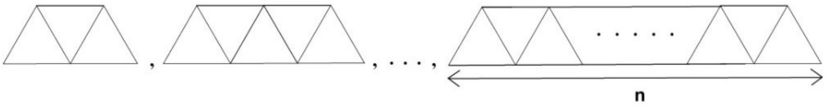
2 points.

Problem 2. Let n be a positive integer. Divide an equilateral triangle of side length n equilateral triangles of side length one. (example shown below for $n = 4$):



Label the small equilateral triangles through which one of the altitudes of the large equilateral triangle passes as $a_1, a_2, \ldots, a_n$ (see illustration above).

Let $f(i)$ denote the number of ways to tile the large equilateral triangle using exactly one of each



such that the triangle a_i is removed.

- If n is even, determine $f(2) + f(4) + \cdots + f(2k) + \cdots + f(n)$.
- Prove that $f(1) + f(2) + \cdots + f(n) \geqslant 2^{n-2}$, for all positive integers n.

(Karlo Jokoš)

First Solution. First Part

Suppose there is a tiling with triangle a_{2k} missing, we tile the triangle starting with the largest tile, we can put it at three places (left, right or down), and we are left with a triangle of side length $n - 1$. We repeat that step repeatedly, choosing to put the tile left, right or down, until we are left with the triangle a_{2k}. But then we had to choose down $n - 2k$ times, and left and right both the same number of times, so $\frac{n-1-(n-2k)}{2} = \frac{2k-1}{2} \notin \mathbb{N}$, which is impossible, hence $f(2k) = 0$ and the sum is 0.

Alternative proof of first part

Colour all the triangles pointing upward black and the others white. The full triangle has n more black triangles than white triangles. Each of the $n - 1$ shapes we are filling the big triangle with has one more black than white triangles, so the last unfilled triangle must be black. This is not the case for even numbers, so $f(2k) = 0$ and the sum is 0.

Regardless of the way the student have proved the first part, they should be awarded **4 points.**

Second Part

Let S_n be the sum we are bounding, and assume that the inequality holds for $1, 2, \ldots, n$.

Let us prove the $n + 1$ case, there are three places where we can put the largest tile, at the bottom of the big triangle, to the left or to the right.

In the first case there are S_n ways to tile the rest of the triangle, in the second case if we put the second largest tile on the right, we end up with a triangle of side length $n-1$ so there are at least S_{n-1} ways to tile the triangle in that case, and the same bound holds for the case where the largest tile is to the right. So we have $S_{n+1} \geq S_n + 2S_{n-1} \geq 2^{n-2} + 2 \cdot 2^{n-3} = 2^{n-1}$

So we have to check the base case for $n = 1, 2$ and we get

$$S_1 = 1 \geq \frac{1}{2}, \quad S_2 = 1 \geq 1$$

6 points.

Second Solution. Suppose there is a tiling with triangle a_k missing, we tile the triangle starting with the largest tile, we can put it at three places (left, right or down) and we are left with a triangle of side length $n-1$. We repeat that step repeatedly, choosing to put the tile left, right or down, until we are left with the triangle a_k. But then we had to choose down $n-k$ times, and left and right both the same number of times so $\frac{n-1-(n-k)}{2} = \frac{k-1}{2}$.
So we count the number of $n-1$-tuples with entries from {down, left, right}, such that there are $n-k$ downs, and $\frac{k-1}{2}$ lefts and rights. So $f(k) = \binom{n-1}{n-k}\binom{k-1}{\frac{k-1}{2}}$

6 points.

First Part

For even k we get that $f(k) = 0$ so if n is even the sum over the even numbers is 0.

1 point.

Second Part

The sum is equal to

$$\sum_{k \text{ odd}} \binom{n-1}{n-k}\binom{k-1}{\frac{k-1}{2}} \geq \sum_{k \text{ odd}} \binom{n-1}{n-k} = \frac{1}{2}\sum \binom{n-1}{n-k} = 2^{n-2}$$

3 points.

Problem 3. Determine the largest positive integer n for which there exist positive integers a and q such that

$$q^6 \le n \quad \text{and} \quad \left|\sqrt{2} - \frac{a}{q}\right| \le \frac{1}{\sqrt{n}}.$$

(Miroslav Marinov)

First Solution. Assume $q \ge 2$. Then

$$\left|\sqrt{2} - \frac{a}{q}\right| \le \frac{1}{\sqrt{n}} \le \frac{1}{q^3} \le \frac{1}{8}.$$

In particular,

$$\frac{a}{q} \le \sqrt{2} + \frac{1}{8},$$

so

$$\sqrt{2} + \frac{a}{q} \le 2\sqrt{2} + \frac{1}{8} < 3.$$

On the other hand, using the fact that $|2q^2 - a^2| \ge 1$ (since $\sqrt{2}$ is irrational and hence $2q^2 - a^2 \ne 0$), we obtain

$$\frac{1}{q^2} \le \frac{|2q^2 - a^2|}{q^2} = \left|\sqrt{2} - \frac{a}{q}\right|\left|\sqrt{2} + \frac{a}{q}\right| < \frac{3}{q^3}.$$

Therefore $q < 3$. Since q is a positive integer, it remains to consider only $q = 1$ and $q = 2$.

Case 1: $q = 1$. We would need

$$\left|\sqrt{2} - a\right| \le \frac{1}{8}.$$

However, since $1.4 < \sqrt{2} < 1.5$, the distance from $\sqrt{2}$ to the nearest integer is greater than 0.4, which is larger than 1/8. (Alternatively, we can proceed directly to $q = 2$ by arguing that if we find a possible n with $q = 2$, then the case $q = 1$ is automatically covered by the case $q = 2$, since any integer can be written as a fraction with denominator 2.)

Case 2: $q = 2$. For $n \ge 2^6$ we must have

$$\left|\sqrt{2} - \frac{a}{2}\right| \le \frac{1}{8}.$$

If $a \ne 3$, then

$$\left|\sqrt{2} - \frac{a}{2}\right| \ge \left|\sqrt{2} - 1\right| > \frac{1}{8},$$

so the only possible value is $a = 3$.

7 points.

It remains to compute the largest n such that

$$\left|\sqrt{2} - \frac{3}{2}\right| \le \frac{1}{\sqrt{n}}.$$

Equivalently,

$$n \le \frac{1}{\left(\frac{3}{2} - \sqrt{2}\right)^2} = \left(\frac{3}{2} + \sqrt{2}\right)^2 = 68 + 48\sqrt{2}.$$

Since $68 + 48\sqrt{2} < 136$ and $68 + 48\sqrt{2} > 135$, the largest possible integer value is

$$\boxed{135}.$$

3 points.

Second Solution. (Alternative proof of $q \leq 2$)
We will find the largest integer n when $q \leqslant 2$ in the same order as in the first solution.

3 points.

We have

$$\begin{aligned} \left|\sqrt{2} - \frac{a}{q}\right| \leq \frac{1}{n} \leq \frac{1}{q^3} \\ \implies |\sqrt{2}q^3 - aq^2| \leq 1 \\ \implies \sqrt{2}q - \frac{1}{q^2} \leq a \leq \sqrt{2}q + \frac{1}{q^2} \end{aligned}$$

Squaring gives

$$2q^2 - \frac{2\sqrt{2}}{q} + \frac{1}{q^4} \leqslant a^2 \leqslant 2q^2 + \frac{2\sqrt{2}}{q} + \frac{1}{q^4}$$

Now assume for contradiction that $q \geqslant 3$ and notice that we have $\frac{2\sqrt{2}}{q} - \frac{1}{q^4} < \frac{2\sqrt{2}}{q} + \frac{1}{q^4} \leq \frac{2\sqrt{2}}{3} + \frac{1}{81} < \frac{29}{30} + \frac{1}{30} = 1$. Hence $2q^2 - 1 < a^2 < 2q^2 + 1$. Since a and q are integers, we must have $a^2 = 2q^2$, but this is impossible, as $\sqrt{2}$ is irrational.

7 points.

Problem 4. Find all positive integers n with the following property:
For every positive integer d which divides n, there exists a positive integer k which divides n such that

$$d + n \mid dn + k.$$

(Ivan Novak)

Solution. Note that $n = 1$ is obviously a solution. Now consider $n > 1$.
Taking $d = n$ yields $2n \mid n^2 + k$, which gives us $k = n$ and $2n \mid n^2 + n$, which implies that n must be odd.
We'll now prove that n must be a prime power.
Note that $d \mid d + n$ for any divisor d of n, so if $d + n \mid dn + k$, it follows that $d \mid k$.
We can rewrite the condition as

$$1 + \frac{n}{d} \mid n + \frac{k}{d}.$$

Now note that $n \equiv -d \pmod{1 + \frac{n}{d}}$, so the condition is equivalent to

$$1 + \frac{n}{d} \mid \frac{k}{d} - d.$$

If $\frac{n}{d} \geqslant d$, then $|\frac{k}{d} - d| \leqslant \frac{n}{d} < 1 + \frac{n}{d}$, so we must have $\frac{k}{d} = d$, or $k = d^2$.
This means that for any divisor $d \leqslant \sqrt{n}$ of n, d^2 also divides n.

2 points.

Now suppose that n is not a power of a prime. Let $n = ab$ with $a > b$ coprime and greater than 1. Then b^2 divides $n = ab$, but then b divides a, a contradiction. Thus, n is a power of a prime.

1 point.

Let $n = p^m$ for some positive integer m.
Let $d = p^j$ with $m \geqslant j > \frac{m}{2}$. Then there exists p^ℓ with $j \leqslant \ell \leqslant m$ such that

$$p^j + p^m \mid p^{m+j} + p^\ell,$$

which is equivalent to

$$p^{m-j} + 1 \mid p^{m-\ell+j} + 1.$$

Recall the well known fact that if $a \geqslant 2$, x and y are positive integers such that $a^x + 1$ divides $a^y + 1$, then $\frac{y}{x}$ is an odd integer. Hence $m - \ell + j$ must be an odd number times $m - j$, i.e. there exists a nonnegative integer r with

$$m - \ell + j = (2r + 1)(m - j),$$

or equivalently

$$2r(m - j) = 2j - \ell \leqslant j.$$

Now take $j = \lfloor m/2 \rfloor + 1$. We get $2r(\lceil m/2 \rceil - 1) = 2\lfloor m/2 \rfloor + 2 - \ell$. For $r = 0$ this doesn't work as then we'd have $\ell > m$. For $r > 0$ the left hand side is at least $m - 2$, and the right hand side is at most $m/2 + 1$, so we have $m - 2 \leqslant m/2 + 1$, or $m \leqslant 6$.

4 points.

For $m = 5$, we must have

$$2r(5 - j) = 2j - \ell.$$

Take $j = 3$. We get $4r = 6 - \ell \leqslant 3$, impossible. Thus, $m = 5$ is also not a solution.

1 point.

For $m \in \{1, 2, 3, 4, 6\}$, we can easily check that for any $j > m/2$ we can find r and ℓ satisfying the condition

$$2r(m - j) = 2j - \ell.$$

Thus, the set of solutions is

$$\{p^m \mid p \text{ an odd prime }, m \in \{0, 1, 2, 3, 4, 6\}\}.$$

2 points.

PROBLEMS AND SOLUTIONS OF THE 14TH EUROPEAN MATHEMATICAL CUP

13th December 2025 - 21th December 2025

Senior Category

Problem 1. Let $k \geqslant 2$ be an integer. Let m and n be coprime positive integers with exactly k positive divisors such that $m < n$.

For $i \in \{1, \ldots, k\}$, denote by f_i and d_i the i-th smallest divisor of m and n, respectively. Suppose that

$$d_i - f_i \mid n - m$$

for all $i \in \{2, \ldots, k\}$. Prove that $d_i \geqslant f_i$ for all $i \leqslant \frac{k}{2}$.

(Ivan Novak)

First Solution. We have by the condition of the problem:

$$f_1 < f_2 < \cdots < f_k \ \ \textit{and} \ \ d_1 < d_2 < \cdots < d_k$$

Pairing up the divisors, we get:

$$n = d_{k+1-j}d_j \ \ \textit{and} \ \ m = f_{k+1-j}f_j, \ \ \forall j \in \{1, 2, \cdots, k\}$$

1 point.

Suppose for the sake of contradiction that there is $i \leq \dfrac{k}{2}$ such that $d_i < f_i$. So, we have:

$$j = k + 1 - i \geq k + 1 - \frac{k}{2} \geq \frac{k}{2} + 1 \geq 2$$

The last inequality implies $f_i < f_j$ and $d_j > f_j$, otherwise $m > n$ which is impossible.

1 point.

Setting $a = f_i - d_i$ and $b = d_j - f_j$, we see that both a and b are positive integers and by the condition of the problem we have:

$$a \mid n - m \ \ \textit{and} \ \ b \mid n - m$$

We have now: $n - m = d_i d_j - f_i f_j = (d_i - f_i)d_j + (d_j - f_j)f_i = -ad_j + bf_i$. The last equation gives:

$$a \mid f_i b \ \ \textit{and} \ \ b \mid d_j a$$

3 points.

But since $gcd(m, n) = 1$, then we get:

$$1 = gcd(f_t, d_j) = gcd(f_i, f_i - d_j) = gcd(f_i, a)$$

1 point.

We deduce that a divides b and by symmetry that b divides a, hence $a = b$.

1 point.

Returning to the initial equation and plugging this result gives:

$$n - m = a(f_i - d_j) = a(f_i - f_j - a)$$

But $a(f_i - f_j - a) < 0$ because $f_i - f_j \leq 0$ and $a > 0$, so $n - m < 0$ which is a contradiction, thus such i does not exist which solves the problem.■

3 points.

Second Solution. For the sake of contradiction, suppose $f_i > d_i$ for some $i \leq \frac{k}{2}$. Then, since

$$n = d_i \cdot d_{k+1-i} > f_i \cdot f_{k+1-i} = m$$

1 point.

we must have $d_{k+1-i} > f_{k+1-i}$.

1 point.

We notice that:

$$\begin{aligned} & d_i - f_i \mid n - m \\ \implies & d_i - f_i \mid n - m - \frac{n}{d_i} \cdot (d_i - f_i) \\ \implies & d_i - f_i \mid \frac{n \cdot f_i}{d_i} - \frac{m \cdot f_i}{f_i} \\ \implies & d_i - f_i \mid \frac{n}{d_i} - \frac{m}{f_i} \\ \implies & d_i - f_i \mid d_{k+1-i} - f_{k+1-i} \end{aligned}$$

4 points.

This gives us

$$d_i - f_i \mid d_{k+1-i} - f_{k+1-i} \quad \text{and} \quad d_{k+1-i} - f_{d+k-1} \mid d_i - f_i \textit{ (in a similar way)}$$

1 point.

From which we deduce that

$$f_i - d_i = d_{k+1-i} - f_{k+1-i} \iff \frac{n}{d_i} + d_i = \frac{m}{f_i} + f_i$$

Now since $n > m$ we have:

$$\frac{n}{d_i} + d_i > \frac{m}{d_i} + d_i$$

And from $(a+b)^2 = (a-b)^2 + 4ab$ and noting that since we supposed $d_i < f_i$ we have $\frac{m}{d_i} - d_i > \frac{m}{f_i} - f_i$ we have:

$$\frac{m}{f_i} + f_i < \frac{m}{d_i} + d_i$$

The last two inequalities give us:

$$\frac{m}{d_i} + d_i < \frac{n}{d_i} + d_i = \frac{m}{f_i} + f_i < \frac{m}{d_i} + d_i$$

which is a contradiction. ■

3 points.

Problem 2. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that for every positive integer k, the set $\{f(1), f(2), \ldots, f(k)\}$ contains exactly $f(f(k))$ elements. Prove that

$$f(f(f(k))) = f(k)$$

for every positive integer k.

(Ivan Novak)

First Solution. Note that $f \circ f$ is increasing and $f(f(n+1)) - f(f(n)) \leqslant 1$ for all $n \geqslant 1$.

1 point.

Therefore, the image of $f \circ f$ is either the whole of $\mathbb{N}$, or a set of the form $\{1, 2, \ldots, a\}$ for some $a \geqslant 1$.

If the image is the whole of $\mathbb{N}$, let $a_1, a_2, \ldots$ be a sequence such that a_i is the smallest positive integer satisfying $f(f(a_i)) = i$. Note that the image of f is also $\mathbb{N}$. Also, if $f(x)$ is distinct from $f(1), f(2), \ldots, f(x-1)$, then $f(f(x)) > f(f(x-1))$. This means that x is an element of the sequence $a_1, a_2, \ldots$. From this we can conclude that the image of f is equal to $\{f(a_1), f(a_2), \ldots, \}$.

Now for any $x \in \mathbb{N}$, if $f(a_j) = x$ for some j, we must have $f(x) = j$. Hence, if $f(x) = f(y) = j$ for some positive integers j, y and x, we must also have $f(a_j) = x = y$. Therefore, f is injective. Now it's easy to see that $f(f(n)) = n$, from which the claim follows.

4 points.

Suppose the image of $f \circ f$ is equal to $\{1, \ldots, a\}$ for some $a \geqslant 1$. Then if the image of f contained more than a elements, taking the least n such that $\{f(1), \ldots, f(n)\}$ contains $a+1$ elements would yield a contradiction. Hence, the image of f is equal to $\{1, \ldots, a\}$ as well. However, this means that

$$\{1, \ldots, a\} = f(\mathbb{N}) = f(f(\mathbb{N})) = f(\{1, \ldots, a\}),$$

which means that the restriction of f to the set $\{1, \ldots, a\}$ is a bijection. Now it's easy to see from the condition that $f(f(j)) = j$ for all $j \leqslant a$. For any $j > a$, we must have $f(f(j)) = a$ and $f(j) \leqslant a$. Hence, $f(j)$ is the unique integer $k \leqslant a$ such that $f(k) = a$. Then we have

$$f(f(f(j))) = f(a) = f(f(k)) = k = f(j),$$

which proves the claim.

5 points.

Second Solution. Like in previous solution we notice that $0 \leq f(f(n+1)) - f(f(n)) \leq 1$.

1 point.

Lemma 1: $f(f(a)) = f(f(b)) \implies f(a) = f(b)$
Let k and l respectively be the smallest integers such that $f(k) = f(a)$ and $f(l) = f(b)$. Then we have

$$f(f(k)) = f(f(a)) = f(f(b)) = f(f(l)).$$

1 point.

Now, if $k > l$, we have $f(f(k)) > f(f(l))$ because $f(k) \notin \{f(1), f(2), \ldots, f(k-1)\}$ (we know that $k-1$ exists since $k > l$), so it must be that $f(f(k)) > f(f(l))$, and $f \circ f$ is increasing. Similarly, we solve $l > k$. So we must have $k = l$ which implies

$$f(a) = f(k) = f(l) = f(b).$$

5 points.

We do a proof by induction.

Base $k = 1$: It is obvious that $f(f(1)) = 1$ from problem statement. So we have $f(f(f(1))) = f(1)$.

Assumption for all $k \leq n$: Assume that $f(f(f(k))) = f(k)$.

Step for $k = n+1$: We will split this in 2 cases.

Case 1: $f(f(n+1)) = f(f(n))$.
Using the lemma, we have $f(n+1) = f(n)$. And then we have $f(f(f(n+1))) = \underbrace{f(f(f(n))) = f(n)}_{Assumption} = f(n+1)$.

1 point.

Case 2: $f(f(n+1)) = f(f(n)) + 1$.
If $f(f(n)) = n$, this is trivial since $f(f(f(n+1))) = f(f(f(n)) + 1) = f(n+1)$.
Assume that $f(f(n)) < n$. This implies $f(f(n+1)) \leq n$ so we can use the assumption of induction for selecting $k = f(f(n+1))$, so we get $f(f(f(f(f(n+1))))) = f(f(f(n+1)))$. Using the lemma twice, we get $f(f(f(n+1))) = f(n+1)$.

2 points.

Third Solution. Let n be the biggest integer such that $f(f(n)) = n$ if it exists. Otherwise since we know $f(f(1)) = 1$ it implies that it must hold for all $n \in \mathbb{N}$ so we have $f(f(f(k))) = f(k)$ for all $k \in \mathbb{N}$.

1 point.

By the fact that n is the biggest such integer, we must have $f(f(n+1)) = f(f(n)) = n$ because, as in previous solutions, we have $0 \leq f(f(n+1)) - f(f(n)) \leq 1$.

1 point.

By the definition of n and the condition from problem we have $f(f(i)) = i$ for all $i \leq n$ this gives $f(f(f(i))) = f(i)$ which means that $f(i)$ is a fixed point of $f \circ f$ and since n is the biggest fixed point of $f \circ f$ by our definition we have $f(i) \leq n$.

1 point.

We will prove that $f(m) = f(n)$ for all $m > n$. Assume that $m > n$ is the first integer such that $f(m) \neq f(n)$ this means that

$$f(n) = f(n+1) = \cdots = f(m-1)$$

Case 1: $f(f(m)) = f(f(n))$.
We can proceed to prove it as in the **Lemma 1 of Second Solution** that we must have $f(m) = f(n)$. However, due to many more assumptions, easier proofs also work.

3 points.

Case 2: $f(f(m)) = f(f(n)) + 1 = n + 1$ Considering $f(f(f(m))) = f(n+1) \leq n < n+1 = f(f(m))$ we must have $f(m) < m$. But now we get

$$f(f(m)) \in \{f(1), f(2), \ldots, f(m-1)\} = \{f(1), f(2), \ldots, f(n)\}$$

. And since we know that all of those numbers are less than n (as we have proven). We get $n + 1 = f(f(n+m)) < n$, which is a contradiction. ■

4 points.

Problem 3. Let ABC be an acute triangle with circumcircle ω. Let the angle bisector of $\angle B$ intersect AC, ω, and the parallel to AB from C in D, E and F respectively. Let X be the intersection of ω and the circumcircle of triangle ΔDCF and let Y be a point on CF such that $YF = YD$. The line XF intersects ω and DY in T and P respectively. The circumcircle of triangle $\triangle TDE$ meets the lines PF and EY in R and S.
Prove that the circumcircles of triangles $\triangle PRS$ and $\triangle BDC$ are internally tangent.

(Yasser Merabet)

Solution. Define the point R as the intersection of the circumcircle of triangle ΔTDE with the line PF, and the point S as the intersection of the circumcircles of triangles ΔTDE with the line EY . We will show that the circumcircles of triangles ΔPRS and ΔBDC are internally tangent which solves the problem. To do so, define the point Z as the intersection of the lines PB and CF.

Let $\measuredangle$ denote directed angles throughout this solution, (signed mod 180°). The notation (AB) means the line which passes through the points A and B and the notation (ABC) means the circumcircle of triangle ΔABC.
Claim 1: $(XB) \parallel (RD)$
Proof: We have:

$$\underbrace{\measuredangle EDR = \measuredangle ETR}_{RTDE \text{ cyclic}} = \underbrace{\measuredangle ETX = \measuredangle EBX}_{XTBE \text{ cyclic}}$$

Which proves claim 1.

2 points.

Claim 2: $XDBP$ is cyclic
Proof: Note that:

$$\underbrace{\measuredangle CBF = \measuredangle FBA}_{BF \text{ is the angle bisector of } \measuredangle CBA} \overset{(AB)\parallel(CF)}{=} \measuredangle BFC \overset{YD=YF}{=} \measuredangle YDF$$

hence, $(DY) \parallel (BC)$. Now, in one hand, we have:

$$\begin{aligned} &\measuredangle DPX + \measuredangle FYP + \measuredangle PFY = 0 \\ &\measuredangle DPX + \measuredangle ABC + \angle PFY = 0 \quad (DY) \parallel (BC) \end{aligned}$$

So, we get:

$$\measuredangle DPX = \measuredangle CBA + \measuredangle YFP \tag{1}$$

In other hand, we have:

$$\begin{aligned} \measuredangle CBA &= \measuredangle DBX + \measuredangle XBA + \measuredangle CBE \\ &= \measuredangle DBX + \measuredangle XCA + \measuredangle CBF \quad (ABCX \text{ is cyclic}) \\ &= \measuredangle DBX + \measuredangle XFD + \measuredangle DFC \quad (DCFX \text{ is cyclic}) \\ &= \measuredangle DBX + \measuredangle XFC \\ &= \measuredangle DBX + \measuredangle PFY \end{aligned}$$

So, we get:

$$\measuredangle DBX = \measuredangle CBA + \measuredangle YFP \tag{2}$$

Combining the two previous results ((1) *and* (2)), we deduce that the quadrilateral $XDBP$ is cyclic.

2 points.

Notice that:

$$\measuredangle DCZ = \measuredangle DCF = \measuredangle DXF = \measuredangle DXP = \measuredangle DBP = \measuredangle DBZ$$

which implies $DCZB$ is cyclic, so $Z \in (BDC)$. Now, we have to show that $Z \in (PRY)$.

1 point.

Claim 3: $FRDZ$ is cyclic
Proof: We have:

$$\measuredangle FRD = \measuredangle PRD \overset{Claim\ 1}{=} \measuredangle PXB \overset{Claim\ 2}{=} \angle PDB \overset{(DY)\parallel(BC)}{=} \angle CBD \overset{DCZB\ cyclic}{=} \measuredangle CZD = \measuredangle FZD$$

which proves claim 3.

1 point.

Claim 4: $ZPRY$ cyclic
Proof: We have:

$$\begin{aligned}\measuredangle PRZ &= \measuredangle PRD + \measuredangle DRZ\\ &= \measuredangle PXB + \measuredangle DFZ \quad (\textit{claim } 1 \textit{ and claim } 3)\\ &= \measuredangle PDB + \measuredangle DFY \quad (\textit{claim } 2)\\ &= \measuredangle YDF + \measuredangle DFY\\ &= \measuredangle FYD\\ &= \measuredangle PYD\end{aligned}$$

which shows the desired claim.

2 points.

Before we finish, we show that $S \in (PRY)$. To do so, notice that:

$$\measuredangle YSR = \measuredangle ESR \overset{ETSR\ cyclic}{=} \measuredangle ETR \overset{EDTR\ cyclic}{=} \measuredangle EDR = \measuredangle FDR \overset{Claim\ 3}{=} \measuredangle FZR = \measuredangle YZR \overset{Claim\ 4}{=} \measuredangle YPR$$

Hence, we deduce that the points R, P, S and Y are concyclic and from what we did before, we find that the circumcircles of triangles $\triangle PRS$ and $\triangle BDC$ pass through the point Z.

1 point.

We finish the proof by taking the homothety centered at Z with ratio $\dfrac{ZC}{ZY}$, we get that C must be sent to Y and B must be sent to P (as $(BC) \parallel (PY)$), but Z belongs to (DBC) and (PRY), thus the circumcircles of triangles ΔDBC and ΔPRS are internally tangent which solves the problem.■

1 point.

Problem 4. Determine all sequences of positive real numbers $a_1, a_2, a_3, \dots$, such that for each positive integer n the following equality holds:

$$a_n + \max(a_{n+1}, a_{n+2}) = \frac{1}{\min(a_n, a_{n+1})}.$$

(Ivan Novak)

Solution. The only constant sequence that is a solution is the sequence satisfying $a_n = \frac{1}{\sqrt{2}}$ for all n. Also, one can easily check that any sequence which contains two consecutive elements equal to $\frac{1}{\sqrt{2}}$ is constant. Now consider a nonconstant sequence $(a_n)_n$ satisfying the conditions.

Now suppose that $a_n \geqslant \frac{1}{\sqrt{2}}$ for some n. If $a_{n+1} > \frac{1}{\sqrt{2}}$, then the right hand side of the equality

$$a_n + \max(a_{n+1}, a_{n+2}) = \frac{1}{\min(a_n, a_{n+1})}$$

is at most $\sqrt{2}$, and the left hand side is greater than $\sqrt{2}$, which is a contradiction.

Thus, if $a_n \geqslant \frac{1}{\sqrt{2}}$, then $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$.

Now suppose that $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$. Then $a_n + a_{n+2} = \frac{1}{a_{n+1}}$. Furthermore, since $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$, one has $a_{n+3} \leqslant \frac{1}{\sqrt{2}}$ and from the recurrence for $n+1$ one obtains

$$a_{n+1} + a_{n+2} = \frac{1}{a_{n+1}},$$

so $a_n = a_{n+1} = \frac{1}{\sqrt{2}}$ and the sequence is constant, a contradiction.

We have thus proved that if $a_n \geqslant \frac{1}{\sqrt{2}}$, then $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$.

1 point.

Now suppose a_n, a_{n+1}, a_{n+2} are all $\leqslant \frac{1}{\sqrt{2}}$. Then the left hand side of the assertion is at most $\sqrt{2}$, and the right hand side is at least $\sqrt{2}$, with equality holding if $a_n = \frac{1}{\sqrt{2}} = a_{n+1}$, which would yield a constant sequence, a contradiction.

Thus, exactly every third element of the sequence is $\geqslant \frac{1}{\sqrt{2}}$, and the others are $\leqslant \frac{1}{\sqrt{2}}$.

Now if $a_n = \frac{1}{\sqrt{2}}$ and $a_{n+1} < \frac{1}{\sqrt{2}}$, then the right hand side is greater than $\sqrt{2}$, and the left hand side is at most $\sqrt{2}$, a contradiction. Thus, we may also assume that $\frac{1}{\sqrt{2}}$ does not occur in the sequence.

Now suppose $a_n > \frac{1}{\sqrt{2}}$. Then $a_{n+3} > \frac{1}{\sqrt{2}}$.

2 points.

If $a_{n+1} < a_{n+2}$, then from the assertion for $n+1$ one has

$$a_{n+1} + a_{n+3} = \frac{1}{a_{n+1}},$$

and from the assertion for $n+2$ one has

$$a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}.$$

From here, it follows that $a_{n+1} = a_{n+2}$, a contradiction.

If $a_{n+1} > a_{n+2}$, one has $a_{n+1} + a_{n+3} = \frac{1}{a_{n+2}}$ and $a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}$, so $a_{n+1} = a_{n+2}$, a contradiction.

Thus, $a_{n+1} = a_{n+2}$ whenever $a_n > \frac{1}{\sqrt{2}}$.

2 points.

Now we also have $a_n + a_{n+1} = \frac{1}{a_{n+1}}$ and $a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}$, so $a_n = a_{n+3}$ and the sequence is periodic with period 3, and of the form

$$(, \dots, \frac{1}{x} - x, x, x, \dots)$$

for some $x < \frac{1}{\sqrt{2}}$, with possibly two initial members of the sequence being $< \frac{1}{\sqrt{2}}$ not being equal to $\frac{1}{x} - x$.

Now suppose that $a_n = \frac{1}{x} - x$, $a_{n+1} = x$ for some $x < \sqrt{2}$, and suppose that a_{n-1} exists.

Then $a_{n-1} + a_n = \frac{1}{a_{n-1}}$, so $a_{n-1} = x$. This means that even the first part of the sequence is periodic.

2 points.

In conclusion, the answer is all periodic sequences of length 3 of the form

$$(, \dots, \frac{1}{x} - x, x, x, \dots),$$

where $x < \frac{1}{\sqrt{2}}$, and the constant sequence $a_n = \frac{1}{\sqrt{2}}$.

1 point.

Now suppose $a_n \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+1} > \frac{1}{\sqrt{2}}$. Then $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$ and $a_n + a_{n+1} = \frac{1}{a_n}$

Now suppose that $a_n \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$. Then the right hand side is at least $\sqrt{2}$, so $\max(a_{n+1}, a_{n+2})$ is at least $\frac{1}{\sqrt{2}}$, which means that $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$, or $a_n = a_{n+1} = \frac{1}{\sqrt{2}}$ and $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$.

2 points.