The 6th Junior Balkan Mathematical Olympiad

SHORTLISTED PROBLEMS
AND
SOLUTIONS

Târgu Mureş, 2002

ALGEBRA

A1-D. *A student plays a computer game. The computer provides him with 2002 positive distinct numbers randomly choosen. The game rules allows him to do the following operations:*

- *take two of the given numbers, double one of them add the second number and keep the sum;*
- *next, choose two other numbers from the remaining ones, double one of them and add the second; then multiply the sum with the previous one and keep the result;*
- *repet the above procedure until all the 2002 given numbers are used.*

The student wins the game if the last product he finally obtains is maximal.

Find, with proof, the winning strategy of the game.

Solution. Let $a_1 < a_2 < \cdots < a_{2002}$ be the given numbers, and let P be the maximal value of the final product. The number P has the form

$$P = (2a_{i_1} + a_{i_2})(2a_{i_3} + a_{i_4}) \cdots (2a_{i_{2001}} + a_{i_{2002}}),$$

where $a_{i_1}, a_{i_2}, \ldots, a_{i_{2002}}$ is a permutation of the given numbers.

First, remark that if $a > b$ then $2a + b > 2b + a$. Hence, the product increases when in a pair the greatest number is doubled. It follows that in P we must have $a_{i_k} > a_{i_{k+1}}$, for any $k = 1, 3, 5, , \ldots, 1001$.

Next, we prove that P contains the factor $2a_{2002} + a_1$. Indeed, when assuming the contrary, P will contain a factor of the form

$$(2a_{2002} + b)(2c + a_1).$$

The following inequality then holds

$$(2a_{2002} + b)(2c + a_1) < (2a_{2002} + a_1)(2c + b),$$

the proof of which being a simple reduction to

$$(a_{2002} - c)(b - a_1) > 0,$$

which is obvious.

The same argument than works with $\frac{P}{2a_{2002}+a_1}$ which should be maximal for $a_2, a_3 \ldots, a_{2001}$, a.s.o. Hence P maximal is given by the formula

$$P = (2a_{2002} + a_1)(2a_{2001} + a_2) \cdots (2a_{1002} + a_{1001}),$$

whence the corresponding strategy of the play.

A2-E. *All positive integers are arranged in patern as it is shown bellow*

1	3	6	10	15	...
2	5	9	14	...	
4	8	13	...		
7	12	...			
11	...				
...					

Find the number of the column and the number of the row where 2002 is put.

Solution. Let a the number of the column and b the number of the row where 2002 is put.

It is easy to see that the n-th number of the first row equals $\frac{n(n+1)}{2}$. Since $\frac{62\cdot 63}{2} = 1953$ and $\frac{63\cdot 64}{2} = 2016$, and $1953 < 2002 < 2016$, we conclude that $a + b = 64$. It follows that $b = 2016 - 2002 + 1 = 15$ and $a = 64 - 15 = 49$.

A3-D. *Given positive numbers a, b, c, prove that*

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(a+c)} \geq \frac{27}{2(a+b+c)^2}.$$

Solution. Apply the GM-AM mean inequality to the right term. One obtains:

$$\left(\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(a+c)}\right)^3 \geq \frac{27}{abc(a+b)(b+c)(c+a)}.$$

On the other side, by the same inequality, we have

$$\left(\frac{a+b+c}{3}\right)^3 \geq abc,$$

and

$$\left(\frac{2(a+b+c)}{3}\right)^3 = \left(\frac{(a+b)+(b+c)+(c+a)}{3}\right)^3 \geq (a+b)(b+c)(c+a).$$

These imply by multiplication

$$\frac{1}{abc(b+c)(c+a)(a+b)} \geq \frac{3^3 \cdot 3^3}{2^3(a+b+c)^6},$$

which concludes the proof.

Alternative solution. The Cauchy-Schwartz inequality provides

$$[(a+b)+(b+c)+(c+a)]\left[\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(a+c)}\right]\geq\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)^2,$$

which implies

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(a+c)}\geq\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)^2\frac{1}{2(a+b+c)}.$$

It will thus enough to prove that

$$\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)^2\frac{1}{2(a+b+c)}\geq\frac{27}{2(a+b+c)^2},$$

which is the same as

$$(a+b+c)\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)^2\geq 27.$$

However we have that

$$3(x^2+y^2+z^2)\geq(x+y+z)^2$$

for all positive numbers x, y, z. For $a=\sqrt{x}, b=\sqrt{y}, c=\sqrt{z}$ we have

$$a+b+c\geq\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{3}.$$

To end the proof it will thus be enough to show that

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\geq 9,$$

which is equivalent to the HM-AM mean inequality.

Alternative solution; non-elementary. First observe that

$$\sum\frac{1}{b(a+b)}\geq\sum\frac{1}{c(a+b)},$$

where the summation is circular. Indeed, the above inequality reduces to

$$\sum ac^3\geq abc\sum a,$$

which can be obtained from the Cauchy inequality by the following trick

$$\sum(\sqrt{a})^2\sum\left(\frac{b}{\sqrt{a}}\right)^2\geq(a+b+c)^2.$$

The given inequality reduces thus to the analogous for $\sum \frac{1}{c(a+b)}$. It can be easily derived by means of the Jensen inequality for the convex function $f(x) = \frac{1}{x(s-x)}$, applied to a, b, c, where $s = a + b + c$.

A4-M. *Let a, b, c be positive real numbers such that $abc = \frac{9}{4}$. Prove that the following inequality holds*

$$a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

Solution. We shall use the inequality $x^3 + y^3 \geq xy(x+y)$, valid for positive numbers x, y. It can be easily shown it is eqiuvalent to the obvious $(x-y)^2(x+y) \geq 0$.

Thus

$$a^3 + b^3 + c^3 \geq ab(a+b) + c^3 \geq 2\sqrt{abc^3(a+b)} = 2\sqrt{\frac{9}{4}c^2(a+b)} = 3c\sqrt{a+b}.$$

Analogously, $a^3+b^3+c^3 \geq 3a\sqrt{b+c}$ and $a^3+b^3+c^3 \geq 3b\sqrt{c+a}$. Adding these inequalities we get the announced one.

It is easy to see that the inequality is strict, as equality implies $a + b = c = 0$.

A4.1-D (Committee's variant for A3). *Let a, b, c be positive real numbers such that $abc = 2$. Prove that the following inequality holds*

$$a^3 + b^3 + c^3 \geq a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}.$$

When does the equality holds.

Solution. By the CS-inequality

$$(a+b+c)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^2$$

and

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2.$$

By using them succesively we can write

$$\begin{aligned} a^3 + b^3 + c^3 &\geq \frac{(a^2+b^2+c^2)(a+b+c)}{3} = \frac{(a^2+b^2+c^2)((b+c)+(a+c)+(a+b))}{6} \\ &\geq \frac{(a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b})^2}{6}. \end{aligned} \tag{1}$$

For $x = a\sqrt{b+c} + b\sqrt{a+c} + c\sqrt{a+b}$, we can write using in two steps the AM-GM inequality

$$x^3 \geq 27abc\sqrt{8abc} = 27 \cdot 8,$$

that is $x \geq 6$ which used in (1) under the form $\frac{x^2}{6} \geq x$, conclude the proof.

A5-D. *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

Solution. We shall make use of the inequality $\frac{a^3}{b^2} \geq \frac{a^2}{b} + a - b$, easily reduced to the obvious one $(a-b)^2(a+b) \geq 0$. In the same way

$$\frac{b^3}{c^2} \geq \frac{b^2}{c} + b - c, \quad \frac{c^3}{a^2} \geq \frac{b^2}{c} + b - c.$$

The given inequality is then obtained by the addition of the previous ones.

A6-M. *If for real numbers $a_1, a_2, a_3, a_4, a_5, a_6$, $a_1 \neq 0$, we have the relations $a_1a_6 + a_3a_4 = 2a_2a_5$ and $a_1a_3 \geq a_2^2$, show that $a_4a_6 \leq a_5^2$. When does equality holds?*

Solution. Let $a \geq 0$ given by $a_1a_3 = a_2^2 + a$. Multiplying the first given relation by a_4 one obtaines

$$a_1a_6a_4 + a_3a_4^2 = 2a_2a_5a_4, \quad \text{or } a_6a_4 = \frac{2a_2a_5a_4 - a_3a_4^2}{a_1}.$$

Use $a_3 = \frac{a_2^2+a}{a_1}$ in the previous relation, gives

$$a_6a_4 - a_5^2 = -\frac{(a_1a_5 - a_2a_4)^2 + aa_4^2}{a_1^2},$$

which is nonpositive.

A7-M. *We are given 2002 positive integers, a_i, $i = 1, 2, \ldots, 2002$ such that*

$$\frac{1}{a_1^3} + \frac{1}{a_2^3} + \cdots + \frac{1}{a_{2002}^3} \geq \frac{1}{2}$$

Prove that at least three of them must be equal.

Solution. It can easily be seen that $a_i \neq 1$, and that:

$$\frac{1}{n^3} = \frac{1}{n^2} \cdot \frac{1}{n} \leq \frac{1}{n^2-1} \cdot \frac{1}{n} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right),$$

for any positive integer n. If in the given sum, there were not more than two equal summands, then

$$\begin{aligned} \frac{1}{2} &= \frac{1}{a_1^3} + \frac{1}{a_2^3} + \cdots + \frac{1}{a_{2002}^3} \\ &\leq 2\left(\frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{1002^2}\right) \\ &\leq \frac{1}{2} - \frac{1}{1002} + \frac{1}{1003} = \frac{1}{2} - \frac{1}{1002 \cdot 1003}, \end{aligned}$$

which is a contradiction.

Thus, there must be at least three mutually equal numbers between the given ones.

NUMBER THEORY

N1-M. *Find all positive integers N which have the following properties*

- *N has exactly 16 divisors $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = N$;*
- *the divisor with number d_5 is equal to $(d_2 + d_4)d_6$.*

Solution. Observe first that N has no more than 4 prime distinct divisors and $d_2 = 2$.

From the conditions in the hypothesis we must have $2 + d_4 \geq d_5 \geq 7$, implying $d_4 \geq 5$. As $d_4 < d_5 \leq 2 + d_4$ we should have $d_5 = 1 + d_4$ (1) or $d_5 = 2 + d_4$ (2).

If the case (1) holds, we will have $d_6 = 2 + d_4$ which implies N multiple of 3, thus $d_3 = 3$. As $6|N$ we must have $d_4 = 6$, implying $d_5 = 7$, $d_6 = 8$, thus $4|N$ and $d_4 = 4$, a contradiction.

It remains (2): $d_5 = 2 + d_4$. We consider the following cases:

- If $4|N$, as $d_4 \geq 5$ we have $d_3 = 4$ implying $8|N$. As $d_6 \geq 8$ we must have $8 \in \{d_4, d_5, d_6\}$. All of these cases conduct to a contradiction. For

if $d_4 = 8$ we must have $d_5 = 10$, thus $5|N$ and $d_4 = 5$, impossible;

if $d_5 = 8$ we will have $d_4 = 6$, thus $3|N$, consequently $d_3 = 3$, impossible;

if $d_6 = 8$ then $d_5 = 7$ implying $d_4 = 5$ and $10|N$. But $d_7 = (2+5)8 = 56 > 10$, a contradiction.

Since N is not divisible by 4 we conclude that d_3 is prime.

- If $3|N$, we deduce $d_3 = 3$. Because $6|N$ and $d_4 \geq 6$, we must have $d_4 = 6$, thus $d_5 = 8$ implying $4|N$, impossible.

Therefore 3 does not divide N and we conclude $d_3 \geq 5$ and $d_4 \geq 7$.

As N and $2 + d_4$ are not multiples of 4 we deduce that d_4 is odd. As $2 + d_4$ and d_4 are not divisible by 3 we obtain that $d_4 = 3k + 2$ for some integer k and as it is odd $d_4 = 6l + 5$

for some l. As $d_5 \leq 16$ we must have $7 \leq d_4 \leq 14$. The only possibility is $d_4 = 11$ and $d_3 = 13$. As $2d_3|N$ and $2d_3 \geq d_4$ we infer $d_3 \geq 6$. As d_3 is prime and $d_3 < 11$ we conclude $d_3 = 7$. We colect $N = 2 \cdot 7 \cdot 11 \cdot 13 = 2002$.

GEOMETRY

G1-M. *Let ABC be a triangle, G be its barycenter and A_1, B_1, C_1 be the midpoints of the sides BC, AC, AB respectively. The parallel line to BB_1 drawn through A_1 meets B_1C_1 at the point F. Prove that the triangles ABC and FA_1A are similar, in that order, if and only if the quadrilateral AB_1GC_1 is cyclic.*

Solution. Let G be the centroid of the triangle ABC, $AA_1 = m_a, BB_1 = m_b$ and $CC_1 = m_c$. The quadrilaterals CBC_1F and CC_1AF are parallelograms.

The point D belongs to the ray $(GA_1$ such that A_1 is the midpoint of the segment GD. The quadrilateral $BGCD$ is a parallelogram with $BG = DC = \frac{2}{3}m_b$, $BD = GC = \frac{2}{3}m_c$ and $GD = \frac{2}{3}m_a$.

If the points A, B_1, G, C_1 are concyclic, then the points A, B, D, C are also concyclic. We have

$$\begin{aligned} m(\angle GAB_1) = m(\angle GC_1B_1) = m(\angle GCA_1), m(\angle BAD) = m(\angle BCD) \Rightarrow m(\angle BAC) \\ = m(\angle DCG) \\ m(\angle ACB) = m(\angle AB_1C_1) = m(\angle AGC_1) = m(\angle CGD), m(\angle ADC) = m(\angle ABC). \end{aligned}$$

Hence, the triangles CDG and ABC are similar. But, the triangles CDG and FA_1A are similar too. So, the triangles ABC and FA_1A are similar.

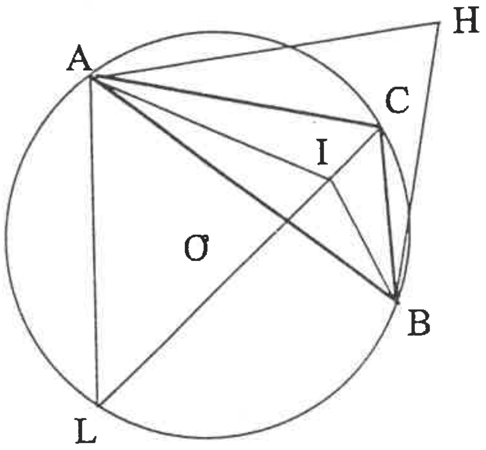
Let the triangles ABC and FA_1A are similar. Because the triangles FA_1A and CDG are similar too, we obtain that the triangles ABC and CDG are similar. So,

$$m(\angle ACB) = m(\angle CGD) = m(\angle AGC_1) = m(\angle AB_1C_1).$$

Hence, the quadrilateral AB_1GC_1 is a cyclic.

G2-M. *Let ABC be a triangle, H its orthocenter, I be the incenter and O the circumcenter. The line CI meets the circumcircle at point L. It is known that $AB = IL$ and $AH = OH$. Find the measures of the angles of ABC.*

Solution. Denote $a = \angle CAB$, $b = \angle ABC$, $c = \angle ACB$. Since $\angle IAL = \angle AIL = \frac{1}{2}(a+c)$, we have $AL = IL = AB = BL$. Hence $\angle ALB = 60°$, $\angle ACB = 120°$, $\angle AOB = 120°$.



From $\angle AHB = 180° - \angle ACB = 60°$ we obtain that A, O, B and H are concyclic. Thus we have $\angle AHO = \angle ABO = 30°$. From $AH = HO$ we obtain $\angle HAO = \angle AOH = 75°$.

Since $\angle HAC = \angle BAO = 30°$ we have $\angle HAO = 60° + a = 75°$ and $a = 15°$.

Finally $a = 15°$, $b = 45°$, $c = 120°$.

G3-D. *Let ABC be a triangle, S its area and D, E, F be arbitrary interior points on the sides BC, CA, AB respectively. The perpendicular lines at points D, E, F on the sides BC, CA, AB intersect the circumcircle in pairs of points $(D_1, D_2), (E_1, E_2), (F_1, F_2)$ respectively. Prove that:*

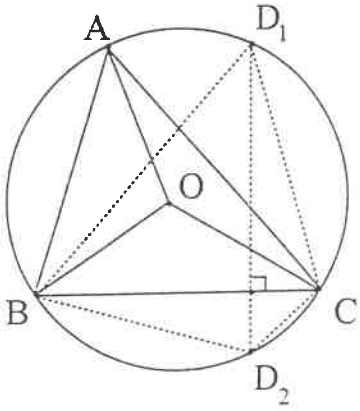
$$|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1C \cdot E_1A - E_2C \cdot E_2A| + |F_1A \cdot F_1B - F_2A \cdot f_2B| > 4S.$$

Solution.We will prove the result in two steps. The first one is a general result.

Lemma. Let AB and D_1D_2 be a pair of perpendicular chords in a circle of center O. Then:

$$|S_{D_1AB} - S_{D_2AB}| = 2S_{AOB},$$

where S_{XYZ} denotes the area of the triangle XYZ.



Indeed, let D_1' be the symmetrical of D_1 with respect to AB. Then $\angle D_1'AB = \angle D_1AB = \angle D_1D_2B = 90° - \angle ABD_2$. It follows that AD_1' is perpendicular to BD_2. Let B' the antipode of B. We infer that $B'D_2$ is parallel to AD_1', and AB' is parallel to D_1D_2. Hence, the quadrilateral $AB'D_2D_2'$ is a parallelogram and $D_2D_1' = AB' = 2OO'$, where O' is the projection of O on AB. We obtain:

$$S_{D_2AB} - S_{D_1AB} = \frac{AB \cdot D_2T}{2} - \frac{D_1T \cdot AB}{2} = \frac{AB}{2}(D_2T - D_1T)$$
$$= \frac{AB}{2} \cdot D_1D_2 = \frac{AB}{2} \cdot 2OO' = AB \cdot OO' = 2S_{AOB}$$

Remark. The lemma can be easily proved by trigonometric arguments, also. Indeed, if we denote $\angle D_2AB = \alpha$, $\angle D_2BA = \beta$, $\angle AD_2B = \gamma$ and $OA = R$, by the sine theorem we easily arrive at:

$$S_{D_2AB} = 2R^2 \sin\alpha \sin\beta \sin\gamma$$

and

$$S_{D_1AB} = 2R^2 \cos\alpha \cos\beta \sin\gamma$$

wence

$$S_{D_2AB} - S_{D_1AB} = 2R^2 \sin\gamma(\sin\alpha\sin\beta - \cos\alpha\cos\beta) = 2R^2 \sin 2\gamma = 2S_{AOB}.$$

Returning to the proof of the given result, we will apply the lemma succesively to the pairs of perpendicular chords (BC, D_1D_2), (CA, E_1E_2) and (AB, F_1F_2). We have

$$|D_1B \cdot D_1C - D_2B \cdot d_2C| \geq |D_1B \cdot D_1C - D_2B \cdot d_2C||\sin A|$$
$$= |D_1B \cdot D_1C \sin A - D_2B \cdot d_2C \sin A|$$
$$= 2|\frac{D_1B \cdot D_1C \sin A}{2} - \frac{D_2B \cdot d_2C \sin A}{2}|.$$
$$= 2|S_{D_1BC} - S_{D_2BC}|$$

Therefore by the lemma:

$$|D_1B \cdot D_1C - D_2B \cdot d_2C| \geq S_{BOC}.$$

It is clear that $\angle A = \angle BD_1C = 180° - \angle BD_2C$ and thus $\sin A = \sin\angle BD_1C \sin\angle BD_2C$.

In a similar way, we can prove the inequalities $|E_1A \cdot E_1C - E_2A \cdot E_2C| \geq 4S_{AOC}$ and $|F_1A \cdot F_1B - F_2A \cdot F_2B \geq 4S_{AOB}|$, which added give the desired result.

Equality is impossible since the equalities $\sin A = \sin B = \sin C = 1$ are simultanouslly impossible.

Remark. In fact, the left quantity in the inequality is constant for an acute triangle. It can been shown (not difficult) that it equals $4R^2(\sum \cos A) = 4R(R + r)$. This easily implies that the inequality can be improved by the lower bound to $\frac{8}{\sqrt{3}}S$.

G4-E. *Let ABC be an isosceles triangle such that $AC = BC$ and let P be a point on that arc (AB) of the circumcircle, which does not contain C. The perpendicular line from C on PB intersects PB at D. Prove that $PA + PB = 2PD$.*

Solution. From the first theorem of Ptolemy we have that

$$PA \cdot BC + PB \cdot AC = PC \cdot AB,$$

thus

$$PA + PB = \frac{PC \cdot AB}{AC}.$$

Therefore it will be enough to prove that

$$2PD = \frac{PC \cdot AB}{AC}.$$

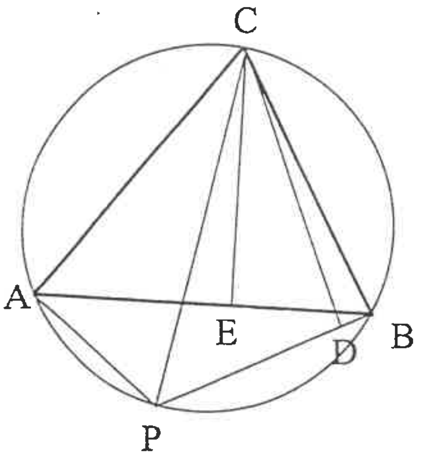
Since $CE \perp AB$, it will be equivalent to prove

$$2PD = \frac{2AE \cdot PC}{AC},$$

which is similar to

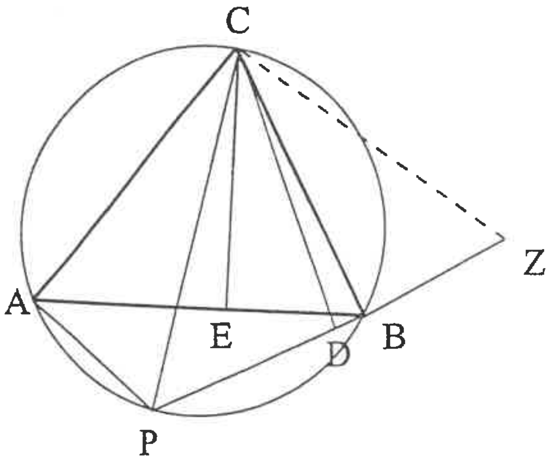
$$\frac{PD}{PC} = \frac{AE}{AC}.$$

The last equality means $\cos \angle CPB = \cos \angle CAB$, obviously true.



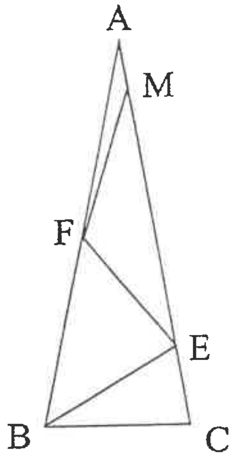
Alternative solution. We extend PB with the segment $BZ = AP$. The triangles BCZ and APC are congruent, hence $CP = CZ$ and the triangle CPZ is isosceles with $CD \perp AZ$. Thus D is the midpoint of PZ, hence

$$2PD = PZ = PB + BZ = PA + PB$$



G5-M. *Let ABC be an isosceles triangle such that $AB = AC$ and $A = 20°$. On the side AC we take a point D such that $AD = BC$. Find the angle $\angle BDC$.*

Solution. From the isosceles triangle ABC we easily find $B = \angle C = 80°$. Let E be the point on AC such that $\angle EBC = 60°$. It follows $\angle BEC = 180° - 100° = 80°$. That is $BE = BC$. Therefore $\angle ABE = 80° - 20° = 60°$. On the side AB take the point F such that $BF = BE = BC$. The triangle BEF is equilateral, so that $\angle BEF = \angle BFE = 60°$ and $FE = BE = BC$.

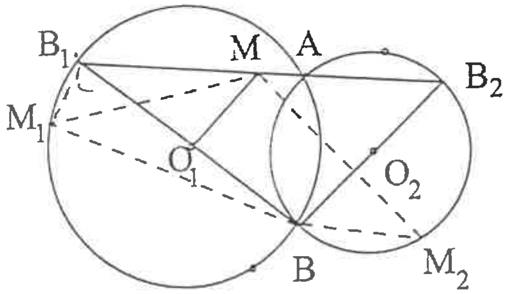


Let M be the point on AC such that $\angle FEM = 180° - 140° = 40°$. It follows $\angle FME = 40°$, thus $FM = FE = BC$. As $\angle AFM = 20°$ we infer $\angle AFM = \angle FAM$ and $AM = MF = BC$. Therefore M coincides with D. Since $FB = FM$ implies $\angle FMB = \angle FBM = \frac{180° - 160°}{2} = 10°$, we conclude $\angle FDB = 10°$. Finally $\angle BMC = 40° - 10° = 30°$.

Comment. The Problem Selection Committee considers the problem to involve known ideas.

G6-M. *Two circles C_1 and C_2 of diferent radii intersect at A and B. Let B_1 and B_2 respectively the simetric point of B with respect to the centers of C_2 and C_2 respectively and M the midpoint of the segment B_1B_2. Take points M_1 on C_1 and M_2 on C_2 such that arc(AB_1M_1) =arc$(ABM_2) < 180^\circ$. Show that $\angle MM_1B = \angle MM_2B$.*

Solution.



Obviously points B_1, A, B_2 are colinear. We have

$$MO_1 = \frac{BB_2}{2} = AO_2 \text{ and } MO_2 = \frac{BB_1}{2} = AO_1,$$

implying that the triangles MO_1A and MO_2A are congruent and therefore $\angle MO_1A = \angle MO_2A$. We conclude that the triangles MO_1M_1 and MO_2M_2 are also congruent which implies $MM_1 = MM_2$.

From the hypothesis we obtain $\angle ABM_1 = \frac{a}{2}$ and $\angle ABM_2 = 180^\circ - \frac{a}{2}$. This gives $\angle ABM_2 + \angle ABM_1 = 180^\circ$ and therefore the points M_1, B and M_2 are colinear. Thus the triangle MM_1M_2 is isosceles and $\angle MM_1B = \angle MM_2B$.

G7-D. *Suppose that $ABCD$ is a convex quadrilateral, where $AB = AD$ and $BC = CD$. Points K, L, L_1, and K_1 are chosen in sequence on AB, BC, CD and DA sides so that the quadrilateral KLL_1K_1 is a rectangle. Then. suppose that a rectangle $MNPQ$ is inscribed in a triangle BLK , where $M \in KB$, $N \in BL$, $P, Q \in LK$, and similarly a rectangle $M_1N_1P_1Q_1$ is inscribed in a triangle DK_1L_1 where $M_1 \in DK_1$, $N_1 \in DL_1$ and $P_1, Q_1 \in L_1K_1$.*

Let $2S$ be the area of the quadrangle $ABCD$ and $2S_1, S_2$ and S_3 in sequence be the areas of rectangles $KLL_1K_1, MNPQ$ and $M_1N_1P_1Q_1$.

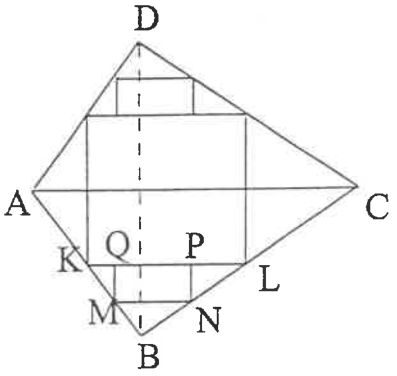
Find the highest possible value of the following expression

$$\frac{2S_1 + S_2 + S_3}{2S}.$$

Solution: The quadrangle $ABCD$ is symmetric with respect to the diagonal AC, so that it would be enough to look at the triangle ABC which includes a half of the rectangle KLL_1K_1 and the rectangle $MNPQ$. If these two rectangles are cut out of the triangle

ABC, we are left with a triangle BMN (height x) which is similar to a triangle BAC and to two pairs of right-angled triangles, which, if adequately connected, can form two more triangles similar to the triangle ABC, the heights of which we denote by y and z respectively. If P_1, P_2, and P_3 are the areas of these triangles (similar to the triangle ABC) and whose heights are x, y, and z respectively, and if S is the area of the triangle ABC, whose height is $x + y + z$, then we get:

$$\frac{P_1}{S} = \frac{x^2}{(x+y+z)^2}, \quad \frac{P_2}{S} = \frac{y^2}{(x+y+z)^2}, \quad \frac{P_3}{S} = \frac{z^2}{(x+y+z)^2}.$$



Due to the symmetry of the quadrilateral $ABCD$, to find the maximum of the quotient

$$\frac{2S_1 + S_2 + S_3}{2S}$$

is equivalent to the finding of the maximum value of the expression

$$\frac{S_1 + S_2}{S}.$$

But,

$$\frac{S_1+S_2}{S} = \frac{S-(P_1+P_2+P_3)}{S} = 1 - \left(\frac{P_1}{S} + \frac{P_2}{S} + \frac{P_3}{S}\right) = \frac{2(xy+yz+zx)}{(x+y+z)^2}.$$

Since $a^2+b^2 \geq 2ab,\ b^2+c^2 \geq 2bc,\ c^2+a^2 \geq 2ca$ we infere $a^2+b^2+c^2 \geq (ab+bc+ca)$, i.e. $(a+b+c)^2 \geq 3(ab+bc+ca)$. Thus:

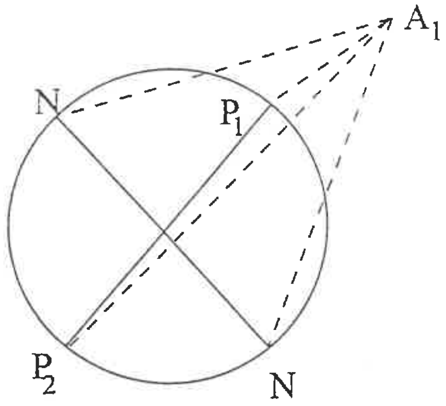
$$\frac{S_1+S_2}{S} = \frac{2(xy+yz+zx)}{(x+y+z)^2} \leq \frac{2}{3}.$$

Equation holds if and only if $x = y = z$.

G8-E. *Let $A_1, A_2, \ldots A_{2002}$ be arbitrary points in a plane. Prove that for any unit circle in the plane, and any rectangle inscribed in it, there are three vertices M, N and P of the rectangle, such that:*

$$MA_1 + MA_2 + \cdots + MA_{2002} + NA_1 + \cdots + NA_{2002} + PA_1 + \cdots + PA_{2002} \geq 6006.$$

Solution. Let us notice any diameter MN of any circle in the plane. Then $2 = MN \leq MA_i + NA_i$ for each $i \in 1, \ldots 2002$. Thus, $MA_1 + MA_2 + \cdots + MA_{2002} + NA_1 + NA_2 + \cdots + NA_{2002} \geq 4004$. Now, let us take another diameter P_1P_2 (different from MN) diameter. Then, we have $P_1A_1 + P_1A_2 + \cdots + P_1A_{2002} + P_2A_1 + P_2A_2 + \cdots + P_2A_{2002} \geq 4004$. The point P is the one of P_1 or P_2 for which the sum is as follows $P_1A_1 + P_1A_2 + \cdots \div P_1A_{2002} \geq 2002$ or $P_2A_1 + P_2A_2 + \cdots + P_2A_{2002} \geq 2002$.



Thus, M, N and P are the three asked points.