39-th Balkan Mathematical Olympiad

– Algebra

A1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x(x + f(y))) = (x + y)f(x),$$

for all $x, y \in \mathbb{R}$.

A2 Let $k > 1$ be a real number, $n \geqslant 3$ be an integer, and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ be positive real numbers. Prove that

$$\frac{x_1 + kx_2}{x_2 + x_3} + \frac{x_2 + kx_3}{x_3 + x_4} + \cdots + \frac{x_n + kx_1}{x_1 + x_2} \geqslant \frac{n(k + 1)}{2}.$$

Ilija Jovcheski

A3 Let a, b, c, d be non-negative real numbers such that

$$\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} + \frac{1}{d + 1} = 3.$$

Prove that

$$3(ab + bc + ca + ad + bd + cd) + \frac{4}{a + b + c + d} \leqslant 5.$$

Vasile Cîrtoaje and Leonard Giugiuc

A4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$f(f(x)) + f(f(y)) = f(x + y)f(xy),$$

for all $x, y \in \mathbb{R}$.

A5 Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(y(f(x))^3 + x) = x^3 f(y) + f(x)$$

for all $x, y > 0$.

Proposed by Jason Prodromidis, Greece

A6 Determine all functions $f : \mathbb{R}^2 \to \mathbb{R}$ for which

$$f(A) + f(B) + f(C) + f(D) = 0,$$

whenever A, B, C, D are the vertices of a square with side-length one.

Ilir Snopce

– Combinatorics

C1 There are 100 positive integers written on a board. At each step, Alex composes 50 fractions using each number written on the board exactly once, brings these fractions to their irreducible form, and then replaces the 100 numbers on the board with the new numerators and denominators to create 100 new numbers.

Find the smallest positive integer n such that regardless of the values of the initial 100 numbers, after n steps Alex can arrange to have on the board only pairwise coprime numbers.

C2 Alice is drawing a shape on a piece of paper. She starts by placing her pencil at the origin, and then draws line segments of length one, alternating between vertical and horizontal segments. Eventually, her pencil returns to the origin, forming a closed, non-self-intersecting shape. Show that the area of this shape is even if and only if its perimeter is a multiple of eight.

C3 Find the largest positive integer k for which there exists a convex polyhedron $\mathcal{P}$ with 2022 edges, which satisfies the following properties:

-The degrees of the vertices of $\mathcal{P}$ don't differ by more than one, and
-It is possible to colour the edges of $\mathcal{P}$ with k colours such that for every colour c, and every pair of vertices (v_1, v_2) of $\mathcal{P}$, there is a monochromatic path between v_1 and v_2 in the colour c.

Viktor Simjanoski

C4 Consider an $n \times n$ grid consisting of n^2 until cells, where $n \geq 3$ is a given odd positive integer. First, Dionysus colours each cell either red or blue. It is known that a frog can hop from one cell to another if and only if these cells have the same colour and share at least one vertex. Then, Xanthias views the colouring and next places k frogs on the cells so that each of the n^2 cells can be reached by a frog in a finite number (possible zero) of hops. Find the least value of k for which this is always possible regardless of the colouring chosen by Dionysus.

Proposed by Tommy Walker Mackay, United Kingdom

C5 Given is a cube of side length 2021. In how many different ways is it possible to add somewhere on the boundary of this cube a $1 \times 1 \times 1$ cube in such a way that the new shape can be filled in with $1 \times 1 \times k$ shapes, for some natural number k, $k \geq 2$?

–	Geometry
G1	Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let t_A and t_B be the tangents to ω at A and B respectively, which meet at X. Let Y be the foot of the perpendicular from O onto the line segment CX. The line through C parallel to line AB meets t_A at Z. Prove that the line YZ passes through the midpoint of the line segment AC. *Proposed by Dominic Yeo, United Kingdom*
G2	Let ABC be a triangle with $AB > AC$ with incenter I. The internal bisector of the angle BAC intersects the BC at the point D. Let M the midpoint of the segment AD, and let F be the second intersection point of MB with the circumcircle of the triangle BIC. Prove that AF is perpendicular to FC.
G3	Let ABC a triangle and let ω be its circumcircle. Let E be the midpoint of the minor arc BC of ω, and M the midpoint of BC. Let V be the other point of intersection of AM with ω, F the point of intersection of AE with BC, X the other point of intersection of the circumcircle of FEM with ω, X' the reflection of V with respect to M, A' the foot of the perpendicular from A to BC and S the other point of intersection of XA' with ω. If $Z \in \omega$ with $Z \neq X$ is such that $AX = AZ$, then prove that S, X' and Z are collinear.
G4	Let ABC be a triangle and let the tangent at B to its circumcircle meet the internal bisector of the angle A at P. The line through P parallel to AC meets AB at Q. Assume that Q lies in the interior of segment AB and let the line through Q parallel to BC meet AC at X and PC at Y. Prove that PX is tangent to the circumcircle of the triangle XYC.
G5	Let ABC be a triangle with circumcircle ω, circumcenter O, and orthocenter H. Let K be the midpoint of AH. The perpendicular to OK at K intersects AB and AC at P and Q, respectively. The lines BK and CK intersect ω again at X and Y, respectively. Prove that the second intersection of the circumcircles of triangles KPY and KQX lies on ω. *Stefan Lozanovski*
G6	Let ABC be a triangle with $AB < AC$ and let D be the other intersection point of the angle bisector of $\angle A$ with the circumcircle of the triangle ABC. Let E and F be points on the sides AB and AC respectively, such that $AE = AF$ and let P be the point of intersection of AD and EF. Let M be the midpoint of BC. Prove that AM and the circumcircles of the triangles AEF and PMD pass through a common point.
–	Number Theory
N1	Let n be a positive integer. What is the smallest sum of digits that $5^n + 6^n + 2022^n$ can take?
N2	Let a, b and n be positive integers with $a > b$ such that all of the following hold:

i. a^{2021} divides n,
ii. b^{2021} divides n,
iii. 2022 divides $a - b$.

Prove that there is a subset T of the set of positive divisors of the number n such that the sum of the elements of T is divisible by 2022 but not divisible by 2022^2.

Proposed by Silouanos Brazitikos, Greece

N3 For every natural number x, let $P(x)$ be the product of the digits of the number x. Is there a natural number n such that the numbers $P(n)$ and $P(n^2)$ are non-zero squares of natural numbers, where the number of digits of the number n is equal to (a) 2021 and (b) 2022?

N4 A hare and a tortoise run in the same direction, at constant but different speeds, around the base of a tall square tower. They start together at the same vertex, and the run ends when both return to the initial vertex simultaneously for the first time. Suppose the hare runs with speed 1, and the tortoise with speed less than 1. For what rational numbers q is it true that, if the tortoise runs with speed q, the fraction of the entire run for which the tortoise can see the hare is also q?
