

STUDENTSKA RUBRIKA

r-inscribable quadrilaterals

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Abstract. *In this paper we characterize convex quadrilaterals that are inscribable in a rectangle, i.e. they are r-inscribable. We also study the problem of finding the rectangle of minimum area and the one of maximum area, if the convex quadrilateral is r-inscribable.*

Key words: *quadrilaterals*

1. Introduction

In the Book IV of “Elements” Euclid gives the following definitions [3]:

Definition 1. *A rectilinear figure is said to be inscribed in a rectilinear figure when the respective (vertices) angles of the inscribed figure lie on the respective sides of the one in which it is inscribed.*

Definition 2. *A rectilinear figure is said to be circumscribed about a rectilinear figure when the respective sides of the circumscribed figure pass through the respective (vertices) angles of the one about which it is circumscribed.*

Anyway, in Book IV, devoted to the properties of polygons inscribed and circumscribed to a circle and to the construction of regular polygons, and in the whole work, Euclid does not use these definitions and does not study the problems concerning polygons circumscribed to the other polygons. The reason is, according to Artmann ([1]), that the content of the Book IV must have been written by another author and that Euclid inserted it in the “Elements”, omitting some parts on topics that he thought were not fundamental.

In [5] an old problem concerning triangles, proposed in [3] and [6] has been studied and completely solved. It had been proved that for any given triangle T the set F of equilateral triangles circumscribed to T is not empty. Moreover, if A , B and C are the vertices of the triangle T , such that $|AB| \geq |AC| \geq |BC|$, among the triangles of the set F there exists one of maximum area if and only if the median of the side \overline{AB} with the side \overline{BC} forms an angle smaller than $5\pi/6$.

For convex quadrilaterals we can study an analogous problem. Let Q be a convex quadrilateral. A rectangle R is said to be *circumscribed* to Q if each side of R contains one and only one vertex of Q . A quadrilateral is said to be *r-inscribable* if there exists a rectangle that is circumscribed to it (Figure 1).

In this paper we:

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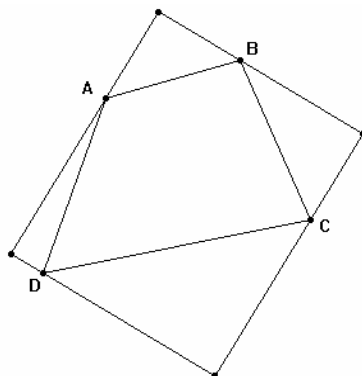


Figure 1.

- a) give a necessary and sufficient condition for Q to be r-inscribable (Section 2);
- b) when Q is r-inscribable, we prove that there does not exist a circumscribed rectangle to Q of minimum area and we find necessary and sufficient conditions so that there exists one of maximum area (Section 3).

The topic developed in this paper can be useful for teachers because they can get several elementary ideas for teaching. Teachers with the help of software (as Cabri, Cinderella or Sketchpad, for example), may discover properties that are shown in the paper making the topic more interesting for the students.

2. r-inscribable quadrilaterals

Let Q be a convex quadrilateral with vertices A, B, C, D . In this section we find a necessary and sufficient condition for Q to be r-inscribable.

Let α and β be two consecutive angles of Q so that their sum is greater than or equal to the sum of any two consecutive angles of Q , and let \overline{AB} be their common side so that $\angle BAD = \alpha$ and $\angle ABC = \beta$. If $\alpha \geq \beta$, then α is an obtuse angle or a right angle, so we get $\angle BCD = \gamma \leq \alpha$ and $\angle ADC = \delta \leq \beta$. If α is a right angle, Q is a rectangle and, of course, it is r-inscribable. Therefore we shall consider only the case when α is an obtuse angle.

Let us prove that: Q is r-inscribable if and only if $\alpha + \beta < 3\pi/2$.

If R is a circumscribed rectangle to Q , one vertex P of R lies on the semicircle Γ with the diameter \overline{AB} , being external to Q . If $\angle PAB = \alpha'$ and $\angle PBA = \beta'$, then $\alpha + \alpha' + \beta + \beta' < 2\pi$, and since $\alpha' + \beta' = \pi/2$, we get $\alpha + \beta < 3\pi/2$.

Vice versa, let Q be a convex quadrilateral with $\alpha + \beta < 3\pi/2$.

The point P of semicircle Γ is the vertex of a circumscribed rectangle to Q if and only if the straight lines PA and PB and the perpendiculars to PA and PB passing through C and D respectively, are all external to Q .

Let A' and B' be the intersection points of Γ with the straight lines AD and BC , respectively. Let A'' and B'' be the intersection points of Γ with the perpendiculars

to DC passing through A and B, respectively (Figure 2). $A' \neq A$ because α is an obtuse angle, while $B' \neq B$ if and only if β is an obtuse angle. Moreover, $A'' = A$ and $B'' = B$ if and only if AB and CD are parallel to each other; if $A'' \neq A$, then $B'' = B$ and if $B'' \neq B$, then $A'' = A$, because the perpendiculars to DC passing through A and B are parallel to each other (Figure 2).

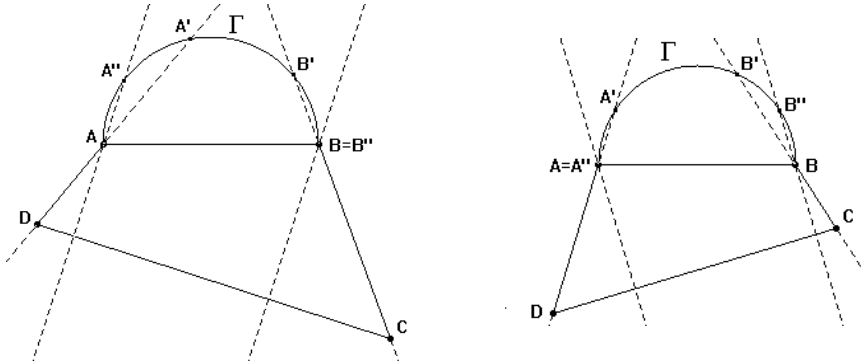


Figure 2.

Observe that the point P of Γ is such that the straight lines PA and PB are external to Q if and only if P does not belong either to the arc AA' or to the arc BB' .

Let us prove that a point P of Γ is such that the perpendicular to PA passing through D and the perpendicular to PB passing through C are external to Q if and only if P does not belong either to the arc AA'' or to the arc BB'' .

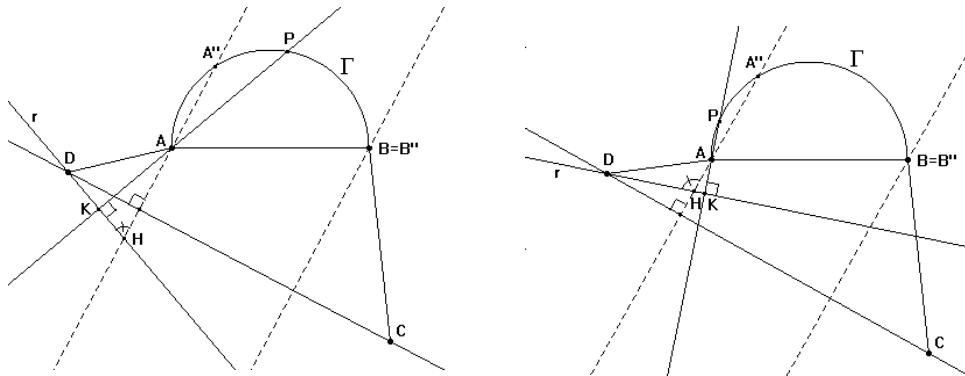


Figure 3. 3a.

3b

Let $A'' \neq A$ (the case $B'' \neq B$ is analogous to this one), then $B = B''$. Let P be a point of Γ , r be perpendicular to AP through D. Let H and K be the intersection points of r with the straight lines AA'' and AP, respectively. If $P = A''$, then r coincides with CD and it is not external to Q . If $P \neq A''$, then $H \neq K$ and A, H, K are the

vertices of the right triangle, with $\angle AKH = \pi/2$; then $\angle AHK < \pi/2$. It follows that if P does not belong to the arc AA'' , then $\angle DHA = \angle AHK < \pi/2$ and r is external to Q (Figure 3a); if P belongs to the arc AA'' , then $\angle DHA = \pi - \angle AHK > \pi/2$ and r is not external to Q (Figure 3b).

Therefore, if $\bar{\Gamma}$ is the arc obtained from Γ without the points of the arcs AA' , BB' , AA'' , BB'' , the point P of Γ is the vertex of the rectangle circumscribed to Q if and only if $P \in \bar{\Gamma}$ (Figure 4).

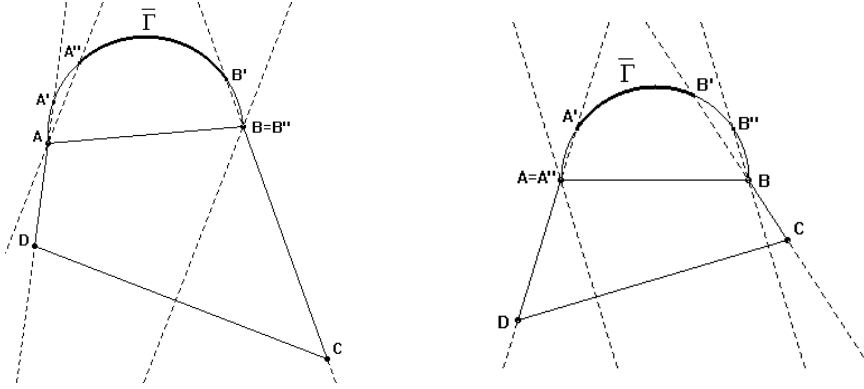


Figure 4.

Observe that $\bar{\Gamma}$ is the locus of the points P of Γ such that:

$$\max\{\angle B'AB, \angle B''AB\} < \angle PAB < \min\{\angle A'AB, \angle A''AB\}.$$

It is easy to prove that $\angle A'AB = \pi - \alpha$, $\angle A''AB = 3\pi/2 - \alpha - \delta$, $\angle B'AB = \beta - \pi/2$, $\angle B''AB = \beta + \gamma - \pi$, where $\angle B'AB = 0$ if $\beta < \pi/2$ and $\angle B''AB = 0$ if $\beta + \gamma < \pi$.

In the end we have to prove that $\bar{\Gamma}$ is not empty. In fact:

$$\begin{aligned} \beta - \pi/2 < \pi - \alpha & \quad \text{because} \quad \alpha + \beta < 3\pi/2; \\ \beta - \pi/2 < 3\pi/2 - \alpha - \delta & \quad \text{because} \quad \alpha + \beta + \delta < 2\pi; \\ \beta + \gamma - \pi < \pi - \alpha & \quad \text{because} \quad \alpha + \beta + \gamma < 2\pi; \\ \beta + \gamma - \pi < 3\pi/2 - \alpha - \delta & \quad \text{because} \quad \alpha + \beta + \gamma + \delta < 5\pi/2. \end{aligned}$$

Then, the arc $\bar{\Gamma}$ is not empty and it is open. Observe that if $\alpha + \beta < 3\pi/2$ then the sum of any two consecutive angles of Q is less than $3\pi/2$. It follows that:

Theorem 1. *A convex quadrilateral is r -inscribable if and only if the sum of any two of its consecutive angles is less than $3\pi/2$.*

Let us apply this result to a particular types of convex quadrilaterals:

- Any **parallelogram** is r -inscribable because the sum of any two consecutive angles of the parallelogram is equal to π ;
- Any quadrilateral with perpendicular diagonals, for example a **rhombus** or a **kite**, is r -inscribable because the straight lines through the vertices of the quadrilateral and parallel to the diagonals determine a rectangle circumscribed to it.

- Any *right trapezium* is *r-inscribable* because the sum of the angles on the minor base is less than $3\pi/2$.
- An *isosceles trapezium* is *r-inscribable* if and only if the angle that the minor base forms with an oblique side is less than $3\pi/4$.

3. Area of circumscribed rectangles to an r-inscribable quadrilateral

Let Q be a convex quadrilateral r-inscribable. We study the problem of the existence of the rectangle of minimum area and the one of maximum area, among the circumscribed rectangles to Q .

Let A, B, C, D be the vertices of Q so that the angle in A is bigger than or equal to any other angle of Q and, if O is the intersection point of the diagonals \overline{AC} and \overline{BD} , the angle $\vartheta = \angle AOB$ is a right angle or an obtuse angle (Figure 5). Then, if $\alpha = \angle DAB$, it is $\alpha \geq \pi/2$.

First let us find the area of any rectangle R circumscribed to Q .

Let P_1, P_2, P_3, P_4 be the vertices of R and suppose that P_1 lies on the semicircle Γ with the diameter \overline{AB} being external to Q . Moreover, let $\overline{P_1P_4}$ and $\overline{P_1P_2}$ be the sides of R that contain the vertices A and B of Q , respectively.

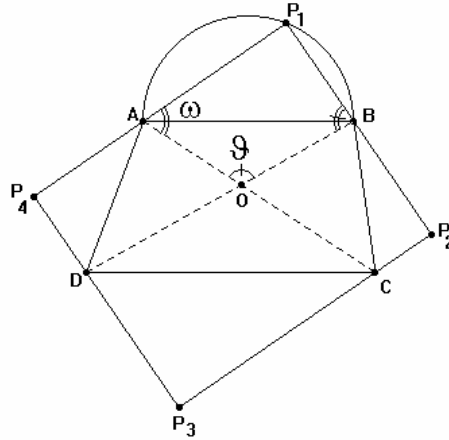


Figure 5.

If $\angle P_1AO = \omega$, then $\angle P_1BO = 3\pi/2 - \vartheta - \omega$. Note that when R varies, P_1 describes an open arc Γ_1 of Γ and then ω describes the interval $\langle \omega', \omega'' \rangle$, with $\omega' = \angle P'AO$, $\omega'' = \angle P''AO$, and P', P'' endpoints of Γ_1 . Moreover, if $d = |AC|$ and $d' = |BD|$, then:

$$|P_1P_2| = d \sin \omega, \quad |P_1P_4| = d' \sin(3\pi/2 - \vartheta - \omega).$$

The area S of the rectangle \mathbf{R} is

$$S = dd' \sin \omega \cdot \sin(3\pi/2 - \vartheta - \omega) = \frac{1}{2} dd' [\cos(2\omega - 3\pi/2 + \vartheta) - \cos(3\pi/2 - \vartheta)].$$

Now let us see if there exists a rectangle of minimum area and a rectangle of maximum area among the rectangles \mathbf{R} circumscribed to \mathbf{Q} .

Among the rectangles \mathbf{R} circumscribed to \mathbf{Q} there exists one of minimum area S if and only if there exists $\omega \in \langle \omega', \omega'' \rangle$ such that $\cos(2\omega - 3\pi/2 + \vartheta) = -1$, which implies $\omega = 5\pi/4 - \vartheta/2$. But, since $5\pi/4 - \vartheta/2 \leq \pi$ because $\vartheta \geq \pi/2$, we can say that:

Theorem 2. *Among the rectangles circumscribed to \mathbf{Q} there does not exist one of minimum area.*

Among the rectangles \mathbf{R} circumscribed to \mathbf{Q} there exists one of maximum area S if and only if there exists $\omega \in \langle \omega', \omega'' \rangle$ such that $\cos(2\omega - 3\pi/2 + \vartheta) = 1$, which implies $\omega = 3\pi/4 - \vartheta/2$. Note that for $\omega = 3\pi/4 - \vartheta/2$ we have

$$\angle P_1BO = 3\pi/2 - \vartheta - \omega = 3\pi/4 - \vartheta/2.$$

Then we can say that: Among the rectangles circumscribed to \mathbf{Q} , the one of maximum area, if it exists, is the one such that $\angle P_1AO = \angle P_1BO = 3\pi/4 - \vartheta/2$.

Observe that for $\omega = 3\pi/4 - \vartheta/2$ it is $|P_1P_2| = d \sin(3\pi/4 - \vartheta/2)$ and $|P_1P_4| = d' \sin(3\pi/4 - \vartheta/2)$, then $|P_1P_2| = |P_1P_4|$ if and only if $d = d'$. Therefore:

Theorem 3. *The rectangle circumscribed to \mathbf{Q} of maximum area, if it exists, is a square if and only if \mathbf{Q} is a quadrilateral with equal diagonals.*

Quadrilaterals with equal diagonals are, for example, the rectangles and the isosceles trapeziums.

Let us now try to construct the rectangle, which we'll denote with $\overline{\mathbf{R}}$, such that $\angle P_1AO = \angle P_1BO = 3\pi/4 - \vartheta/2$.

There exists the point P_1 of Γ , different from A and B , such that $\angle P_1AO = \angle P_1BO = 3\pi/4 - \vartheta/2$, if and only if $0 < 3\pi/4 - \vartheta/2 - \angle BAC < \pi/2$, i.e. if and only if:

$$\pi/4 - \vartheta/2 < \angle BAC < 3\pi/4 - \vartheta/2.$$

Since $\vartheta \geq \pi/2$, it is $\pi/4 - \vartheta/2 \leq 0$ and $3\pi/4 - \vartheta/2 \leq \pi/2$, if ϑ is a right angle the previous relation is still true, while if ϑ is an obtuse angle, the previous relation becomes:

$$\angle BAC < 3\pi/4 - \vartheta/2. \quad (1)$$

Then, *it is possible to construct the rectangle $\overline{\mathbf{R}}$ if and only if ϑ is a right angle or ϑ is an obtuse angle and (1) is true.*

If (1) is true, P_1 is the intersection point of Γ with the ray with the origin in A , that lies on the half-plane with the origin in the straight line AB and not containing \mathbf{Q} , such that $\angle P_1AO = 3\pi/4 - \vartheta/2$. Moreover, $\overline{\mathbf{R}}$ is constructed drawing the perpendicular to P_1B through C , the perpendicular to P_1A through D , and

determining the remaining vertices. Let $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$ be the sides of $\overline{\mathbf{R}}$ that contain the points B, C, D, respectively.

Let (1) be true. Let us now see if $\overline{\mathbf{R}}$ is circumscribed to \mathbf{Q} . We know that $\overline{\mathbf{R}}$ is circumscribed to \mathbf{Q} if and only if the following straight lines are external to \mathbf{Q} : P_1A and P_1B ; the perpendicular to P_1A passing through D and the perpendicular to P_1B passing through C.

Let us prove that P_1A and P_1B are external to \mathbf{Q} if and only if ϑ is a right angle or ϑ is an obtuse angle and the following conditions are true:

$$\angle ACB > \vartheta/2 - \pi/4 \tag{2}$$

$$\angle ADB > \vartheta/2 - \pi/4. \tag{3}$$

In fact, if ϑ is a right angle, we have $\angle P_1AO = \angle P_1BO = \pi/2$; then P_1A and P_1B are respectively parallel to BD and AC and they are external to \mathbf{Q} .

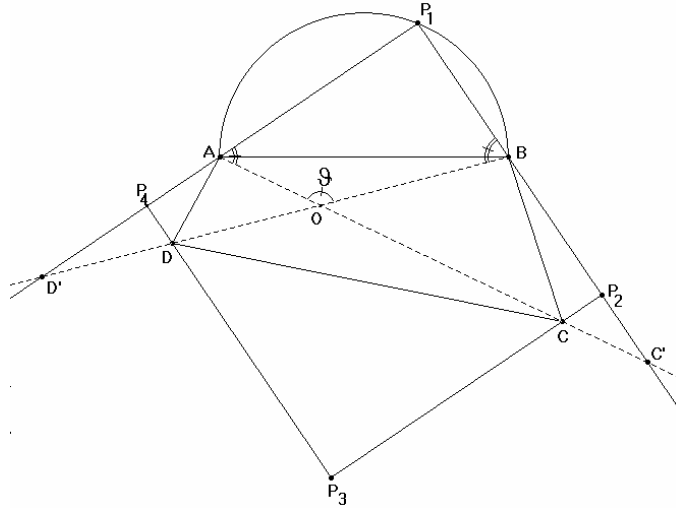


Figure 6.

If ϑ is an obtuse angle, the straight lines P_1A and BD have the common point D' on the half-plane with the origin AB and containing \mathbf{Q} because they form, with P_1B , the angles $\angle BP_1A = \pi/2$ and $\angle DBP_1 = 3\pi/4 - \vartheta/2 < \pi/2$ whose sum is less than π ; analogously, P_1B and AC have their common point C' on the half-plane with the origin AB containing \mathbf{Q} (Figure 6). Since $\angle P_1D'B = \angle P_1C'A = \vartheta/2 - \pi/4$, P_1A and P_1B are external to \mathbf{Q} if and only if the conditions (2) and (3) are true.

Let us consider now the straight line P_3P_4 passing through D and perpendicular to P_1A . Let us prove that it is external to \mathbf{Q} .

In fact, P_3P_4 is external to \mathbf{Q} if and only if $\angle BDC < \angle BDP_3$ and $\angle DAP_4$ is an acute angle. Since $\angle BDP_3 = \angle DBP_1 = 3\pi/4 - \vartheta/2$, it is $\angle BDC < \angle BDP_3$ if and only if $\angle BDC < 3\pi/4 - \vartheta/2$. But, since $\angle ACD > \pi/4 - \vartheta/2$ because $\pi/4 - \vartheta/2 \leq 0$, then

$\angle BDC = \pi - \vartheta - \angle ACD < 3\pi/4 - \vartheta/2 = \angle BDP_3$. Moreover, $\angle DAP_4$ is an acute angle because $\angle DAP_1$ is an obtuse angle, since $\alpha \geq \pi/2$. Then P_3P_4 is external to Q .

Finally, let us consider the straight line P_2P_3 passing through C and perpendicular to P_1B . Let us prove that it is external to Q if and only if:

$$\angle ACB < \vartheta/2 + \pi/4. \quad (4)$$

In fact, P_2P_3 is external to Q if and only if $\angle ACD < \angle ACP_3$ and $\angle CBP_2$ is an acute angle. Since $\angle ACP_3 = \angle CAP_1 = 3\pi/4 - \vartheta/2$, we have $\angle ACD < \angle ACP_3$ if and only if $\angle ACD < 3\pi/4 - \vartheta/2$. But, since $\angle BDC > \pi/4 - \vartheta/2$ because $\pi/4 - \vartheta/2 \leq 0$, then $\angle ACD = \pi - \vartheta - \angle BDC < 3\pi/4 - \vartheta/2 = \angle ACP_3$. Moreover, $\angle CBP_2$ is an acute angle if $\angle P_1BC$ is an obtuse angle. But in the quadrilateral P_1BCA we have $\angle P_1BC + \pi/2 + 3\pi/4 - \vartheta/2 + \angle ACB = 2\pi$, which implies $\angle P_1BC = 3\pi/4 + \vartheta/2 - \angle ACB$; then $\angle P_1BC > \pi/2$ if and only if $3\pi/4 + \vartheta/2 - \angle ACB > \pi/2$, from where follows $\angle ACB < \vartheta/2 + \pi/4$.

It follows that P_2P_3 is external to Q if and only if (4) is true.

Observe that if ϑ is a right angle the condition (4) is true.

Then we can conclude that *the rectangle \overline{R} is circumscribed to Q if and only if ϑ is a right angle, or ϑ is an obtuse angle and the conditions (2), (3), (4) are true.*

So we have proved:

Theorem 4. *If the convex quadrilateral Q is r-inscribable (that is if the sum of any two consecutive angles of Q is less than $3\pi/2$), then there exists a rectangle of maximum area among those circumscribed to Q if and only if ϑ is a right angle or ϑ is an obtuse angle and the following conditions are true:*

$$\angle BAC < 3\pi/4 - \vartheta/2 \quad (5)$$

$$\angle ACB > \vartheta/2 - \pi/4 \quad (6)$$

$$\angle ADB > \vartheta/2 - \pi/4 \quad (7)$$

$$\angle ACB < \vartheta/2 + \pi/4. \quad (8)$$

Let us apply this result to particular cases of convex quadrilaterals.

- Let Q be a **quadrilateral with perpendicular diagonals**. We know that Q is r-inscribable. Because of the previous theorem we can state that \overline{R} is the rectangle of maximum area circumscribed to Q . In particular, this property is true for **rhombi** and **kites**.
- Let Q be an **isosceles trapezium**. We know that Q is r-inscribable if and only if $\alpha = \angle DAB < 3\pi/4$. Let us prove that if Q is r-inscribable, then \overline{R} is the rectangle of maximum area that is circumscribed to it (Figure 7b).

In fact, if $\angle BAC = \angle DBA = \pi/2 - \vartheta/2 < 3\pi/4 - \vartheta/2$, then (5) is true. Moreover, if $\angle ACB < \pi/2 < \vartheta/2 + \pi/4$, then (8) is also true. Since $\angle ADB = \angle ACB = \pi - \angle BAC - \angle ABC$, we have:

$$\angle ADB = \pi/2 + \vartheta/2 - \alpha > \pi/2 + \vartheta/2 - 3\pi/4 = \vartheta/2 - \pi/4,$$

and then (6) and (7) are also true.

Observe that the vertex P_1 of \overline{R} is such that

$$\angle P_1AB = \angle P_1AO - \angle BAC = 3\pi/4 - \vartheta/2 - (\pi/2 - \vartheta/2) = \pi/4,$$

and then it is the intersection point of Γ with the perpendicular bisector of the segment \overline{AB} . Moreover, since an isosceles trapezium has equal diagonals, we can state that $\overline{\mathbf{R}}$ is a square.

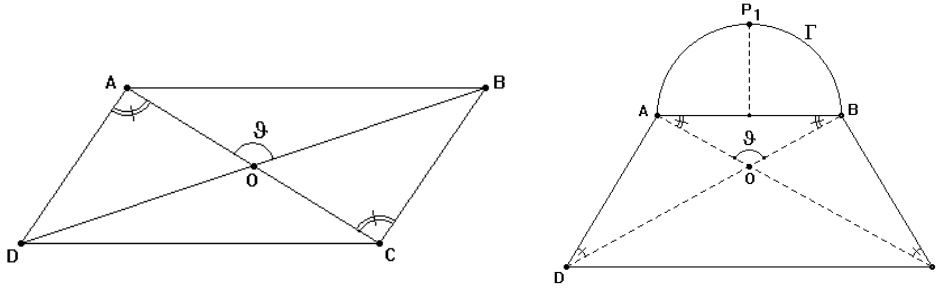


Figure 7. 7a.

7b

- Let Q be a parallelogram (Figure 7a). We know that Q is r-inscribable. Let us prove that $\overline{\mathbf{R}}$ is the circumscribed rectangle to Q of maximum area if and only if

$$\vartheta/2 - \pi/4 < \angle DAC < \vartheta/2 + \pi/4. \tag{9}$$

In fact, if $\angle ACB = \angle DAC$ and $\angle ADB = \vartheta - \angle DAC$, then $\angle ADB > \vartheta/2 - \pi/4$ if and only if $\angle DAC < \vartheta/2 + \pi/4$. Thus (8) is true if and only if (7) is true. Moreover, if $\pi/2 < \angle DAB = \angle DAC + \angle BAC$, then $\angle BAC < 3\pi/4 - \vartheta/2$ if and only if $\angle DAC > \vartheta/2 - \pi/4$, i.e. (5) is true if and only if (6) is true.

Then, *there exists a rectangle of maximum area among those circumscribed to Q if and only if (9) is true.*

Note that *if Q is a rectangle, there exists a rectangle of maximum area among those circumscribed to Q , because $\angle DAC = \vartheta/2$ and then (9) is true.*

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