42$^{\text{nd}}$ Balkan Mathematical Olympiad
Sarajevo, Bosnia and Herzegovina

April 27, 2025
Language: *English*

Problem 1. An integer $n > 1$ is called *good* if there exists a permutation $a_1, a_2, a_3, \ldots, a_n$ of the numbers $1, 2, 3, \ldots, n$, such that:

- a_i and a_{i+1} have different parities for every $1 \leq i \leq n-1$;
- the sum $a_1 + a_2 + \cdots + a_k$ is a quadratic residue modulo n for every $1 \leq k \leq n$.

Prove that there exist infinitely many good numbers, as well as infinitely many positive integers which are not good.

Remark: Here an integer x is considered a quadratic residue modulo n if there exists an integer y such that $x \equiv y^2 \pmod{n}$.

Problem 2. Let $\triangle ABC$ be an acute-angled triangle with orthocentre H and let D be an arbitrary interior point on side BC. Suppose E and F are points on the segments AB and AC respectively such that the quadrilaterals $ABDF$ and $ACDE$ are cyclic, and let BF and CE intersect at P. Let L be the point of line HA such that LC is tangent to the circumcircle of triangle $\triangle PBC$ at point C. Let lines BH and CP intersect at X. Prove that D, L and X are collinear.

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(x + yf(x)) + y = xy + f(x + y).$$

Problem 4. There are n cities in a country, where $n \geq 100$ is an integer. Some pairs of cities are connected by direct (two-way) flights. For two cities A and B we define:

- a *path* between A and B as a sequence of distinct cities $A = C_0, C_1, \ldots, C_k, C_{k+1} = B$, $k \geq 0$, such that there are direct flights between C_i and C_{i+1} for every $0 \leq i \leq k$;
- a *long path* between A and B as a path between A and B such that no other path between A and B has more cities;
- a *short path* between A and B as a path between A and B such that no other path between A and B has fewer cities.

Assume that for any pair of cities A and B in the country, there exist a long path and a short path between them that have no cities in common (except A and B). Let F be the total number of pairs of cities in the country that are connected by direct flights. In terms of n, find all possible values of F.

Time: 4 hours and 30 minutes.
Each problem is worth 10 points.

42^{nd} Balkan Mathematical Olympiad
Sarajevo, Bosnia and Herzegovina

April 27, 2025
Language: *Macedonian*

Задача 1. Цел број $n > 1$ е *добар* ако постои пермутација $a_1, a_2, a_3, \ldots, a_n$ на броевите $1, 2, 3, \ldots, n$, таква што:

- a_i и a_{i+1} имаат различна парност за секој $1 \leq i \leq n-1$;
- збирот $a_1 + a_2 + \cdots + a_k$ е квадратен остаток по модул n за секој $1 \leq k \leq n$.

Докажете дека постојат бесконечно многу добри броеви и бесконечно многу позитивни цели броеви кои не се добри.

Забелешка: За цел број x велиме дека е квадратен остаток по модул n ако постои цел број y таков што $x \equiv y^2 \pmod{n}$.

Задача 2. Нека ABC е остроаголен триаголник со ортоцентар H и D е внатрешна точка на страната BC таква што постојат точки E и F на отсечките AB и AC, соодветно, за кои четириаголниците $ABDF$ и $ACDE$ се тетивни. Правите BF и CE се сечат во точка P. Точката L на правата HA е таква што LC е тангента на опишаната кружница за триаголникот PBC низ точката C. Правите BH и CP се сечат во точка X. Докажете дека D, L и X се колинеарни.

Задача 3. Најдете ги сите функции $f : \mathbb{R} \to \mathbb{R}$ такви што, за секои реални броеви x и y,

$$f(x + yf(x)) + y = xy + f(x + y).$$

Задача 4. Во една држава има n градови, каде $n \geq 100$ е цел број. Некои парови градови се поврзани со (двонасочни) авиолинии. За два града A и B дефинираме:

- *пат* помеѓу A и B е низа од различни градови $A = C_0, C_1, \ldots, C_k, C_{k+1} = B$, со $k \geq 0$, таква што има авиолинија помеѓу C_i и C_{i+1} за секој $0 \leq i \leq k$;
- *долг пат* помеѓу A и B е пат помеѓу A и B таков што нема друг пат помеѓу A и B со повеќе градови;
- *краток пат* помеѓу A и B е пат помеѓу A и B таков што нема друг пат помеѓу A и B со помалку градови.

За секои два града A и B во државата, постојат долг пат и краток пат помеѓу нив кои немаат заеднички град (изземајќи ги A и B). Нека F е вкупниот број авиолинии во државата. Зависно од n, одредете ги сите можни вредности на F.

Време: 4 саати и 30 минути.
Секоја задача вреди 10 поени.

Problem 1. An integer $n > 1$ is called *good* if there exists a permutation $a_1, a_2, a_3, \ldots, a_n$ of the numbers $1, 2, 3, \ldots, n$, such that:

- a_i and a_{i+1} have different parities for every $1 \leqslant i \leqslant n-1$;
- the sum $a_1 + a_2 + \cdots + a_k$ is a quadratic residue modulo n for every $1 \leqslant k \leqslant n$.

Prove that there exist infinitely many good numbers, as well as infinitely many positive integers which are not good.

Remark: Here an integer x is considered a quadratic residue modulo n if there exists an integer y such that $x \equiv y^2 \pmod{n}$.

Solution

We will split the problem into two parts - the first one proving there are infinitely many numbers that are not good, and the second part proving there are infinitely many good numbers.

Infinitely many numbers are not good

Proof #1

We will show that all numbers $n = 4^m$ with $m \in \mathbb{Z}^+$ are not *good*. Indeed, consider the last sum in the given condition

$$a_1 + a_2 + \cdots + a_n = 1 + 2 + \cdots + n = \frac{4^m\left(4^m + 1\right)}{2}$$

Suppose that there exists $x \in \mathbb{Z}$ such that

$$\frac{4^m\left(4^m+1\right)}{2} \equiv x^2 \pmod{4^m} \Longleftrightarrow 4^m \equiv 2x^2 \pmod{2 \cdot 4^m} \Longleftrightarrow 2 \cdot 4^m \mid 4^m - 2x^2$$

This means that $4^m \mid 2x^2$, that is $2^{2m-1} \mid x^2$, so $2^m \mid x$. Let $x = c \cdot 2^m$ with $c \in \mathbb{Z}$. Thus

$$4^m \equiv 2\left(2^m c\right)^2 \equiv 2c^2 \cdot 4^m \equiv 0 \pmod{2 \cdot 4^m}$$

this implies that $4^m \equiv 0 \pmod{2 \cdot 4^m}$, which is not true. This proves the second part of the problem, i.e. that there are infinitely many numbers that are not good.

Proof #2.1

We will show that all numbers $n = 4m$ with $m \in \mathbb{Z}^+$ are not *good*. Assume otherwise and let $a_k = 2$ for some $1 \leqslant k \leqslant n$.

Let $S_i = a_1 + a_2 + \cdots + a_i$ (if $i < 1$, S_i is the empty sum). Let $S_{k-1} \equiv x^2 \pmod{4m}$ and $S_k \equiv y^2 \pmod{4m}$. Thus $x^2 + 2 \equiv y^2 \pmod{4m}$, which means that x and y have the same parity. Now $4m \mid (x-y)(x+y)+2$, but since $4 \mid (x-y)(x+y)$, we get $4 \mid 2$, a contradiction.

Proof #2.2

We will show that all numbers $n = 2^m$ with $m \in \mathbb{Z}^+$, $m > 3$ are not *good*. For the sake of contradiction, assume that n is good.

Lemma. Let $n = 2^m$ with $m \in \mathbb{Z}^+$, $m > 3$ and let r be an odd quadratic residue modulo n. Then $r \equiv 1 \pmod 8$.

Proof. Since r is a quadratic residue, we know that $r \equiv t^2 \pmod{2^m}$ for some odd integer t. Then we have that $2^m \mid r - t^2$, and because $m > 3$, we have that $8 \mid r - t^2$. Since t is odd, $t^2 \equiv 1 \pmod 8$, so we get that $r \equiv 1 \pmod 8$. □

Claim. Let $n = 2^m$ with $m \in \mathbb{Z}^+$, $m > 3$ and let r be a quadratic residue modulo n. If $v_2(r) \leqslant m - 3$ then $r = 4^a \cdot (8b + 1)$ for some nonnegative integers a and b.

Proof. If r is odd, from the previous lemma we have that $r = 8b + 1$ ($a = 0$) for some integer b. If $r = 2^c r_1$ for some $1 \leqslant c \leqslant m - 3$ and odd r_1, we get that $2^m \mid r - k^2$ for some integer k. That is, we have $2^m \mid 2^c r_1 - k^2$. Let $k^2 = 2^{2t} k_1^2$ for some nonnegative integer t and odd integer k_1. Since $2^c \mid k^2$, we get $2t \geqslant c$. If $2t > c$, it follows that $v_2(2^c r_1 - k^2) = c < m$, a contradiction. Therefore $c = 2t$ and so $v_2(r)$ is even. Now we have that $2^m \mid 2^c r_1 - 2^c k_1^2$, thus $2^{m-c} \mid r_1 - k_1^2$. Since $m - c \geqslant 3$, we have that $8 \mid r_1 - k_1^2$ and because k_1 is odd we get $r_1 \equiv 1 \pmod 8$. □

Assume 2^m with $m > 3$ is good with some permutation $a_1, a_2, \ldots, a_{2^m}$ and let $a_i = 2$, for some $i > 1$ (from the claim we know that 2 is not a quadratic residue modulo 2^m). Consider the following cases:

Case 1. If $2^{m-2} \mid a_1 + a_2 + \cdots + a_{i-1}$. Let $a_1 + a_2 + \cdots + a_{i-1} = 2^{m-2}c$ for some integer c. Then $v_2(a_1 + a_2 + \ldots + a_i) = v_2(2^{m-2}c + 2) = 1$ and from the claim this is not a quadratic residue, a contradiction.

Case 2. If $2^{m-2} \mid a_1 + a_2 + \cdots + a_i$. Let $a_1 + a_2 + \cdots + a_i = 2^{m-2}c$ for some integer c. Then $v_2(a_1 + a_2 + \ldots + a_{i-1}) = v_2(2^{m-2}c - 2) = 1$ and from the claim this is not a quadratic residue, a contradiction.

Case 3. Otherwise, the claim implies $a_1 + a_2 + \cdots + a_{i-1} = 4^{k_1}(8l_1 + 1)$ and $a_1 + a_2 + \cdots + a_i = 4^{k_2}(8l_2 + 1)$ for some nonnegative integers k_1, k_2, l_1, l_2. Then we have $4^{k_1}(8l_1 + 1) + 2 = 4^{k_2}(8l_2 + 1)$. Looking at the equation modulo 4, we get that at least one of k_1, k_2 is 0. If exactly one of k_1, k_2 is equal to 0 we get a contradiction modulo 2. Therefore $k_1 = k_2 = 0$ and thus $8l_1 + 3 = 8l_2 + 1$, which is impossible.

Infinitely many numbers are good

Proof #1

Now let $n = p$ be a prime number of the form $4k+3, k \in \mathbb{Z}$. Consider the numbers

$$1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2, p-1^2, p-2^2, \ldots, p-\left(\frac{p-1}{2}\right)^2.$$

Clearly, in this sequence, no two numbers are congruent modulo p. Indeed, suppose that there is $i^2 \equiv j^2 \pmod{p}$ with $1 \leqslant i < j \leqslant \frac{p-1}{2}$ then $p \mid (j-i)(j+i)$. But $0 < j+i < p$, $0 < j-i < p$, a contradiction. From there, it follows that the first $\frac{p-1}{2}$ numbers have distinct remainders in modulo p. We reason similarly for the last $\frac{p-1}{2}$ numbers. Next suppose that there is $i^2 \equiv p - j^2 \pmod{p}$ with $1 \leqslant i, j \leqslant \frac{p-1}{2}$ then $p \mid i^2 + j^2$. According to the well known properties of quadratic residues modulo a prime $p = 4k+3$, we conclude that $p \mid i$ and $p \mid j$, which is also a contradiction. Thus, the claim is proved.
Notice that for $1 \leqslant i \leqslant \frac{p-1}{2}$, two numbers i^2 and $p - i^2$ have different parity remainders in modulo p (since the sum of the two remainders is p, which is odd). Consider the remainder of $1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2$ when divided by p. We denote by $a_1, a_2, \ldots, a_m$ the odd remainders and by $b_1, b_2, \ldots, b_n$ the even remainders; note that $m + n = \frac{p-1}{2}$. Finally, consider the following permutation:

$$a_1, p-a_1, a_2, p-a_2, \ldots, a_m, p-a_m, p, b_1, p-b_1, b_2, p-b_2, \ldots, b_n, p-b_n$$

Obviously, according to the above arguments, two consecutive numbers in the above permutation have different parity, and the sum of any first i numbers in the permutation is either congruent to 0 or congruent to some number in $\{1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2\}$, which is clearly a quadratic residue modulo p. Thus, the constructed permutation as above satisfies the given conditions. Since there are infinitely many primes of the form $p = 4k+3$, we have proved that there are infinitely many good numbers as well.

Proof #2

Let n be an odd integer. We will prove that the number $2n$ is good. Consider the numbers:

$$1, 3+n, 5, 7+n, 9, 11+n, \ldots, 4n-1+n.$$

It can be easily proven that no two numbers in this sequence are congruent modulo $2n$. Since there are $2n$ numbers in the sequence, they form a complete residue system modulo $2n$. Also, note that the sum of the first k ($1 \leqslant k \leqslant 2n$) numbers in the sequence is a quadratic residue modulo $2n$ (it is a quadratic residue modulo n as it is congruent to $1 + 3 + \ldots + 2k - 1 = k^2$ modulo n and since $x^2 + n \equiv (x+n)^2 \pmod{2n}$ for all integers x, it is also a quadratic residue modulo $2n$). The parity condition is also satisfied (even after reduction by modulo $2n$, since $2n$ is even). Finally, taking the numbers modulo $2n$ gives the desired permutation.

Proof #3

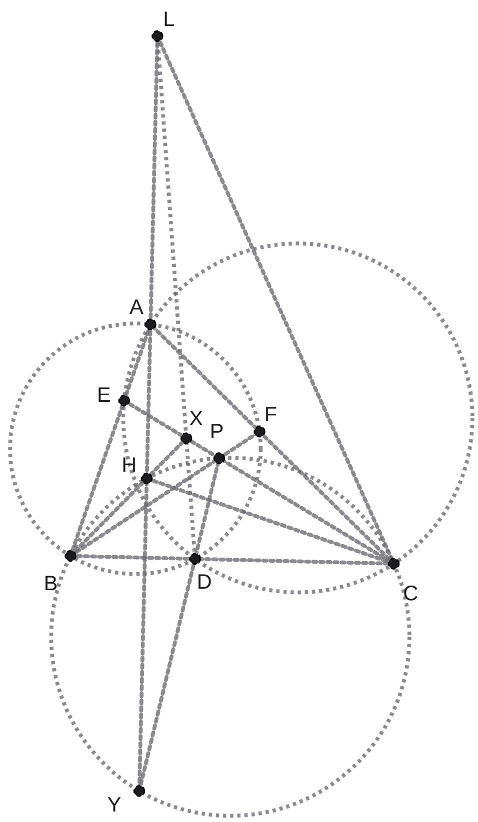
Let $p > 2$ be a prime number of the form $p = 3k + 2, k \in \mathbb{Z}$. We will prove that the number $n = 2p$ is good. Consider the numbers:

$$1^3, 2^3, 3^3, \ldots, (2p)^3.$$

It is well known that if $p \equiv 2 \pmod 3$ then the above numbers form a complete residue system modulo p. It can easily be proven that they also form a complete residue system modulo $2p$ (by also taking parity into account). Now, since $1^3+2^3+\ldots+k^3 = (1+2+\ldots+k)^2$, we get that the sum of the first k ($1 \leqslant k \leqslant 2p$) numbers in the sequence modulo p is a quadratic residue modulo p. The parity condition is also satisfied (even after reduction modulo $n = 2p$, since $2p$ is even). Finally, taking the numbers modulo n gives the desired permutation.

Problem 2. Let $\triangle ABC$ be an acute-angled triangle with orthocentre H and let D be an arbitrary interior point on side BC. Suppose E and F are points on the segments AB and AC respectively such that the quadrilaterals $ABDF$ and $ACDE$ are cyclic, and let BF and CE intersect at P. Let L be the point of line HA such that LC is tangent to the circumcircle of triangle $\triangle PBC$ at point C. Let lines BH and CP intersect at X. Prove that D, L and X are collinear.

Solution 1



We have $\angle PCD = \angle ECD = \angle EAD$ and $\angle PBD = \angle FBD = \angle FAD$, therefore

$$\angle BPC = 180^\circ - \angle EAD - \angle FAD = 180^\circ - \angle BAC = \angle BHC,$$

meaning that $BHPC$ is cyclic.
We also have $\angle PFD = \angle BFD = \angle BAD = \angle PCD$ showing that $DPFC$ is cyclic. Then, using that $BAFD$ is also cyclic, we have

$$\angle DPC = \angle DFC = \angle ABC.$$

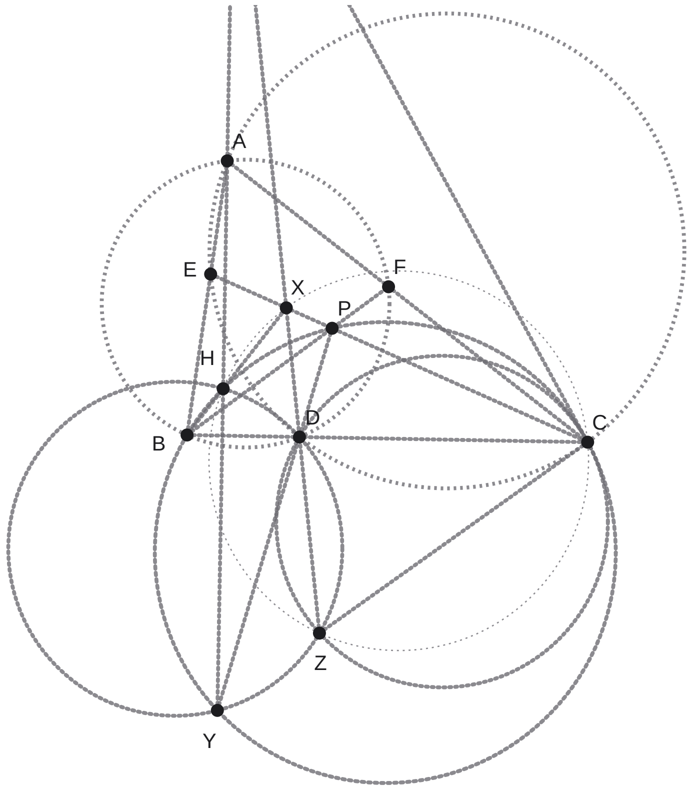
Let Y be the point of intersection of AH with the circumcircle of $BHPC$. Then

$$\angle YPC = \angle YHC = \angle ABC = \angle DPC$$

showing that D belongs on YP.
Finally, applying Pascal's theorem on the hexagon $BHYPCC$, we get that $X = BH \cap PC, L = HY \cap CC$ and $D = YP \cap CB$ are collinear, as required.

Solution 2

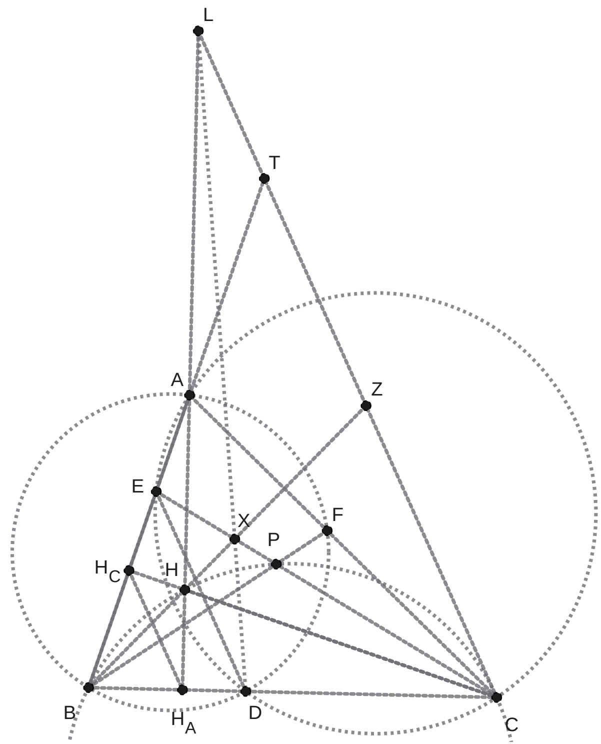
As in Solution 1, we introduce the point Y and prove that the quadrilateral $BHPC$ is cyclic and that points Y, D, P are collinear. Define point Z as the second intersection of (CXH) with the line DX.
Since $\angle HZD = \angle HZX = \angle HCX = \angle HCP = \angle HYP = \angle HYD$, we get the points H, D, Z, Y are concyclic. We also know that $\angle DZC = \angle XZC = \angle XHC = \angle BYC =$



$\angle LCB$, where the last equality holds because LC is tangent to $(BHPC)$. This implies that the circles (BYC) and (ZDC) share a common tangent at the point C.
Finally, applying the radical axis theorem to the circles $(ZDC), (HDZY)$ and $(BHPC)$, we conclude that the line XD passes through the point L, finishing the proof.

Solution 3

Let H_A and H_C be the feet of the altitudes from A and C, respectively. Also, let BA and BH meet CL at Z and T respectively.
We prove that $BCPH$ is cyclic as in Solution 1. CH_AHC_A and $CDEA$ are cyclic, so $\angle BH_AH_C = \angle BDE = \angle BAC$. Since CL is tangent to $(BHPC)$, we have $\angle LCB = 180^\circ - \angle BHC = \angle BAC$. We obtained $\angle BH_AH_C = \angle BDE = \angle BCL$, so $H_CH_A \parallel ED \parallel LC$.
Projecting from C, we obtain $(B, X; H, Z) = (B, E; H_C, T)$. From $H_CH_A \parallel ED \parallel TC$ we have $(B, E; H_C, T) = (B, D; H_A, C)$, which can be seen by applying Thales' theorem or by projecting from infinity.
If we denote the intersection of LD and BZ as X', projecting from L we get $(B, X'; H, Z) = (B, D; H_A, C)$.
Combining the above, we have $(B, X; H, Z) = (B, D; H_A, C) = (B, X'; H, Z)$. It is well-known that, with 3 fixed points and a fixed cross-ratio, the fourth point is uniquely determined. This implies $X \equiv X'$ and we are done.



Solution 4

Same as in Solution 3, we prove that $H_CH_A \parallel ED \parallel l$, where l is the tangent to $(BHPC)$ at C. Now define L as the intersection of AH and DX. Apply Desargues's theorem on triangles $\triangle BH_AH_C$ and $\triangle XLC$. Since LH_A, CH_C and BX are concurrent at H, we obtain that the intersection of BH_C and CX which is E, the intersection of BH_A and LX which is D and the intersection of LC and H_AH_C are collinear. However, since $DE \parallel H_AH_C$, LC is also parallel to these lines, therefore it coincides with the tangent and we are done.

Solution 5

Let H_A, H_B, H_C to be the feet of the altitudes. We again obtain that $BHCP$ is cyclic and that $DE \parallel LC$. We want to prove LC, AH and DX are concurrent. Applying Trigonometric Ceva's theorem to $\triangle AED$, we need to prove:

$$\frac{\sin\angle ABH}{\sin\angle HBC}\cdot\frac{\sin\angle LDB}{\sin\angle LDE}\cdot\frac{\sin\angle CED}{\sin\angle CEB}=\frac{\cos\angle BAC}{\cos\angle ACB}\cdot\frac{\sin\angle LDC}{\sin\angle DLC}\cdot\frac{\sin\angle CAD}{\sin\angle ADB}.$$

From law of sines in $\triangle ADC$ and $\triangle ADB$, the last expression is equal to:

$$\frac{\cos\angle BAC}{\cos\angle ACB}\cdot\frac{LC}{CD}\cdot\frac{CD\cdot\sin\angle ACB\cdot AD}{AB\cdot\sin\angle ABC\cdot AD}.$$

However, $LS=\frac{H_AC}{\cos\angle BAC}=\frac{AC\cdot\cos\angle ACB}{\cos\angle BAC}$ and we get:

$$\frac{\cos BAC}{\cos ACB}\cdot\frac{AC\cdot\cos ACB}{\cos BAC}\cdot\frac{\sin ACB}{AB\cdot\sin ABC}=1.$$

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f(x + yf(x)) + y = xy + f(x + y).$$

Solution 1.

Let $P(x, y)$ denote the given relation. If there is an $a \in \mathbb{R}$ such that $f(a) = 0$, then $P(a, y)$ gives that $y = ay + f(a + y)$, and so f must be linear. Then we can easily check and get that the only linear solutions are $f(x) = x$ and $f(x) = 2 - x$ ($x \in \mathbb{R}$).
Now suppose that $f(x) \neq 0$ for all real numbers x. From $P(x - y, y)$ we get that:

$$f(x - y + yf(x - y)) = -y^2 + y(x - 1) + f(x).$$

Since $f(t) \neq 0$ for all real numbers t, it follows that $-y^2 + y(x - 1) + f(x) \neq 0$ for all real numbers x, y, and so, its discriminant (as a polynomial in y) must be negative. That is, $(x - 1)^2 + 4f(x) < 0$, which gives us

$$f(x) < -\frac{(x-1)^2}{4} \leqslant 0$$

for all real numbers x. Since $(x + 1)^2 \geqslant 0$ implies that $-\frac{(x-1)^2}{4} \leqslant x$, we see that

$$f(x) < -\frac{(x-1)^2}{4} \leqslant x$$

for all real numbers x. Now from $P(x, y)$ for $y > 0$ and $x \in \mathbb{R}$, we get that

$$xy - y + f(x + y) = f(x + yf(x)) < x + yf(x) < x - y\frac{(x-1)^2}{4}$$

and so

$$f(x + y) < x + y - y(x + \frac{(x-1)^2}{4}) = x + y - y\frac{(x+1)^2}{4}.$$

Setting $x = -y$ above, we get that:

$$f(0) < -y\frac{(-y+1)^2}{4}.$$

for all positive real numbers y. Letting $y \to +\infty$ above, we reach a contradiction. Hence, the only solutions in this functional equation are $f(x) = x$ and $f(x) = 2 - x$.

Solution 2.

Let $P(x, y)$ denote the given relation. Similarly to the first solution, if a root exists ($f(a) = 0$ for any a), we get that the function is linear and that the two solutions are $f(x) = x$ and $f(x) = 2 - x$. Assertion $P(x, c - x)$ gives us the following relation:

$$f(x + (c - x)f(x)) = (c - x)(x - 1) + f(c) = -x^2 + (c + 1)x + (f(c) - c)$$

The right hand side of the expression is a quadratic equation in x with the discriminant $\Delta = \Delta(c) = (c+1)^2 + 4(f(c) - c) = (c - 1)^2 + 4f(c)$. Therefore, if there exists a c such that $(c - 1)^2 + 4f(c) \geqslant 0$, the quadratic equation has a real solution which implies the existence of a root, in which case we are done.
If $f(1) = 0$, then we found a root and are done. If $f(1) = 1$, then by taking $c = 1$ we obtain that $\Delta(1) = 4$, implying the existence of a root. We now check the case when $f(1) = -1$. From the assertion $P(1 - x, x)$, we obtain:

$$f(1 - x + xf(1 - x)) = -x^2 - 1$$

Plugging in $x = 1$, in the above assertion, we obtain that $f(f(0)) = -2$. Now plugging in $x = 1 - f(0)$ in the above assertion we get that $f(f(0) + (1 - f(0))f(f(0))) = -(1 - f(0))^2 - 1$, simplifying and utilizing $f(f(0)) = -2$ we obtain $f(3f(0) - 2) = -f(0)^2 + 2f(0) - 2$. Note that if $f(0) \geqslant 0$, we have that $\Delta(0) = 1 + 4f(0) > 0$, implying the existence of a root, so assume that $f(0) < 0$. Now using $c = 3f(0) - 2$ for our discriminant value, we obtain $\Delta(3f(0) - 2) = (3f(0) - 3)^2 + 4f(3f(0) - 2) = 9(f(0) - 1)^2 + 4(-f(0)^2 + 2f(0) - 2) = 5f(0)^2 - 10f(0) + 1 > 0$, implying the existence of a root, and resolving the case when $f(1) = -1$.

Now assume that $f(1) \notin \{0, 1, -1\}$. From $P(1, y)$, we obtain the relation that $f(1 + yf(1)) = f(1 + y)$. As $f(1) \neq 0$, we can inductively show that $f(1 + yf(1)^k) = f(1 + y)$ for all $k \in \mathbb{Z}$. Since $f(1) \notin \{1, -1\}$, there exists an unbounded sequence a_n such that $f(a_n)$ is constant. Namely, one can take $a_n = 1 + f(1)^{2n}$ if $|f(1)| > 1$, and $a_n = 1 + f(1)^{-2n}$ if $|f(1)| < 1$, both times it holds that $f(a_n) = f(2)$. The value of the discriminant along this sequence is $\Delta(a_n) = (a_n - 1)^2 + 4f(a_n) = (a_n - 1)^2 + 4f(2)$, and since a_n is unbounded this there exists n where the value of the discriminant is positive, yielding our root. This finishes the problem.

Solution 3.

Let $P(x, y)$ denote the given relation. As in the previous solutions, if a root exists, then we are done. From $P(1, y)$ we obtain $f(1 + yc) = f(1 + y)$, where we have put $c = f(1)$. From the substitution $P(1 + x, y)$, we get:

$$f(1 + x + yf(1 + x)) + y = (1 + x)y + f(1 + x + y) \tag{1}$$

Substituting $P(1+cx, cy)$ instead, we obtain

$$f(1+cx+cyf(1+cx))+cy=(1+cx)cy+f(1+cx+cy) \tag{2}$$

Note that

$$f(1+cx+cyf(1+cx))=f(1+cx+cyf(1+x))=f(1+c(x+yf(1+x)))=f(1+x+yf(1+x))$$

and that $f(1+cx+cy)=f(1+c(x+y))=f(1+x+y)$. By subtracting (1) and (2) we obtain $c^2xy=xy$ for all x, y, concluding that $c^2=1$. From here, one can proceed in numerous ways (some of which have been highlighted in the previous solutions) to finish the problem.

Solution 4. (by Stefan Šebez)

Let $P(x, y)$ denote the given relation. Putting $P(0, x+y)$ gives us:

$$f((x+y)f(0))+x+y=0+f(x+y)$$

Subtracting this identity from the relation $P(x, y)$ yields:

$$P(x+yf(x))-P((x+y)f(0))=xy+x$$

Suppose that $f(x)\neq f(0)$ for some $x\in\mathbf{R}$ (thus in particular $x\neq 0$). Then letting y equal $x(f(0)-1)/(f(x)-f(0))$ makes the left-hand side vanish, so that $x(y+1)=0$ and $y=-1$. We conclude that, for an arbitrary $x\in\mathbf{R}$, either $f(x)=f(0)$ or $f(x)=x(1-f(0))+f(0)$. Consider the values $f(x)$ and $f(xf(0))$. They are related by $P(0, x)$:

$$f(xf(0))=f(x)-x$$

Fix some $x\neq 0$. Then, for this x, (at least) one of four possible cases holds:

- Case $f(x)=f(0)$ and $f(xf(0))=f(0)$
- Case $f(x)=f(0)$ and $f(xf(0))=xf(0)(1-f(0))+f(0)$
- Case $f(x)=x(1-f(0))+f(0)$ and $f(xf(0))=f(0)$
- Case $f(x)=x(1-f(0))+f(0)$ and $f(xf(0))=xf(0)(1-f(0))+f(0)$

The first case implies that $x=0$, a contradiction. The second gives $f(0)^2-f(0)-1=0$. The third gives $f(0)=0$ and the fourth $f(0)\in\{0,2\}$.
It is now clear that $f(0)=0$ implies $f(x)=x$ for all x, and that $f(0)=2$ implies $f(x)=2-x$ for all x. We check that these two functions indeed satisfy the starting equation. If, on the other hand, $f(0)\notin\{0,2\}$, then the second case holds for all $x\neq 0$ and hence $f(x)=f(0)$ for all x. However, this is a contradiction with $P(0, x)$. Thus there are no more solutions.

Problem 4. There are n cities in a country, where $n \geqslant 100$ is an integer. Some pairs of cities are connected by direct (two-way) flights. For two cities A and B we define:

- a *path* between A and B as a sequence of distinct cities $A = C_0, C_1, \ldots, C_k, C_{k+1} = B$, $k \geqslant 0$, such that there are direct flights between C_i and C_{i+1} for every $0 \leqslant i \leqslant k$;
- a *long path* between A and B as a path between A and B such that no other path between A and B has more cities;
- a *short path* between A and B as a path between A and B such that no other path between A and B has fewer cities.

Assume that for any pair of cities A and B in the country, there exist a long path and a short path between them that have no cities in common (except A and B). Let F be the total number of pairs of cities in the country that are connected by direct flights. In terms of n, find all possible values of F.

Solution

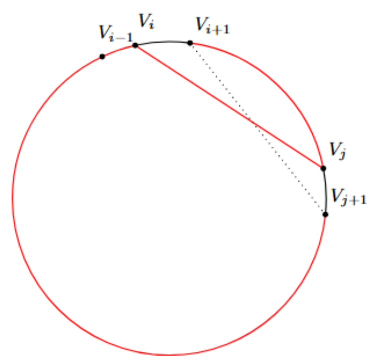
Use the obvious graph interpretation. We show that any such graph is one of the following: the full graph K_n, the circular graph C_n, and for n even, the bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.
First, we show that these graphs satisfy the condition.

- For K_n, we can choose any long path and the short path is be the edge.
- For C_n, we have exactly two paths between any two vertices, and one of them has at most as many vertices as the other.
- For $K_{\frac{n}{2},\frac{n}{2}}$, if the vertices are on different sides, the short path is the edge. Otherwise, take any long path. We observe that it alternates between the sides and begins and ends on one side. Therefore, there is a vertex on the other side that doesn't appear in the long path. Additionally, there is a short path that passes through this vertex.

Next, we show that only these graphs work for n large enough.
The graph is clearly connected, as any two vertices belong to a path. Consider a longest path in the graph. Let p be its length and denote the vertices in the path by $V_1, V_2, \ldots, V_p$ in the corresponding order. We can assume that this path is the long path between V_1 and V_p that has a corresponding short path through other vertices. We show that the edge V_1V_p belongs to the graph. If the edge doesn't exist, the short path has length at least two, implying that there is a vertex X different from $V_i, i \in \{1, \ldots, p\}$ such that there exists an edge from V_1 to X. Then the path $XV_1V_2\ldots V_p$ has length $p+1$, which gives a contradiction.
Next we show that $p = n$, i.e. that the cycle $V_1 \ldots V_p$ contains all the vertices. If there exists another vertex A connected with an edge to a vertex V_i, then the path $AV_iV_{i+1}\ldots V_{i-1}$ has length $p+1$, which gives a contradiction. Since the graph is connected, the cycle contains all vertices.

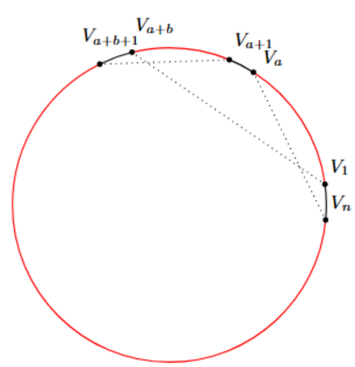
For two vertices of the graph, we say that they have distance r if there are exactly $r-1$ vertices between them on a side of the cycle. Observe that they also have distance $n-r$.
If we relabel the vertices by $A_1, A_2, \ldots, A_n$ in such a way that we know the graph has $n-1$ of the edges $A_iA_{i+1}, i \in \{1, \ldots, n\}$ (where $A_{n+1} = A_1$), then it also has the last one. This is shown same as before.
Next, we show that if we have an edge between V_i and V_j, then we also have an edge between V_{i+1} and V_{j+1}. Assume $i < j$. Consider the path

$$V_{i+1}V_{i+2}\ldots V_jV_iV_{i-1}\ldots V_{j+1}$$



of length n. As before, we conclude that there is an edge between V_{i+1} and V_{j+1}. Repeating this, we get that if we have an edge between two vertices at distance r, then we have edges between any two vertices at distance r.
Define S as the set of numbers $1 \leqslant r \leqslant n-1$ such that the graph has the edges of distance r. Note that $1, n-1 \in S$.
For positive integers a and b with $a+b \leqslant n-1$, consider the ordering

$$V_1, V_{a+b}, V_{a+b-1}, \ldots, V_{a+1}, V_{a+b+1}, V_{a+b+2}, \ldots, V_n, V_a, V_{a-1}, \ldots, V_1.$$

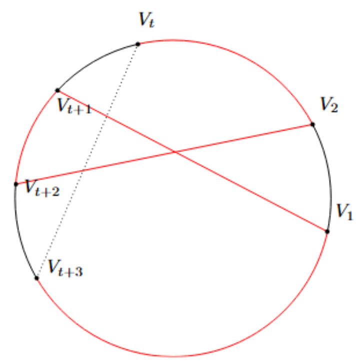


The distance between two consecutive vertices in this ordering is $1, a, b$ or $a+b-1$. This implies that if two numbers from the multiset $\{a, b, a+b-1\}$ belong to S, so does the third one. Now, if $2 \in S$, we take $b = 2$ and easily get that that S contains any number from 1 to $n-1$. This gives us the solution K_n.

Assume now $2 \notin S$. This implies that we do not have two consecutive numbers smaller than $n-2$ in S. But as $2 \notin S$, we also have $n-2 \notin S$, so S doesn't contain two consecutive integers.
If $S = \{1, n-1\}$, we get the solution C_n. Otherwise, there exists $t \in S$ such that $3 \leqslant t \leqslant n-3$. Consider the path

$$V_t V_{t-1} \ldots V_2 V_{t+2} V_{t+1} V_1 V_n \ldots V_{t+3}$$

of length n.



Same as before, we get that there is an edge between V_t and V_{t+3}. Therefore, we have $3 \in S$. Now, taking $b = 3$, we get that any odd number smaller than or equal to $n-1$ lies in S. Since we assumed S doesn't contain consecutive integers, we get that n is even and $S = \{1 \leqslant i \leqslant n-1 \,|\, i \text{ odd}\}$. This gives us the solution $K_{\frac{n}{2},\frac{n}{2}}$.
Finally, the number of edges can be n, $\frac{n(n-1)}{2}$, and if n is even it can also be $\frac{n^2}{4}$.

Remark: Even if n is not big enough, we still characterize all such graphs similarly. The condition was added as at some point we choose a number t between 3 and $n-3$, and this wouldn't make sense for small n and we would need to quickly discuss why those cases also have the same graphs.