24$^{\text{th}}$ Junior Balkan Mathematical Olympiad

September 9-13 2020, Athens, Greece

Shortlisted problems with Solutions

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Note of Confidentiality

The shortlisted problems should be kept strictly confidential until JBMO 2021

Contributing countries

The Organising Committee and the Problem Selection Committee of the JBMO 2020 wish to thank the following countries for contributing problem proposals.

Albania
Bosnia
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ALGEBRA

A 1. Find all triples (a, b, c) of real numbers such that the following system holds:

$$\begin{cases} a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \\ a^2+b^2+c^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2} \end{cases}$$

Solution. First of all if (a, b, c) is a solution of the system then also $(-a, -b, -c)$ is a solution. Hence we can suppose that $abc > 0$. From the first condition we have

$$a+b+c=\frac{ab+bc+ca}{abc}\cdot \qquad (1)$$

Now, from the first condition and the second condition we get

$$(a+b+c)^2-(a^2+b^2+c^2)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2-\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

The last one simplifies to

$$ab+bc+ca=\frac{a+b+c}{abc}\cdot \qquad (2)$$

First we show that $a+b+c$ and $ab+bc+ca$ are different from 0. Suppose on contrary then from relation (1) or (2) we have $a+b+c=ab+bc+ca=0$. But then we would have

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=0,$$

which means that $a=b=c=0$. This is not possible since a, b, c should be different from 0.
Now multiplying (1) and (2) we have

$$(a+b+c)(ab+bc+ca)=\frac{(a+b+c)(ab+bc+ca)}{(abc)^2}.$$

Since $a+b+c$ and $ab+bc+ca$ are different from 0, we get $(abc)^2=1$ and using the fact that $abc>0$ we obtain that $abc=1$. So relations (1) and (2) transform to

$$a+b+c=ab+bc+ca.$$

Therefore,

$$(a-1)(b-1)(c-1)=abc-ab-bc-ca+a+b+c-1=0.$$

This means that at least one of the numbers a, b, c is equal to 1. Suppose that $c=1$ then relations (1) and (2) transform to $a+b+1=ab+a+b \Rightarrow ab=1$. Taking $a=t$ then we have $b=\frac{1}{t}$. We can now verify that any triple $(a,b,c)=\left(t,\frac{1}{t},1\right)$ satisfies both conditions. $t\in\mathbb{R}\setminus\{0\}$. From the initial observation any triple $(a,b,c)=\left(t,\frac{1}{t},-1\right)$ satisfies both conditions. $t\in\mathbb{R}\setminus\{0\}$. So, all triples that satisfy both conditions are $(a,b,c)=\left(t,\frac{1}{t},1\right),\left(t,\frac{1}{t},-1\right)$ and all permutations for any $t\in\mathbb{R}\setminus\{0\}$. □

Comment by PSC. After finding that $abc=1$ and

$$a+b+c=ab+bc+ca,$$

we can avoid the trick considering $(a-1)(b-1)(c-1)$ as follows. By the Vieta's relations we have that a, b, c are roots of the polynomial

$$P(x)=x^3-sx^2+sx-1$$

which has one root equal to 1. Then, we can conclude as in the above solution.

A 2. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 9$ and

$$a_{n+1} = \frac{(n+5)a_n + 22}{n+3}$$

for $n \geqslant 1$.

Find all natural numbers n for which a_n is a perfect square of an integer.

Solution: Define $b_n = a_n + 11$. Then

$$22 = (n+3)a_{n+1} - (n+5)a_n = (n+3)b_{n+1} - 11n - 33 - (n+5)b_n + 11n + 55$$

giving $(n+3)b_{n+1} = (n+5)b_n$. Then

$$b_{n+1} = \frac{n+5}{n+3}b_n = \frac{(n+5)(n+4)}{(n+3)(n+2)}b_{n-1} = \frac{(n+5)(n+4)}{(n+2)(n+1)}b_{n-2} = \cdots = \frac{(n+5)(n+4)}{5 \cdot 4}b_1 = (n+5)(n+4).$$

Therefore $b_n = (n+4)(n+3) = n^2 + 7n + 12$ and $a_n = n^2 + 7n + 1$.

Since $(n+1)^2 = n^2 + 2n + 1 < a_n < n^2 + 8n + 16 = (n+4)^2$, if a_n is a perfect square, then $a_n = (n+2)^2$ or $a_n = (n+3)^2$.

If $a_n = (n+2)^2$, then $n^2 + 4n + 4 = n^2 + 7n + 1$ giving $n = 1$. If $a_n = (n+3)^2$, then $n^2 + 6n + 9 = n^2 + 7n + 1$ giving $n = 8$.

□

Comment. We provide some other methods to find a_n.

Method 1: Define $b_n = \frac{a_n + 11}{n+3}$. Then $b_1 = 5$ and $a_n = (n+3)b_n - 11$. So

$$a_{n+1} = (n+4)b_{n+1} - 11 = \frac{(n+5)a_n + 22}{n+3} = a_n + \frac{2(a_n + 11)}{n+3} = (n+3)b_n - 11 + 2b_n$$

giving $(n+4)b_{n+1} = (n+5)b_n$. Then

$$b_{n+1} = \frac{n+5}{n+4}b_n = \frac{n+5}{n+3}b_{n-1} = \cdots = \frac{n+5}{5}b_1 = n+5.$$

Then $b_n = n+4$, so $a_n = (n+3)(n+4) - 11 = n^2 + 7n + 1$.

Method 2: We have

$$(n+3)a_{n+1} - (n+5)a_n = 22$$

and therefore

$$\frac{a_{n+1}}{(n+5)(n+4)} - \frac{a_n}{(n+4)(n+3)} = \frac{22}{(n+3)(n+4)(n+5)} = 11\left[\frac{1}{n+3} - \frac{2}{n+4} + \frac{1}{n+5}\right].$$

Now define $b_n = \frac{a_n}{(n+4)(n+3)}$ to get

$$b_{n+1} = b_n + 11\left[\frac{1}{n+3} - \frac{2}{n+4} + \frac{1}{n+5}\right]$$

which telescopically gives

$$\begin{aligned} b_{n+1} &= b_1 + 11\left[\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \cdots + \left(\frac{1}{n+3} - \frac{2}{n+4} + \frac{1}{n+5}\right)\right] \\ &= b_1 + 11\left(\frac{1}{4} - \frac{1}{5} - \frac{1}{n+4} + \frac{1}{n+5}\right) = \frac{9}{20} + \frac{11}{20} - \frac{11}{(n+4)(n+5)} \end{aligned}$$

We get $b_{n+1} = (n+4)(n+5) - 11$ from which it follows that $b_n = (n+3)(n+4) - 11 = n^2 + 7n + 1$.

A 3. Find all triples of positive real numbers (a, b, c) so that the expression

$$M = \frac{(a+b)(b+c)(a+b+c)}{abc}$$

gets its least value.

Solution. The expression M is homogeneous, therefore we can assume that $abc = 1$. We set $s = a+c$ and $p = ac$ and using $b = \frac{1}{ac}$, we get

$$M = \left(a + \frac{1}{ac}\right)\left(\frac{1}{ac} + c\right)\left(a + \frac{1}{ac} + c\right) = (a + p^{-1})(c + p^{-1})(s + p^{-1}).$$

Expanding the right-hand side we get

$$M = ps + \frac{s^2}{p} + 1 + \frac{2s}{p^2} + \frac{1}{p^3}.$$

Now by $s \geq 2\sqrt{p}$ and setting $x = p\sqrt{p} > 0$ we get

$$M \geq 2x + 5 + \frac{4}{x} + \frac{1}{x^2}.$$

We will now prove that

$$2x + 5 + \frac{4}{x} + \frac{1}{x^2} \geq \frac{11 + 5\sqrt{5}}{2}.$$

Indeed, the latter is equivalent to $4x^3 - (5\sqrt{5} + 1)x^2 + 8x + 2 \geq 0$, which can be rewritten as

$$\left(x - \frac{1+\sqrt{5}}{2}\right)^2 (4x + 3 - \sqrt{5}) \geq 0,$$

which is true. □

Remark: Notice that the equality holds for $a = c = \sqrt{p} = \sqrt[3]{\frac{1+\sqrt{5}}{2}}$ and $b = \frac{1}{ac}$.

C 1. Alice and Bob play the following game: starting with the number 2 written on a blackboard, each player in turn changes the current number n to a number $n+p$, where p is a prime divisor of n. Alice goes first and the players alternate in turn. The game is lost by the one who is forced to write a number greater than $\underbrace{2\dots2}_{2020}$. Assuming perfect play, who will win the game.

Solution. We prove that Alice wins the game. For argument's sake, suppose that Bob can win by proper play regardless of what Alice does on each of her moves. Note that Alice can force the line $2 \to \mathbf{4} \to 6 \to \mathbf{8} \to 10 \to \mathbf{12}$ at the beginning stages of the game. (As each intermediate 'position' from which Bob has to play is a prime power.) Thus the player on turn when the number 12 is written on the blackboard must be in a 'winning position', i.e., can win the game with skillful play. However, Alice can place herself in that position through the following line that is once again forced for Bob: $2 \to \mathbf{4} \to 6 \to \mathbf{9} \to 12$. (This time she is in turn with 12 written on the blackboard.) The obtained contradiction proves our point.

$\square$

Comment by the PSC. Notice that this is a game of two players which always ends in a finite number of moves with a winner. For games like this, a player whose turn is to make a move, may be in position to force a win for him. If not, then the other player is in position to force a win for him.

C 2. Viktor and Natalia bought 2020 buckets of ice-cream and want to organize a *degustation schedule* with 2020 rounds such that:

- In every round, each one of them tries 1 ice-cream, and those 2 ice-creams tried in a single round are different from each other.
- At the end of the 2020 rounds, each one of them has tried each ice-cream exactly once.

We will call a degustation schedule *fair* if the number of ice-creams that were tried by Viktor before Natalia is equal to the number of ice creams tried by Natalia before Viktor.
Prove that the number of fair schedules is strictly larger than $2020!(2^{1010} + (1010!)^2)$.

Solution. If we fix the order in which Natalia tries the ice-creams, we may consider 2 types of fair schedules:
1) Her last 1010 ice-creams get assigned as Viktor's first 1010 ice-creams, and vice versa: Viktor's first 1010 ice-creams are assigned as Natalia's last 1010 ice-creams. This generates $(1010!)^2$ distinct fair schedules by permuting the ice-creams within each group.
2) We divide all ice-creams into disjoint groups of 4, and in each group we swap the first 2 ice-creams with the last 2, which gives us $((2!)^2)^{504} = 2^{1010}$ distinct schedules.
Now, to make the inequality strict, we consider 1 more schedule like 2), but with groups of 2 ice-creams instead of 4.

□

C 3. Alice and Bob play the following game: Alice begins by picking a natural number $n \geqslant 2$. Then, with Bob starting first, they alternately choose one number from the set $A = \{1, 2, \ldots, n\}$ according to the following condition: The number chosen at each step should be distinct from all the already chosen numbers, and should differ by 1 from an already chosen number. (At the very first step Bob can choose any number he wants.) The game ends when all numbers from the set A are chosen.

For example, if Alice picks $n = 4$, then a valid game would be for Bob to choose 2, then Alice to choose 3, then Bob to choose 1, and then Alice to choose 4.

Alice wins if the sum S of all of the numbers that she has chosen is composite. Otherwise Bob wins. (In the above example $S = 7$, so Bob wins.)

Decide which player has a winning strategy.

Solution. Alice has a winning strategy. She initially picks $n = 8$. We will give a strategy so that she can end up with S even, or $S = 15$, or $S = 21$, so she wins.

Case 1: If Bob chooses 1, then the game ends with Alice choosing $2, 4, 6, 8$ so S is even (larger than 2) and Alice wins.

Case 2: If Bob chooses 2, then Alice chooses 3. Bob can now choose either 1 or 3.

Case 2A: If Bob chooses 1, then Alice's numbers are $3, 4, 6, 8$. So $S = 21$ and Alice wins.

Case 2B: If Bob chooses 4, then Alice chooses 1 and ends with the numbers $1, 3, 6, 8$. So S is even and alice wins.

Case 3: If Bob chooses 3, then Alice chooses 2. Bob can now choose either 1 or 4.

Case 3A: If Bob chooses 1, then Alice's numbers are $2, 4, 6, 8$. So S is even and Alice wins.

Case 3B: If Bob chooses 4, then Alice chooses 5. Bob can now choose either 1 or 6.

Case 3Bi: If Bob chooses 1, then Alice's numbers are $2, 5, 6, 8$. So $S = 21$ and Alice wins.

Case 3Bii: If Bob chooses 6, then Alice chooses 1. Then Alice's numbers are $2, 5, 1, 8$. So S is even and Alice wins.

Case 4: If Bob chooses 4, then Alice chooses 5. Bob can now choose either 3 or 6.

Case 4A: If Bob chooses 3, then Alice chooses 6. Bob can now choose either 2 or 7.

Case 4Ai: If Bob chooses 2, then Alice chooses 1 and ends up with $5, 6, 1, 8$. So S is even and Alice wins.

Case 4Aii: If Bob chooses 7, then Alice chooses 8 and ends up with $5, 6, 8, 1$. So S is even and Alice wins.

Case 4B: If Bob chooses 6, then Alice chooses 7. Bob can now choose either 3 or 8.

Case 4Bi: If Bob chooses 3, then Alice chooses 2 and ends up with $5, 7, 2$ and either 1 or 8. So $S = 15$ or $S = 22$ and Alice wins.

Case 4Bii: If Bob chooses 8, then Alice's numbers are $5, 7, 3, 1$. So S is even and Alice wins.

Cases 5-8: If Bob chooses $k \in \{5, 6, 7, 8\}$ then Alice follows the strategy in case $9 - k$ but whenever she had to choose ℓ, she instead chooses $9 - \ell$. If at the end of that strategy she ended up with S, she will now end up with $S' = 4 \cdot 9 - S = 36 - S$. Then S' is even or $S' = 15$ or $S' = 21$ so again she wins.

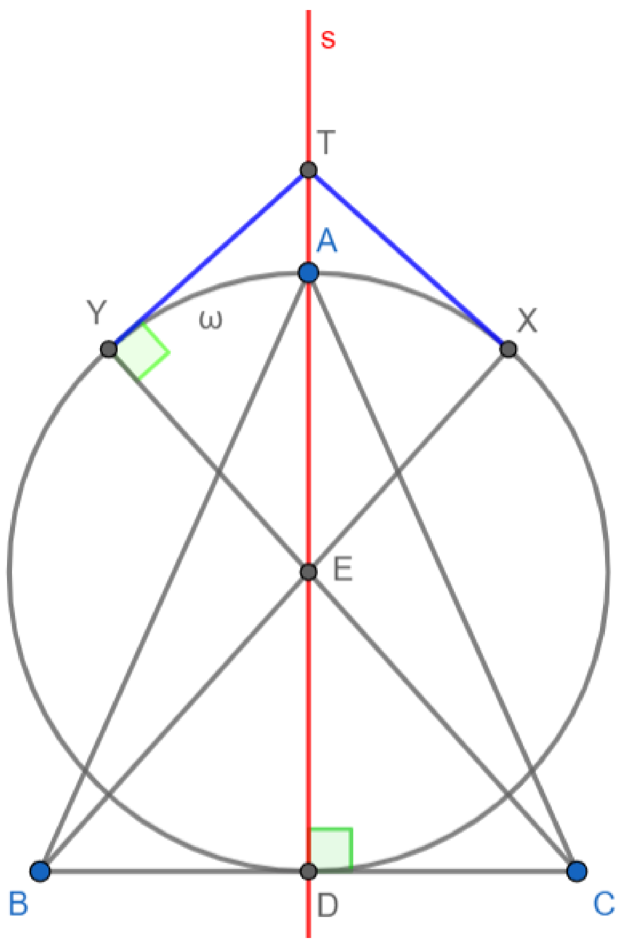
GEOMETRY

G 1. Let $\triangle ABC$ be an acute triangle. The line through A perpendicular to BC intersects BC at D. Let E be the midpoint of AD and ω the the circle with center E and radius equal to AE. The line BE intersects ω at a point X such that X and B are not on the same side of AD and the line CE intersects ω at a point Y such that C and Y are not on the same side of AD. If both of the intersection points of the circumcircles of $\triangle BDX$ and $\triangle CDY$ lie on the line AD, prove that $AB = AC$.

Solution. Denote by s the line AD. Let T be the second intersection point of the circumcircles of $\triangle BDX$ and $\triangle CDY$. Then T is on the line s. Note that $CDYT$ and $BDXT$ are cyclic. Using this and the fact that AD is perpendicular to BC we obtain:

$$\angle TYE = \angle TYC = \angle TDC = 90^\circ$$

This means that EY is perpendicular to TY, so TY must be tangent to ω. We similarly show that TX is tangent to ω. Thus, TX and TY are tangents from T to ω which implies that s is the perpendicular bisector of the segment XY. Now denote by σ the reflection of the plane with respect to s. Then the points X and Y are symmetric with respect to s, so $\sigma(X) = Y$. Also note that $\sigma(E) = E$, because E is on s. Using the fact that BC is perpendicular to s, we see that BC is the reflection image of itself with respect to s. Now note that B is the intersection point of the lines EX and BC. This means that the image of B is the intersection point of the lines $\sigma(EX) = EY$ and $\sigma(BC) = BC$, which is C. From here we see that $\sigma(B) = C$, so s is the perpendicular bisector of BC, which is what we needed to prove.



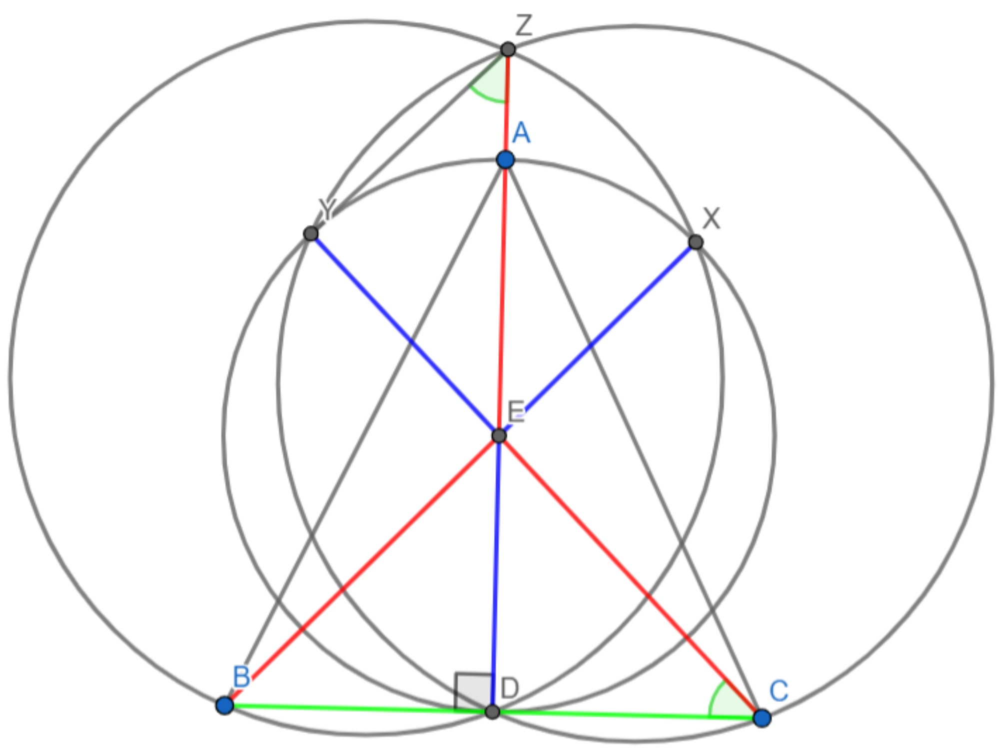
Alternative Solution. Let the circle $\odot CDY$ intersects the line AD at another point Z. Then we have $\angle CED = \angle YEZ$. We also have $ED = EY$ because E is the center of the circle ω. Also note that

$$\angle DCE = \angle DCY = \angle DZY = \angle EZY$$

We conclude that $\triangle CDE$ and $\triangle ZYE$ are congruent. From here we have that $EZ = EC$. Now denote by Z' the other intersection point of AD and $\odot BDX$. In the same way we prove that $EZ' = EB$. By the assumption of the problem, we must have that $Z = Z'$. We now conclude that

$$BE = CE = EZ = EZ'$$

Also, $\angle BDE = 90^\circ = \angle CDE$. Now we see that $\triangle BDE$ and $\triangle CDE$ are congruent (they share the side ED), so $BD = CD$. But D is both the midpoint of BC and the foot of the altitude from A, which means that $AB = AC$.



Alternative Solution. Let $\alpha = \angle BXD$. Denote by T the second intersection point of the circumcircles of $\triangle BDX$ and $\triangle CDY$, which is on AD. We have

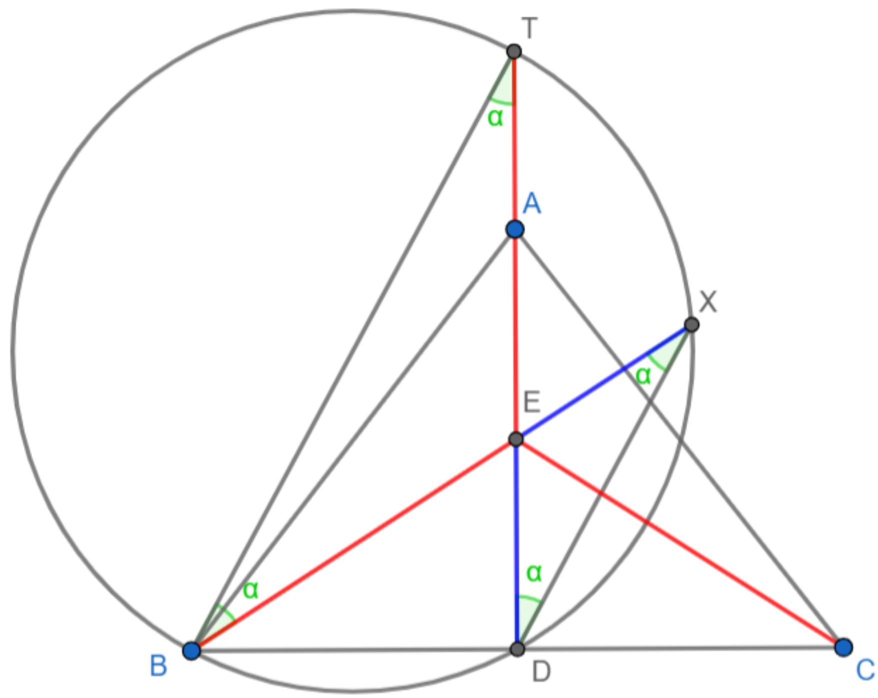
$$ED = EA = EX$$

because E is the center of ω. Now $EX = ED$ implies $\angle EDX = \alpha$. From here we have that $\angle BED = 2\alpha$. Using that $BTXD$ is cyclic we obtain $\angle BTD = \angle BXD = \alpha$. We also have that

$$\angle TBE = 180^\circ - \angle BET - \angle ETB =$$

$$= 180^\circ - (180^\circ - 2\alpha) - \alpha = \alpha = \angle BTE$$

This gives us $BE = TE$. We similarly show that $CE = TE$, and so $BE = CE$. In the same way as in the second solution, this now gives us that $BD = CD$, so D is also the midpoint of BC and we must have $AB = AC$.



Alternative Solution. We can solve the problem using only calculations. Note that the condition of the problem is that E lies on the radical axis of the circumcircles of $\triangle BDX$ and $\triangle CDY$. This gives us $EB \cdot EX = EC \cdot EY$. However, $EX = EY$ because E is the center of ω and this means that $BE = CE$. Now using Pythagoras' theorem we have the following:

$$AB^2 = AD^2 + BD^2 = AD^2 + (BE^2 - DE^2) = AD^2 + (CE^2 - DE^2) = AD^2 + CD^2 = AC^2$$

From here we obtain $AB = AC$.

G 2. Problem: Let $\triangle ABC$ be a right-angled triangle with $\angle BAC = 90^\circ$, and let E be the foot of the perpendicular from A on BC. Let $Z \neq A$ be a point on the line AB with $AB = BZ$. Let (c), (c_1) be the circmucircles of the triangles $\triangle AEZ$ and $\triangle BEZ$, respectively. Let (c_2) be an arbitrary circle passing through the points A and E. Suppose (c_1) meets the line CZ again at the point F, and meets (c_2) again at the point N. If P is the other point of intesection of (c_2) with AF, prove that the points N, B, P are collinear.

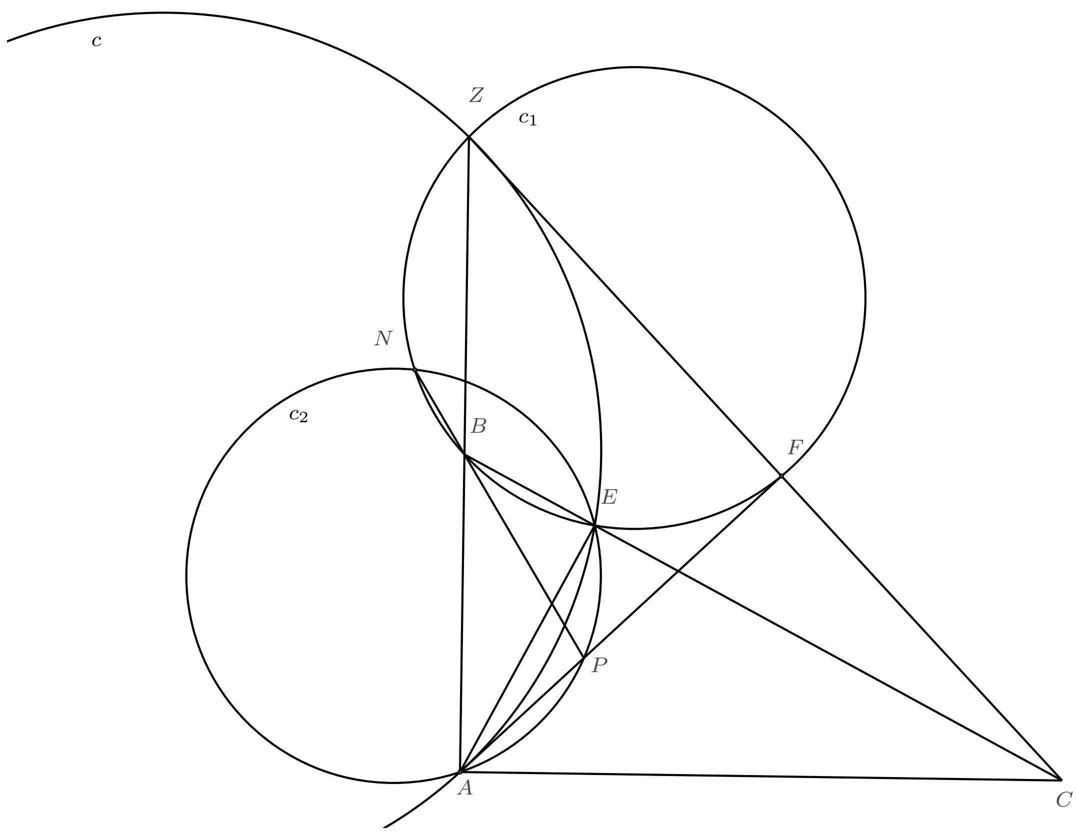
Solution. Since the triangles $\triangle AEB$ and $\triangle CAB$ are similar, then

$$\frac{AB}{EB} = \frac{CB}{AB}.$$

Since $AB = BZ$ we get

$$\frac{BZ}{EB} = \frac{CB}{BZ}$$

from which it follows that the triangles $\triangle ZBE$ and $\triangle CBZ$ are also similar. Since $FEBZ$ is cyclic,



then $\angle BEZ = \angle BFZ$. So by the similarity of triangles $\triangle ZBE$ and $\triangle CBZ$ we get

$$\angle BFZ = \angle BEZ = \angle BZC = \angle BZF$$

and therefore the triangle $\triangle BFZ$ is isosceles. Since $BF = BZ = AB$, then the triangle $\triangle AFZ$ is right-angled with $\angle AFZ = 90^\circ$.

It now follows that the points A, E, F, C are concyclic. Since A, P, E, N are also concyclic, then

$$\angle ENP = \angle EAP = \angle EAF = \angle ECF = \angle BCZ = \angle BZE,$$

where in the last equality we used again the similarity of the triangles $\triangle ZBE$ and $\triangle CBZ$. Since N, B, E, Z are concyclic, then $\angle ENP = \angle BZE = \angle ENB$, from which it follows that the points N, B, P are collinear.

$\square$

G 3. Let $\triangle ABC$ be a right-angled triangle with $\angle BAC = 90^\circ$ and let E be the foot of the perpendicular from A on BC. Let $Z \neq A$ be a point on the line AB with $AB = BZ$. Let (c) be the circumcircle of the triangle $\triangle AEZ$. Let D be the second point of intersection of (c) with ZC and let F be the antidiametric point of D with respect to (c). Let P be the point of intersection of the lines FE and CZ. If the tangent to (c) at Z meets PA at T, prove that the points T, E, B, Z are concyclic.

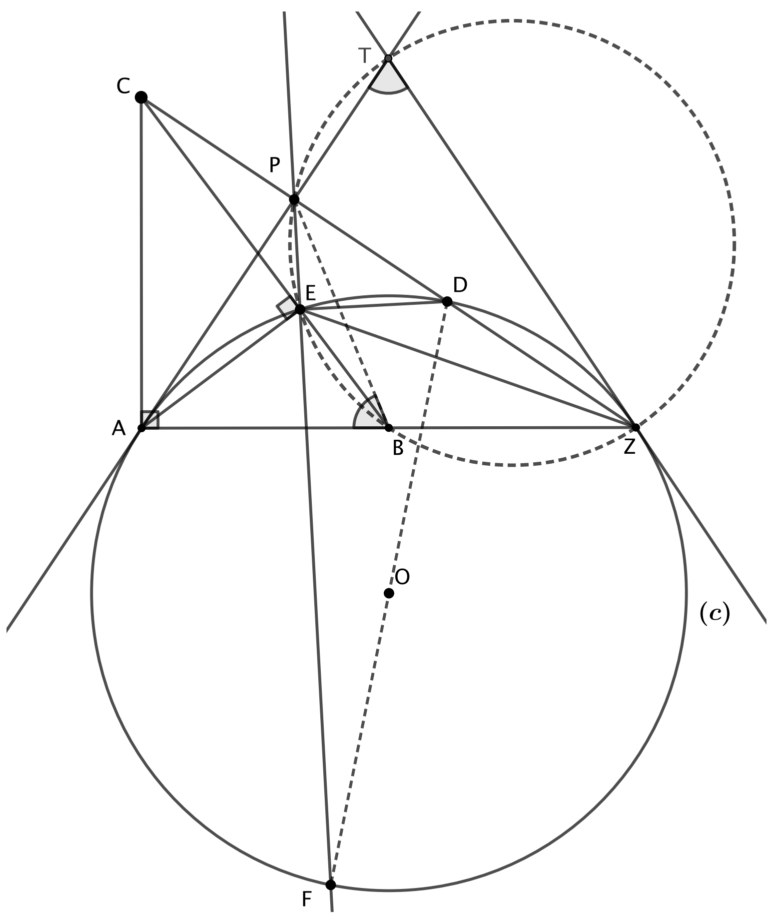
Solution. We will first show that PA is tangent to (c) at A.

Since E, D, Z, A are concyclic, then $\angle EDC = \angle EAZ = \angle EAB$. Since also the triangles $\triangle ABC$ and $\triangle EBA$ are similar, then $\angle EAB = \angle BCA$, therefore $\angle EDC = \angle BCA$.

Since $\angle FED = 90^\circ$, then $\angle PED = 90^\circ$ and so

$$\angle EPD = 90^\circ - \angle EDC = 90^\circ - \angle BCA = \angle EAC.$$

Therefore the points E, A, C, P are concyclic. It follows that $\angle CPA = 90^\circ$ and therefore the triangle $\angle PAZ$ is right-angled. Since also B is the midpoint of AZ, then $PB = AB = BZ$ and so $\angle ZPB = \angle PZB$.



Furthermore, $\angle EPD = \angle EAC = \angle CBA = \angle EBA$ from which it follows that the points P, E, B, Z are also concyclic.

Now observe that

$$\angle PAE = \angle PCE = \angle ZPB - \angle PBE = \angle PZB - \angle PZE = \angle EZB.$$

Therefore PA is tangent to (c) at A as claimed.

It now follows that $TA = TZ$. Therefore

$$\begin{aligned}\angle PTZ &= 180^\circ - 2(\angle TAB) = 180^\circ - 2(\angle PAE + \angle EAB) = 180^\circ - 2(\angle ECP + \angle ACB) \\ &= 180^\circ - 2(90^\circ - \angle PZB) = 2(\angle PZB) = \angle PZB + \angle BPZ = \angle PBA.\end{aligned}$$

Thus T, P, B, Z are concyclic, and since P, E, B, Z are also concyclic then T, E, B, Z are concyclic as required.

$\square$

NT 1. Determine whether there is a natural number n for which $8^n + 47$ is prime.

Solution. The number $m = 8^n + 47$ is never prime.

If n is even, say $n = 2k$, then $m = 64^k + 47 \equiv 1 + 2 \equiv 0 \bmod 3$. Since also $m > 3$, then m is not prime.

If $n \equiv 1 \bmod 4$, say $n = 4k + 1$, then $m = 8 \cdot (8^k)^4 + 47 \equiv 3 + 2 \equiv 0 \bmod 5$. Since also $m > 3$, then m is not prime.

If $n \equiv 3 \bmod 4$, say $n = 4k + 3$, then $m = 8(64^{2k+1} + 1) \equiv 8((-1)^{2k+1} + 1) \equiv 0 \bmod 13$. Since also $m > 13$, then m is not prime.

□

NT 2. Find all positive integers a, b, c and p, where p is a prime number, such that

$$73p^2 + 6 = 9a^2 + 17b^2 + 17c^2.$$

Solution. Since the equation is symmetric with respect to the numbers b and c, we assume that $b \geq c$.

If $p \geq 3$, then p is an odd number. We consider the equation modulo 8. Since,

$$73p^2 + 6 \equiv 79 \equiv 7 \pmod 8,$$

we get that

$$a^2 + b^2 + c^2 \equiv 7 \pmod 8.$$

This cannot happen since for any integer x we have that

$$x^2 \equiv 0, 1, 4 \pmod 8.$$

Hence, p must be an even prime number, which means that $p = 2$. In this case, we obtain the equation

$$9a^2 + 17(b^2 + c^2) = 289.$$

It follows that $b^2 + c^2 \leq 17$. This is possible only for

$$(b, c) \in \{(4, 1),\ (3, 2),\ (3, 1),\ (2, 2),\ (2, 1),\ (1, 1)\}.$$

It is easy to check that among these pairs only the $(4, 1)$ gives an integer solution for a, namely $a = 1$.

Therefore, the given equation has only two solutions,

$$(a, b, c, p) \in \{(1, 1, 4, 2),\ (1, 4, 1, 2)\}.$$

□

NT 3. Find the largest integer k ($k \geq 2$), for which there exists an integer n ($n \geq k$) such that from any collection of n consecutive positive integers one can always choose k numbers, which verify the following conditions:

1. each chosen number is not divisible by 6, by 7 and by 8;
2. the positive difference of any two different chosen numbers is not divisible by at least one of the numbers 6, 7 or 8.

Solution. An integer is divisible by 6, 7 and 8 if and only if it is divisible by their Least Common Multiple, which equals $6 \times 7 \times 4 = 168$.

Let n be a positive integer and let A be an arbitrary set of n consecutive positive integers. Replace each number a_i from A with its remainder r_i ($\mod 168$). The number a_i is divisible by 6 (7 or 8) if and only if its remainder r_i is divisible by 6 (respectively 7 or 8). The difference $|a_i - a_j|$ is divisible by 168 if and only if their remainders $r_i = r_j$.

Choosing k numbers from the initial set A, which verify the required conditions, is the same as choosing k their remainders ($\mod 168$) such that:

1. each chosen remainder is not divisible by 6, 7 and 8;
2. all chosen remainders are different.

Suppose we have chosen k numbers from A, which verify the conditions. Therefore, all remainders are different and $k \leq 168$ (otherwise, there would be two equal remainders).

Denote by $B = \{0, 1, 2, 3, \ldots, 167\}$ the set of all possible remainders ($\mod 168$) and by B_m the subset of all elements of B, which are divisible by m. Compute the number of elements of the following subsets:

$$|B_6| = 168 : 6 = 28, \quad |B_7| = 168 : 7 = 24, \quad |B_8| = 168 : 8 = 21,$$

$$|B_6 \cap B_7| = |B_{42}| = 168 : 42 = 4, \quad |B_6 \cap B_8| = |B_{24}| = 168 : 24 = 7,$$

$$|B_7 \cap B_8| = |B_{56}| = 168 : 56 = 3, \quad |B_6 \cap B_7 \cap B_8| = |B_{168}| = 1.$$

Denote by $D = B_6 \cup B_7 \cup B_8$, the subset of all elements of B, which are divisible by at least one of the numbers 6, 7 or 8. By the Inclusion-Exclusion principle we got

$$|D| = |B_6| + |B_7| + |B_8| - (|B_6 \cap B_7| + |B_6 \cap B_8| + |B_7 \cap B_8|) + |B_6 \cap B_7 \cap B_8| =$$

$$28 + 24 + 21 - (4 + 7 + 3) + 1 = 60.$$

Each chosen remainder belongs to the subset $B \setminus D$, since it is not divisible by 6, 7 and 8. Hence, $k \leq |B \setminus D| = 168 - 60 = 108$.

Let us show that the greatest possible value is $k = 108$. Consider $n = 168$. Given any collection A of 168 consecutive positive integers, replace each number with its remainder ($\mod 168$). Choose from these remainders 108 numbers, which constitute the set $B \setminus D$. Finally, take 108 numbers from the initial set A, having exactly these remainders. These $k = 108$ numbers verify the required conditions. □

NT 4. Find all prime numbers p such that

$$(x+y)^{19}-x^{19}-y^{19}$$

is a multiple of p for any positive integers x, y.

Solution. If $x=y=1$ then p divides

$$2^{19}-2=2(2^{18}-1)=2(2^9-1)(2^9+1)=2\cdot 511\cdot 513=2\cdot 3^3\cdot 7\cdot 19\cdot 73.$$

If $x=2$, $y=1$ then

$$p \mid 3^{19}-2^{19}-1.$$

We will show that $3^{19}-2^{19}-1$ is not a multiple of 73. Indeed,

$$3^{19}\equiv 3^3\cdot(3^4)^4\equiv 3^3\cdot 8^4\equiv 3^3\cdot(-9)^2\equiv 27\cdot 81\equiv 27\cdot 8\equiv 70 \pmod{73}$$

and

$$2^{19}\equiv 2\cdot 64^3\equiv 2\cdot(-9)^3\equiv -18\cdot 81\equiv -18\cdot 8\equiv -144\equiv 2 \pmod{73}.$$

Thus p can be only among 2, 3, 7, 19. We will prove all these work.

- For $p=19$ this follows by Fermat's Theorem as

$$(x+y)^{19}\equiv x+y \pmod{19},\quad x^{19}\equiv x \pmod{19},\quad y^{19}\equiv y \pmod{19}.$$

- For $p=7$, we have that

$$a^{19}\equiv a \pmod{7},$$

  for every integer a. Indeed, if $7\mid a$, it is trivial, while if $7\nmid a$, then by Fermat's Theorem we have

$$7\mid a^6-1\mid a^{18}-1,$$

  therefore

$$7\mid a(a^{18}-1).$$

- For $p=3$, we will prove that

$$b^{19}\equiv b \pmod{3}.$$

  Indeed, if $3\mid b$, it is trivial, while if $3\nmid b$, then by Fermat's Theorem we have

$$3\mid b^2-1\mid b^{18}-1,$$

  therefore

$$3\mid b(b^{18}-1).$$

- For $p=2$ it is true, since among $x+y$, x and y there are 0 or 2 odd numbers.

□

NT 5. The positive integer k and the set A of different integers from 1 to $3k$ inclusive are such that there are no distinct a, b, c in A satisfying $2b = a + c$. The numbers from A in the interval $[1, k]$ will be called *small*; those in $[k+1, 2k]$ – *medium* and those in $[2k+1, 3k]$ – *large*. Is it always true that there are **no** positive integers x and d such that if x, $x+d$ and $x+2d$ are divided by $3k$ then the remainders belong to A and those of x and $x+d$ are different and are:
a) small? b) medium? c) large?
(*In this problem we assume that if a multiple of* $3k$ *is divided by* $3k$ *then the remainder is* $3k$ *rather than* 0.)

Solution.

Solution. A counterexample for a) is $k = 3$, $A = \{1, 2, 9\}$, $x = 2$ and $d = 8$.
A counterexample for c) is $k = 3$, $A = \{1, 8, 9\}$, $x = 8$ and $d = 1$.
We will prove that b) is true.

Suppose the contrary and let x, d have the above properties. We can assume $0 < d < 3k$, $0 < x \leq 3k$ (since for $d = 3k$ the remainders for x and $x+d$ are equal). Hence $0 < x+d < 6k$ and there are two cases:

- If $x + d > 3k$, then since the remainder for $x+d$ is medium we have $4k < x+d \leq 5k$. This means that the remainder of $x+d$ when it is divided by $3k$ is

$$x + d - 3k.$$

  Since x is medium we have $x \leq 2k$ so $d = (x+d) - x > 2k$. Therefore $6k = 4k + 2k < (x+d) + d < 8k$. This means that the remainder of $x+2d$ when it is divided by $3k$ is

$$x + 2d - 6k.$$

  Thus the remainders $(x+2d-6k), (x+d-3k)$ and x are in $[1, 3k]$, they belong to A and

$$2(x + d - 3k) = (x + 2d - 6k) + x,$$

  a contradiction.

- If $x + d \leq 3k$ then as $x+d$ is medium we have $k < x+d \leq 2k$. From the limitations on x, we have $x > k$ so $d = (x+d) - x < k$. Hence $0 \leq x + 2d = (x+d) + d < 3k$. Thus the remainders x, $x+d$ and $x+2d$ are in A and

$$2(x + d) = (x + 2d) + x,$$

  a contradiction.

□

NT 6. Are there any positive integers m and n satisfying the equation

$$m^3 = 9n^4 + 170n^2 + 289 \ ?$$

Solution. We will prove that the answer is no. Note that

$$m^3 = 9n^4 + 170n^2 + 289 = (9n^2 + 17)(n^2 + 17).$$

If n is odd then m is even, therefore $8 \mid m^3$. However,

$$9n^4 + 170n^2 + 289 \equiv 9 + 170 + 289 \equiv 4 \pmod 8,$$

which leads to a contradiction. If n is a multiple of 17 then so is m and hence 289 is a multiple of 17^3, which is absurd. For n even and not multiple of 17, since

$$\gcd(9n^2 + 17, n^2 + 17) \mid 9(n^2 + 17) - (9n^2 + 17) = 2^3 \cdot 17,$$

this gcd must be 1. Therefore $n^2 + 17 = a^3$ for an odd a, so

$$n^2 + 25 = (a + 2)(a^2 - 2a + 4).$$

For $a \equiv 1 \pmod 4$ we have $a + 2 \equiv 3 \pmod 4$, while for $a \equiv 3 \pmod 4$ we have $a^2 - 2a + 4 \equiv 3 \pmod 4$. Thus $(a + 2)(a^2 - 2a + 4)$ has a prime divisor of type $4\ell + 3$. As it divides $n^2 + 25$, it has to divide n and 5, which is absurd.

NT 7. Prove that there doesn't exist any prime p such that every power of p is a palindrome (palindrome is a number that is read the same from the left as it is from the right; in particular, number that ends in one or more zeros cannot be a palindrome).

Solution. Note that by criterion for divisibility by 11 and the definition of a palindrome we have that every palindrome that has even number of digits is divisible by 11.
Since $11^5 = 161051$ is not a palindrome and since 11 cannot divide p^k for any prime other than 11 we are now left to prove that no prime whose all powers have odd number of digits exists.
Assume the contrary. It means that the difference between the numbers of digits of p^m and p^{m+1} is even number. We will prove that for every natural m, the difference is the same even number.
If we assume not, that means that the difference for some m_1 has at least 2 digits more than the difference for some m_2. We will prove that this is impossible.
Let $p^{m_1} = 10^{t_1} \cdot a_1$, $p^{m_2} = 10^{t_2} \cdot a_2$ and $p = 10^h \cdot z$, where

$$1 < a_1, a_2, z < 10$$

This implies that

$$1 < a_1 \cdot z, a_2 \cdot z < 100,$$

which further implies that multiplying these powers of p by p can increase their number of digits by either h or $h + 1$.
This is a contradiction. Call the difference between numbers of digits of consecutive powers d. Number p clearly cannot be equal to 10^d for $d \geq 1$ because 10 is divisible by two primes, but for $d = 0$, we would have that 1 is a prime which is not true.

Case 1. $p > 10^d$. Let $p = 10^d \cdot a$, for some real number a greater than 1. (1)
From the definition of d we also see that a is smaller than 10. (2)
From (1) we see that powering a gives us arbitrarily large numbers and from (2) we conclude that there is some natural power of a, call it b, greater than 1, such that

$$10 < a^b < 100$$

It is clear that p^b has exactly $(b-1)d + 1$ digits more than p has, which is an odd number, but sum of even numbers is even.

Case 2. $p < 10^d$. Let $p = \frac{10^d}{a}$, for some real number a greater than 1. (1)
From the definition of d we also see that a is smaller than 10. (2)
From (1) we see that powering a gives us arbitrarily large numbers and from (2) we conclude that there is some natural power of a, call it b, greater than 1, such that

$$10 < a^b < 100$$

It is clear that p^b has exactly $(b-1)d - 1$ digits more than p has, which is an odd number, but sum of even numbers is even.
We have now arrived at the desired contradiction for both cases and have thus finished the proof.
Alternative solution. Note that the sequence $\{p^n\}$ is periodic $\pmod{10}$. Let the period be d. Also, let $p^d = g$.

Since all powers of p are palindromes, all powers of g are as well. Since $\{g^n\}$ is constant $\pmod{10}$, the leftmost digit of each power of g is equal to some f.
We will prove that the difference between numbers of digits of g^m and g^{m+1} is equal to some r for every natural number m.
This is true due to size reasons. Namely, to add exactly k digits, and yet to have the same leftmost digit, we need to multiply the number by at least $5 \cdot 10^{k-1}$ (if $k = 0$ then it's 1) and by at most $2 \cdot 10^k$ (values depend on the leftmost digit, it can easily be seen that leftmost digit being 1 yields the extremal values). Notice that

$$2 \cdot 10^k < 5 \cdot 10^{k+1-1}$$

Since this inequality has clearly shown that the interval of multipliers which add exactly k digits and leave the leftmost digit the same is disjunct from the same kind of interval for $k + 1$ digits, which implies that no number can belong to both intervals, we have successfully proven the claim.
Clearly, g cannot be equal to 10^r for $r \geq 1$ because a palindrome cannot be divisible by 10, but for $r = 0$ we again cannot have the equality because 1 is not a natural power of a prime.

Case 1. $g > 10^r$. Let $g = 10^r \cdot a$, where a is a real number, $10 > a > 1$. (1) Here, a is less than 10 because if it was not, multiplying by g would add at least $r + 1$ digits, which is impossible.
From (1) we see that powering a gives us arbitrarily large numbers and that there is some natural power of a, call it b, greater than 1, such that

$$10 < a^b < 100$$

Pick smallest such b.
Now we easily see that the difference between numbers of digits of numbers g^{b-1} and g^b is exactly $r+1$.

Case 2. $g < 10^r$. Let $g = \frac{10^r}{a}$, where a is a real number, $10 > a > 1$. (2) Here, a is less than 10 because if it was not, multiplying by g would add at most $r - 1$ digits, which is impossible.
From (2) we see that powering a gives us arbitrarily large numbers and that there is some natural power of a, call it b, greater than 1, such that

$$10 < a^b < 100$$

Pick smallest such b.
Now we easily see that the difference between numbers of digits of numbers g^{b-1} and g^b is exactly $r - 1$.
We have arrived at the desired contradiction for both cases and have thus finished the proof.

Comment. In both solutions, after introducing a, there are multiple ways to finish the problem. In particular, solution 1 and solution 2 could be finished in the same way, but distinct finishes were purposely offered. Third, maybe even the most intuitive finishing argument, could be using non-exact size arguments; namely, just the fact that powers of a grow arbitrarily large is enough to reach the contradiction.

Alternative problem. Find all positive integers n such that every power of n is a palindrome (palindrome is a number that is read the same from the left as it is from the right; in particular, number that ends in one or more zeros cannot be a palindrome).
Suggested difficulty for this problem is hard. Noting why solution 2 doesn't work for $n = 1$ (because

g now could be equal to 1) and saying that $n = 1$ actually works are all necessary modifications to solution 2 to make it work for the alternative problem as well.

NT 8. Find all pairs (p, q) of prime numbers such that

$$1 + \frac{p^q - q^p}{p + q}$$

is a prime number.

Solution. It is clear that $p \neq q$. We set

$$1 + \frac{p^q - q^p}{p + q} = r$$

and we have that

$$p^q - q^p = (r-1)(p+q). \tag{1}$$

From Fermat's Little Theorem we have

$$p^q - q^p \equiv -q \pmod{p}.$$

Since we also have that

$$(r-1)(p+q) \equiv -rq - q \pmod{p},$$

from (1) we get that

$$rq \equiv 0 \pmod{p} \Rightarrow p \mid qr,$$

hence $p \mid r$, which means that $p = r$. Therefore, (1) takes the form

$$p^q - q^p = (p-1)(p+q). \tag{2}$$

We will prove that $p = 2$. Indeed, if p is odd, then from Fermat's Little Theorem we have

$$p^q - q^p \equiv p \pmod{q}$$

and since

$$(p-1)(p+q) \equiv p(p-1) \pmod{q},$$

we have

$$p(p-2) \equiv 0 \pmod{q} \Rightarrow q \mid p(p-2) \Rightarrow q \mid p-2 \Rightarrow q \leq p-2 < p.$$

Now, from (2) we have

$$p^q - q^p \equiv 0 \pmod{p-1} \Rightarrow 1 - q^p \equiv 0 \pmod{p-1} \Rightarrow q^p \equiv 1 \pmod{p-1}.$$

Clearly $\gcd(q, p-1) = 1$ and if we set $k = ord_{p-1}(q)$, it is well-known that $k \mid p$ and $k < p$, therefore $k = 1$. It follows that

$$q \equiv 1 \pmod{p-1} \Rightarrow p-1 | q-1 \Rightarrow p-1 \leq q-1 \Rightarrow p \leq q$$

a contradiction.

Therefore, $p = 2$ and (2) transforms to

$$2^q = q^2 + q + 2.$$

We can easily check by induction that for every positive integer $n \geq 6$ we have $2^n > n^2 + n + 2$. This means that $q \leq 5$ and the only solution is for $q = 5$. Hence the only pair which satisfy the condition is $(p, q) = (2, 5)$.

□

Comment by the PSC. From the problem condition, we get that p^q should be bigger than q^p, which gives

$$q \ln p > p \ln q \iff \frac{\ln p}{p} > \frac{\ln q}{q}.$$

The function $\dfrac{\ln x}{x}$ is decreasing for $x > e$, thus if p and q are odd primes, we obtain $q > p$.