41st **Balkan Mathematical Olympiad**
27th April – 2nd May 2024
Varna, Bulgaria

Language: English

Monday, April 29, 2024

Problem 1. Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E and F respectively. A line through D meets AC and AB at G and H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

Problem 2. Let $n \geq k \geq 3$ be integers. Show that for every integer sequence $1 \leq a_1 < a_2 < \ldots < a_k \leq n$ one can choose non-negative integers $b_1, b_2, \ldots, b_k$, satisfying the following conditions:

(i) $0 \leq b_i \leq n$ for each $1 \leq i \leq k$,

(ii) all the positive b_i are distinct,

(iii) the sums $a_i + b_i$, $1 \leq i \leq k$, form a permutation of the first k terms of a non-constant arithmetic progression.

Problem 3. Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Problem 4. Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $P(x)$ with non-negative real coefficients such that $P(0) = 0$ which satisfy the equality

$$f(f(x) + P(y)) = f(x - y) + 2y$$

for all real numbers $x > y > 0$.

Time is 4 hours and 30 minutes
Each problem is worth 10 points

Problems with Solutions

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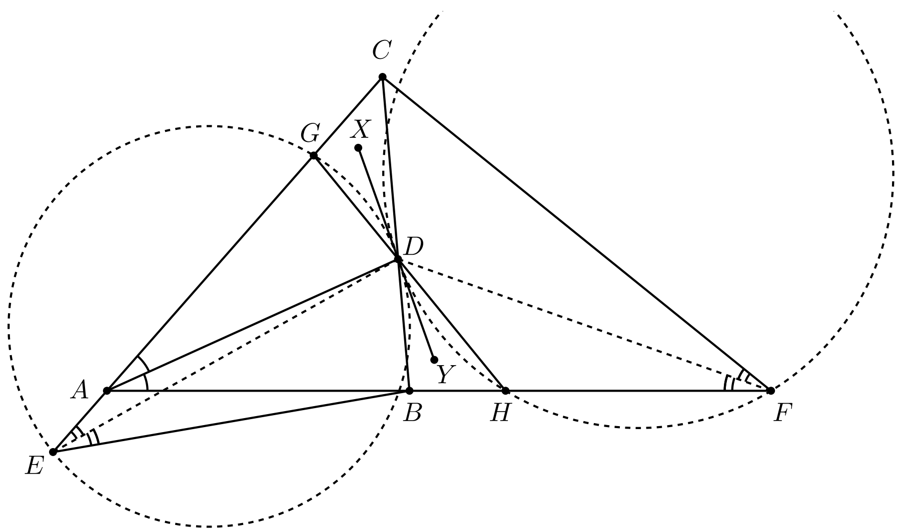
Problem 1. Let ABC be an acute-angled triangle with $AC > AB$ and let D be the foot of the A-angle bisector on BC. The reflections of lines AB and AC in line BC meet AC and AB at points E and F respectively. A line through D meets AC and AB at G and H respectively such that G lies strictly between A and C while H lies strictly between B and F. Prove that the circumcircles of $\triangle EDG$ and $\triangle FDH$ are tangent to each other.

Solution 1. Let X and Y lie on the tangent to the circumcircle of $\triangle EDG$ on the opposite side to D as shown in the figure below. Regarding diagram dependency, the acute condition with $AC > AB$ ensures E lies on extension of CA beyond A, and F lies on extension of AB beyond B. The condition on ℓ means the points lie in the orders E, A, G, C and A, B, H, F.



Using the alternate segment theorem, the condition that $\odot EDG$ and $\odot FDH$ are tangent at D can be rewritten as

$$\measuredangle HFD = \measuredangle YDH.$$

But using the same theorem, we get $\measuredangle YDH = \measuredangle XDG = \measuredangle DEG$. So we can remove G, H from the figure, and it is sufficient to prove that $\measuredangle DEA = \measuredangle DFB$.

The reflection property means that AD and BD are external angle bisectors in $\triangle EAB$ and hence D

is the E-*excentre* of this triangle. Thus DE (internally) bisects $\sphericalangle BEA$, giving

$$\sphericalangle DEA = \sphericalangle DEB.$$

Now observe that the pairs of lines (BE, CE) and (BF, CF) are reflections in BC thus E, F are reflections in BC. Also D is its own reflection in BC. Hence $\sphericalangle DEB = \sphericalangle DFB$ and so

$$\sphericalangle DEA = \sphericalangle DEB = \sphericalangle DFB,$$

as required.

Problem 2. Let $n \geq k \geq 3$ be integers. Show that for every integer sequence $1 \leq a_1 < a_2 < \ldots < a_k \leq n$ one can choose non-negative integers $b_1, b_2, \ldots, b_k$, satisfying the following conditions:

(i) $0 \leq b_i \leq n$ for each $1 \leq i \leq k$,

(ii) all the positive b_i are distinct,

(iii) the sums $a_i + b_i$, $1 \leq i \leq k$, form a permutation of the first k terms of a non-constant arithmetic progression.

Solution 1. Let the resulting progression be $Ans := \{a_k - (k-1), a_k - (k-2), \ldots, a_k\}$ and a_t be the largest number not belonging to Ans. Clearly the set $Ans \setminus \{a_1, a_2, \ldots, a_k\}$ has cardinality t; let its members be $c_1 > c_2 > \cdots > c_t$. Define $b_j := c_j - a_j$ for $1 \leq j \leq t$ or zero otherwise. Since $\{c_j\}$ is decreasing and $\{a_j\}$ is increasing, all b_j are distinct and clearly $b_1 < n$. After we add b_j to a_j we get a permutation of Ans as desired.

Solution 2. Let the resulting progression be $Ans := \{a_k - (k-1), a_k - (k-2), \ldots, a_k\}$.

We proceed with the following reduction. Let δ be the smallest b we used before (in the beginning it is n). While $a_1 \notin Ans$ we map a_1 to the largest element q of $Ans \setminus \{a_1, a_2, \ldots, a_k\}$ and put $\delta_{new} := b_1 := q - a_1$. Now we rearrange the sequence of a-s. We do not touch $Ans \cap \{a_1, a_2, \ldots, a_k\}$ so every b is defined at most once (in the end undefined b-s become zeros). Also $b < \delta$ and δ decreases at each step, because q decreases and a_1 grows, and hence all nonzero b-s are distinct.

Problem 3. Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Solution 1. Obviously we have $a > b$. Let $a = bq + r$, where $0 \leq r < b$. Then

$$3^a \equiv 3^{bq+r} \equiv (-2)^q . 3^r \equiv -2 \pmod{3^b + 2}$$

So 3^b+2 divides $A=(-2)^q.3^r+2$ and it follows that

$$|(-2)^q.3^r+2| \geq 3^b+2 \text{ or } (-2)^q.3^r+2=0.$$

We make case distinction:

1. $(-2)^q.3^r+2=0$. Then $q=1$ and $r=0$ or $a=b$, a contradiction.
2. q is even. Then
$$A=2^q.3^r+2=(3^b+2).k.$$
Consider both sides of the last equation modulo 3^r. Since $b>r$:
$$2 \equiv 2^q.3^r+2=(3^b+2)k \equiv 2k \pmod{3^r},$$
so it follows that $3^r|k-1$. If $k=1$ then $2^q.3^r=3^b$, a contradiction. So $k \geq 3^r+1$, and therefore:
$$A=2^q.3^r+2=(3^b+2)k \geq (3^b+2)(3^r+1) > 3^b.3^r+2$$
It follows that
$$2^q.3^r > 3^b.3^r, \text{ i.e. } 2^q > 3^b, \text{ which implies } 3^{b^2} < 2^{bq} < 3^{bq} \leq 3^{bq+r} = 3^a.$$
Consequently $a > b^2$.
3. If q is odd. Then
$$2^q.3^r-2=(3^b+2)k.$$
Considering both sides of the last equation modulo 3^r, and since $b>r$, we get: $k+1$ is divisible by 3^r and therefore $k \geq 3^r-1$. Thus $r>0$ because $k>0$, and:
$$\begin{aligned} 2^q.3^r-2=(3^b+2)k \geq (3^b+2)(3^r-1), &\text{ and therefore} \\ 2^q.3^r > (3^b+2)(3^r-1) > 3^b(3^r-1) > 3^b\frac{3^r}{2}, &\text{ which shows} \\ &2^{q+1} > 3^b. \end{aligned}$$
But for $q>1$ we have $2^{q+1} < 3^q$, which combined with the above inequality, implies that $3^{b^2} < 2^{(q+1)b} < 3^{qb} \leq 3^a$, q.e.d. Finally, If $q=1$ then $2^q.3^r-2=(3^b+2)k$ and consequently $2.3^r-2 \geq 3^b+2 \geq 3^{r+1}+2 > 2.3^r-2$, a contradiction.

Solution 2. $D=a-b$, and we shall show $D>b^2-b$. We have $3^b+2|3^a+2$, so $3^b+2|3^D-1$. Let $D=bq+r$ where $r<b$. First suppose that $r \neq 0$. We have

$$1 \equiv 3^D \equiv 3^{bq+r} \equiv (-2)^{q+1}3^{r-b} \pmod{3^b+2} \Longrightarrow 3^{b-r} \equiv (-2)^{q+1} \pmod{3^b+2}$$

Therefore

$$3^b + 2 \le |(-2)^{q+1} - 3^{b-r}| \le 2^{q+1} + 3^{b-r} \le 2^{q+1} + 3^{b-1}$$

Hence

$$2 \times 3^{b-1} + 2 \le 2^{q+1} \Longrightarrow 3^{b-1} < 2^q \Longrightarrow \frac{\log 3}{\log 2}(b-1) < q$$

Which yields $D = bq + r > bq > \frac{\log 3}{\log 2} b(b-1) \ge b^2 - b$ as desired. Now for the case $r = 0$, $(-2)^q \equiv 1 \pmod{3^b + 2}$ and so

$$3^b + 2 \le |(-2)^q - 1| \le 2^q + 1 \Longrightarrow 3^{b-1} < 3^b < 2^q \Longrightarrow \frac{\log 3}{\log 2}(b-1) < q$$

and analogous to the previous case, $D = bq + r = bq > \frac{\log 3}{\log 2} b(b-1) \ge b^2 - b$.

Problem 4. Let $\mathbb{R}^+ = (0, \infty)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $P(x)$ with non-negative real coefficients such that $P(0) = 0$ which satisfy the equality

$$f(f(x) + P(y)) = f(x - y) + 2y$$

for all real numbers $x > y > 0$.

Solution 1. Assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ and the polynomial P with non-negative coefficients and $P(0) = 0$ satisfy the conditions of the problem. For positive reals with $x > y$, we shall write $Q(x, y)$ for the relation:

$$f(f(x) + P(y)) = f(x - y) + 2y.$$

1. Step 1. $f(x) \ge x$. Assume that this is not true. Since $P(0) = 0$ and P is with non-negative coefficients, $P(x) + x$ is surjective on positive reals. If $f(x) < x$ for some positive real x, then setting y such that $y + P(y) = x - f(x)$ (where obviously $y < x$), we shall get $f(x) + P(y) = x - y$ and by $Q(x, y)$, $f(f(x) + P(y)) = f(x - y) + 2y$, we get $2y = 0$, a contradiction.

2. Step 2. $P(x) = cx$ for some non-negative real c. We will show $\deg P \le 1$ and together with $P(0) = 0$ the result will follow. Assume the contrary. Hence there exists a positive l such that $P(x) \ge 2x$ for all $x \ge l$. By Step 1 we get

$$\forall x > y \ge l : f(x - y) + 2y = f(f(x) + P(y)) \ge f(x) + P(y) \ge f(x) + 2y$$

   and therefore $f(x - y) \ge f(x)$. We get $f(y) \ge f(2y) \ge \cdots \ge f(ny) \ge ny$ for all positive integers n, which is a contradiction.

3. Step 3. If $c \ne 0$, then $f(f(x) + 2z + c^2) = f(x + 1) + 2(z - 1) + 2c$ for $z > 1$. Indeed by $Q(f(x + z) + cz, c)$, we get

$$f(f(f(x + z) + cz) + c^2) = f(f(x + z) + cz - c) + 2c = f(x + 1) + 2(z - 1) + 2c.$$

On the other hand by $Q(x+z,z)$, we have:

$$f(x)+2z+c^2 = f\left(f\left(x+z\right)+P\left(z\right)\right)+c^2 = f\left(f\left(x+z\right)+cz\right)+c^2.$$

Substituting in the LHS of $Q(f(x+z)+cz,c)$, we get $f(f(x)+2z+c^2) = f(x+1)+2(z-1)+2c$.

4. Step 4. There is x_0, such that $f(x)$ is linear on (x_0,∞). If $c \neq 0$, then by Step 3, fixing $x=1$, we get $f(f(1)+2z+c^2) = f(2)+2(z-1)+2c$ which implies that f is linear for $z > f(1)+2+c^2$. As for the case $c=0$, consider $y,z \in (0,\infty)$. Pick $x > \max(y,z)$, then by $Q(x,x-y)$ and $Q(x,x-z)$ we get:

   $$f(y)+2(x-y) = f(f(x)) = f(z)+2(x-z)$$

   which proves that $f(y)-2y = f(z)-2z$ and there fore f is linear on $(0,\infty)$.

5. Step 5. $P(y)=y$ and $f(x)=x$ on (x_0,∞). By Step 4, let $f(x)=ax+b$ on (x_0,∞). Since f takes only positive values, $a \geq 0$. If $a=0$, then by $Q(x+y,y)$ for $y > x_0$ we get:

   $$2y+f(x) = f(f(x+y)+P(y)) = f(b+cy).$$

   Since the LHS is not constant, we conclude $c \neq 0$, but then for $y > x_0/c$, we get that the RHS equals b which is a contradiction.

   Hence $a>0$. Now for $x > x_0$ and $x > (x_0-b)/a$ large enough by $P(x+y,y)$ we get:

   $$ax+b+2y = f(x)+2y = f(f(x+y)+P(y)) = f(ax+ay+b+cy) = a(ax+ay+b+cy)+b.$$

   Comparing the coefficients before x, we see $a^2=a$ and since $a \neq 0$, $a=1$. Now $2b=b$ and thus $b=0$. Finally, equalising the coefficients before y, we conclude $2=1+c$ and therefore $c=1$.

Now we know that $f(x)=x$ on (x_0,∞) and $P(y)=y$. Let $y > x_0$. Then by $Q(x+y,x)$ we conclude:

$$f(x)+2y = f(f(x+y)+P(y)) = f(x+y+y) = x+2y.$$

Therefore $f(x)=x$ for every x. Conversely, it is straightforward that $f(x)=x$ and $P(y)=y$ do indeed satisfy the conditions of the problem.

Solution 2. Assume that the function $f:\mathbb{R}^+ \to \mathbb{R}^+$ and the polynomial with non-negative coefficients $P(y)=yP_1(y)$ satisfy the given equation. Fix $x=x_0>0$ and note that:

$$f(f(x_0+y)+P(y)) = f(x_0+y-y)+2y = f(x_0)+2y.$$

Assume that $g=0$. Then $f(f(x+y)) = f(x)+2y$ for $x,y>0$. Let $x>0$ and $z>0$. Pick $y>0$. Then:

$$2y+f(x+z) = f(f(x+y+z)) = f(f(x+z+y)) = f(x)+2(z+y).$$

Therefore $f(x+z) = f(x)+2z$ for any $x > 0$ and $z > 0$. Setting $c = f(1)$, we see that $f(z+1) = c+2z$ for all positive z. Therefore if $x, y > 1$ we have that $f(x+y) = c+2(x+y-1) > 1$. This shows that:

$$f(f(x+y)) = c + 2(f(x+y) - 1) = 3c + 4(x+y) - 4.$$

On the other hand $f(x) + 2y = c + 2x + 2y$. Therefore the equality $f(f(x+y)) = f(x) + 2y$ is not universally satisfied.

From now on, we assume that $g \neq 0$. Therefore P is strictly increasing with $P(0) = 0$, $\lim_{y\to\infty} P(y) = \infty$, i.e. g is bijective on $[0, \infty)$ and $P(0) = 0$.

Let $x > 0, y > 0$ and set $u = f(x+y)$, $v = P(y)$. From above, we have $u > 0$ and $v > 0$. Therefore:

$$f(f(u+v) + P(v)) = f(u) + 2v = f(f(x+y)) + 2P(y).$$

On the other hand $f(u+v) = f(f(x+y) + P(y)) = f(x) + 2y$. Therefore we obtain that:

$$f(f(x) + 2y + P(P(y))) = f(f(x+y)) + 2P(y).$$

Since g is bijective from $(0, \infty)$ to $(0, \infty)$ for any $z > 0$ there is t such that $P(t) = z$. Applying this observation to $z = P(P(y)) + 2y$ and setting $x' = x + t$, we obtain that:

$$f(f(x+t+y))+2P(y) = f(f(x'+y))+2P(y) = f(f(x')+P(P(y))+2y) = f(f(x+t)+P(t)) = f(x)+2t.$$

Thus if we denote $h(y) = P(P(y)) + 2y$, then $t = P^{(-1)}(h(y))$ and the above equality can be rewritten as:

$$f(f(x + P^{(-1)}(h(y)) + y)) = f(x) + 2P^{(-1)}(h(y)) - 2P(y) = f(x) + 2P^{(-1)}(h(y)) + 2y - 2y - 2P(y).$$

Let $s(y) = P^{(-1)}(h(y))+y$ and note that since h is continuous and monotone increasing, g is continuous and monotone increasing, then so are $P^{(-1)}$ and consequently $P^{(-1)} \circ h$ and s. It is also clear, that $\lim_{y\to 0} s(y) = 0$ and $\lim_{y\to\infty} s(y) = \infty$. Therefore s is continuously bijective from $[0, \infty)$ to $[0, \infty)$ with $s(0) = 0$.

Thus we have:

$$f(f(x + s(y))) = f(x) + 2s(y) - 2y - 2P(y)$$

and using that s is invertible, we obtain:

$$f(f(x+y)) = f(x) + 2y - 2s^{(-1)}(y) - 2P(s^{(-1)}(y)).$$

Now fix x_0, then for any $x > x_0$ and any $y > 0$ we have:

$$\begin{aligned} f(x) + 2y - 2s^{(-1)}(y) - 2P(s^{(-1)}(y)) &= f(f(x+y)) = f(f(x_0 + x + y - x_0)) \\ &= f(x_0) + 2(x + y - x_0) - 2s^{(-1)}(x + y - x_0) - 2P(s^{(-1)}(x + y - x_0)). \end{aligned}$$

Setting $y = x_0$, we get:

$$f(x) + 2x_0 - 2s^{(-1)}(x_0) - 2P(s^{(-1)}(x_0)) = f(x_0) + 2x - 2s^{(-1)}(x) - 2P(s^{(-1)}(x)).$$

Since this equality is valid for any $x > x_0$ we actually have that:

$$f(x) - 2x + 2s^{(-1)}(x) + 2P(s^{(-1)}(x)) = c \text{ for some fixed constant } c \in \mathbb{R} \text{ and all } x \in \mathbb{R}^+.$$

Let $\phi(x) = -x + 2s^{(-1)}(x) + 2P(s^{(-1)}(x))$. Then $f(x) = x - \phi(x) + c$ and since ϕ is a sum of continuous functions that are continuous at 0. Therefore f is continuous and can be extended to a continuous function on $[0, \infty)$. Back in the original equation we fix $x > 0$ and let y tend to 0. Using the continuity of f and g on $[0, \infty)$ and $P(0) = 0$ we obtain:

$$f(f(x)) = \lim_{y \to 0+} f(f(x) + P(y)) = \lim_{y \to 0+} (f(x - y) + P(y)) = f(x) + P(0) = f(x).$$

Finally, fixing $x = 1$ and varying $y > 0$, we obtain:

$$f(f(1 + y) + P(y)) = f(1) + 2y.$$

It follows that f takes every value on $(f(1), \infty)$. Therefore for any $y \in (f(1), \infty)$ there is z such that $f(z) = y$. Using that $f(f(z)) = f(z)$ we conclude that $f(y) = y$ for all $y \in (f(1), \infty)$.

Now fix x and take $y > f(1)$. Hence

$$f(x) + 2y = f(f(x + y) + P(y)) = f(x + y + P(y)) = x + y + P(y).$$

We conclude $f(x) - x = P(y) - y$ for every x an $y > f(1)$. In particular $f(x_1) - x_1 = f(x_2) - x_2$ for all $x_1, x_2 \in (0, \infty)$ and since $f(x) = x$ for $x \in (f(1), \infty)$, we get $f(x) = x$ on $(0, \infty)$.

Finally, $x + 2y = f(x) + 2y = f(f(x + y) + P(y)) = f(x + y) + P(y) = x + y + P(y)$, which shows that $P(y) = y$ for every $y \in (0, \infty)$.

It is also straightforward to check that $f(x) = x$ and $P(y) = y$ satisfy the equality:

$$f(f(x + y) + P(y)) = f(x + 2y) = x + 2y = f(x) + 2y.$$