Shortlisted problems for the Junior Balkan Mathematics Olympiad

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Contributing Countries:

The Organizing Committee and the Problem Selection Committee of the JBMO 2016 thank the following countries for contributing with problem proposals:

Albania, Azerbaijan, Bosnia and Herzegovina, Bulgaria, Cyprus, FYROM, Greece, Moldova, Montenegro, Serbia

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Chapter 1

Algebra

A1. Let a, b, c be positive real numbers such that $abc = 8$. Prove that

$$\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 6.$$

Solution. We have $ab+4 = \frac{8}{c}+4 = \frac{4(c+2)}{c}$ and similarly $bc+4 = \frac{4(a+2)}{a}$ and $ca+4 = \frac{4(b+2)}{b}$. It follows that

$$(ab+4)(bc+4)(ca+4) = \frac{64}{abc}(a+2)(b+2)(c+2) = 8(a+2)(b+2)(c+2),$$

so that

$$\frac{(ab+4)(bc+4)(ca+4)}{(a+2)(b+2)(c+2)} = 8.$$

Applying AM-GM, we conclude:

$$\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 3 \cdot \sqrt[3]{\frac{(ab+4)(bc+4)(ca+4)}{(a+2)(b+2)(c+2)}} = 6.$$

Alternatively, we can write LHS as

$$\frac{bc(ab+4)}{2(bc+4)} + \frac{ac(bc+4)}{2(ac+4)} + \frac{ab(ca+4)}{2(ab+4)}$$

and then apply AM-GM.

□

A2. Given positive real numbers a, b, c, prove that

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(a+c)^2+4abc}+\frac{8}{(b+c)^2+4abc}+a^2+b^2+c^2 \geq \frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}.$$

Solution. Since $2ab \leq a^2+b^2$, it follows that $(a+b)^2 \leq 2(a^2+b^2)$ and $4abc \leq 2c(a^2+b^2)$, for any positive reals a, b, c. Adding these inequalities, we find

$$(a+b)^2+4abc \leq 2(a^2+b^2)(c+1),$$

so that

$$\frac{8}{(a+b)^2+4abc} \geq \frac{4}{(a^2+b^2)(c+1)}.$$

Using the AM-GM inequality, we have

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2} \geq 2\sqrt{\frac{2}{c+1}}=\frac{4}{\sqrt{2(c+1)}},$$

respectively

$$\frac{c+3}{8}=\frac{(c+1)+2}{8} \geq \frac{\sqrt{2(c+1)}}{4}.$$

We conclude that

$$\frac{4}{(a^2+b^2)(c+1)}+\frac{a^2+b^2}{2} \geq \frac{8}{c+3},$$

and finally

$$\frac{8}{(a+b)^2+4abc}+\frac{8}{(a+c)^2+4abc}+\frac{8}{(b+c)^2+4abc}+a^2+b^2+c^2 \geq \frac{8}{a+3}+\frac{8}{b+3}+\frac{8}{c+3}.$$

□

A3. Determine the number of pairs of integers (m, n) such that

$$\sqrt{n+\sqrt{2016}}+\sqrt{m-\sqrt{2016}} \in \mathbb{Q}.$$

Solution. Let $r=\sqrt{n+\sqrt{2016}}+\sqrt{m-\sqrt{2016}}$. Then

$$n+m+2\sqrt{n+\sqrt{2016}} \cdot \sqrt{m-\sqrt{2016}}=r^2,$$

and

$$(m-n)\sqrt{2106}=\frac{1}{4}(r^2-m-n)^2-mn+2016 \in \mathbb{Q}.$$

Since $\sqrt{2016} \notin \mathbb{Q}$, it follows that $m=n$. Then

$$\sqrt{n^2-2016}=\frac{1}{2}(r^2-2n) \in \mathbb{Q}.$$

Hence, there is some nonnegative integer p such that $n^2-2016=p^2$ and (1) becomes $2n+2p=r^2$.

It follows that $2(n+p)=r^2$ is the square of a rational and also an integer, hence a perfect square. On the other hand, $2016=(n-p)(n+p)$ and $n+p$ is a divisor of 2016, larger than $\sqrt{2016}$. Since $n+p$ is even, so is also $n-p$, and $r^2=2(n+p)$ is a divisor of $2016=2^5 \cdot 3^2 \cdot 7$, larger than $2\sqrt{2016}>88$. The only possibility is $r^2=2^4 \cdot 3^2=12^2$. Hence, $n+p=72$ and $n-p=28$, and we conclude that $n=m=50$. Thus, there is only one such pair.

□

A4. If x, y, z are non-negative real numbers such that $x^2+y^2+z^2=x+y+z$, then show that:

$$\frac{x+1}{\sqrt{x^5+x+1}}+\frac{y+1}{\sqrt{y^5+y+1}}+\frac{z+1}{\sqrt{z^5+z+1}} \geq 3.$$

When does the equality hold?

Solution. First we factor x^5+x+1 as follows:

$$\begin{aligned} x^5+x+1 &= x^5-x^2+x^2+x+1 = x^2(x^3-1)+x^2+x+1 = x^2(x-1)(x^2+x+1)+x^2+x+1 \\ &= (x^2+x+1)(x^2(x-1)+1) = (x^2+x+1)(x^3-x^2+1) \end{aligned}$$

Using the $AM-GM$ inequality, we have

$$\sqrt{x^5+x+1} = \sqrt{(x^2+x+1)(x^3-x^2+1)} \leq \frac{x^2+x+1+x^3-x^2+1}{2} = \frac{x^3+x+2}{2}$$

and since

$$x^3+x+2 = x^3+1+x+1 = (x+1)(x^2-x+1)+x+1 = (x+1)(x^2-x+1+1) = (x+1)(x^2-x+2),$$

then

$$\sqrt{x^5+x+1} \leq \frac{(x+1)(x^2-x+2)}{2}$$

Using $x^2-x+2=\left(x-\frac{1}{2}\right)^2+\frac{7}{4}>0$, we obtain $\frac{x+1}{\sqrt{x^5+x+1}} \geq \frac{2}{x^2-x+2}$ Applying the Cauchy-Schwarz inequality and the given condition, we get

$$\sum_{cyc} \frac{x+1}{\sqrt{x^5+x+1}} \geq \sum_{cyc} \frac{2}{x^2-x+2} \geq \frac{18}{\sum_{cyc}(x^2-x+2)} = \frac{18}{6} = 3$$

which is the desired result.
For the equality both conditions: $x^2-x+2=y^2-y+2=z^2-z+2$(equality in CBS) and $x^3-x^2+1=x^2+x+1$(equality in AM-GM) have to be satisfied.
By using the given condition it follows that $x^2-x+2+y^2-y+2+z^2-z+2=6$, hence $3(x^2-x+2)=6$, implying $x=0$ or $x=1$. Of these, only $x=0$ satisfies the second condition. We conclude that equality can only hold for $x=y=z=0$.
It is an immediate check that indeed for these values equality holds.

□

Alternative solution
Let us present an heuristic argument to reach the key inequality $\frac{x+1}{\sqrt{x^5+x+1}} \geq \frac{2}{x^2-x+2}$.
In order to exploit the condition $x^2+y^2+z^2=x+y+z$ when applying CBS in Engel form, we are looking for $\alpha, \beta, \gamma > 0$ such that

$$\frac{x+1}{\sqrt{x^5+x+1}} \geq \frac{\gamma}{\alpha(x^2-x)+\beta}.$$

After squaring and cancelling the denominators, we get

$$(x+1)^2(\alpha(x^2-x)+\beta)^2 \geq \gamma^2(x^5+x+1)$$

for all $x \geq 0$, and, after some manipulations, we reach to $f(x) \geq 0$ for all $x \geq 0$, where

$$f(x) = \alpha^2 x^6 - \gamma^2 x^5 + (2\alpha\beta - 2\alpha^2)x^4 + 2\alpha\beta x^3 + (\alpha - \beta)^2 x^2 + (2\beta^2 - 2\alpha\beta - \gamma^2)x + \beta^2 - \gamma^2.$$

As we are expecting the equality to hold for $x = 0$, we naturally impose the condition that f has 0 as a double root. This implies $\beta^2 - \gamma^2 = 0$ and $2\beta^2 - 2\alpha\beta - \gamma^2 = 0$, that is, $\beta = \gamma$ and $\gamma = 2\alpha$.
Thus the inequality $f(x) \geq 0$ becomes

$$\alpha^2 x^6 - 4\alpha^2 x^5 + 2\alpha^2 x^4 + 4\alpha^2 x^3 + \alpha^2 x^2 \geq 0, \forall x \geq 0,$$

that is,

$$\alpha^2 x^2 (x^2 - 2x - 1)^2 \geq 0 \ \forall x \geq 0,$$

which is obviously true.
Therefore, the inequality $\frac{x+1}{\sqrt{x^5+x+1}} \geq \frac{2}{x^2-x+2}$ holds for all $x \geq 0$ and now we can continue as in the first solution.

A5. Let x, y, z be positive real numbers such that $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
a) Prove the inequality

$$x+y+z \geq \sqrt{\frac{xy+1}{2}}+\sqrt{\frac{yz+1}{2}}+\sqrt{\frac{zx+1}{2}}.$$

b) (Added by the problem selecting committee) When does the equality hold?

Solution.
a) We rewrite the inequality as

$$\left(\sqrt{xy+1}+\sqrt{yz+1}+\sqrt{zx+1}\right)^2 \leq 2\cdot(x+y+z)^2 \qquad (1)$$

and note that, from CBS,

$$\text{LHS } \leq (\frac{xy+1}{x}+\frac{yz+1}{y}+\frac{zx+1}{z})(x+y+z).$$

But
$$\frac{xy+1}{x}+\frac{yz+1}{y}+\frac{zx+1}{z}=x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2(x+y+z),$$
which proves (1).
b) The equality occurs when we have equality in CBS, i.e. when

$$\frac{xy+1}{x^2}=\frac{yz+1}{y^2}=\frac{zx+1}{z^2}\left(=\frac{xy+yz+zx+3}{x^2+y^2+z^2}\right).$$

Since we can also write

$$\left(\sqrt{xy+1}+\sqrt{yz+1}+\sqrt{zx+1}\right)^2 \leq (\frac{xy+1}{y}+\frac{yz+1}{z}+\frac{zx+1}{x})(y+z+x)=2(x+y+z)^2,$$

the equality implies also

$$\frac{xy+1}{y^2}=\frac{yz+1}{z^2}=\frac{zx+1}{x^2}\left(=\frac{xy+yz+zx+3}{x^2+y^2+z^2}\right).$$

But then $x=y=z$, and since $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$, we conclude that $x=\frac{1}{x}=1=y=z$.
□

Alternative solution to b): The equality condition

$$\frac{xy+1}{x^2}=\frac{yz+1}{y^2}=\frac{zx+1}{z^2}$$

can be rewritten as

$$\frac{y+\frac{1}{x}}{x}=\frac{z+\frac{1}{y}}{y}=\frac{x+\frac{1}{z}}{z}=\frac{x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{x+y+z}=2$$

and thus we obtain the system:

$$\begin{cases} y = 2x - \dfrac{1}{x} \\ z = 2y - \dfrac{1}{y} \\ x = 2z - \dfrac{1}{z} \end{cases}.$$

We show that $x = y = z$. Indeed, if for example $x > y$, then $2x - \frac{1}{x} > 2y - \frac{1}{y}$, that is, $y > z$ and $z = 2y - \frac{1}{y} > 2z - \frac{1}{z} = x$, and we obtain the contradiction $x > y > z > x$. Similarly, if $x < y$, we obtain $x < y < z < x$.

Hence, the numbers are equal, and as above we get $x = y = z = 1$.

Chapter 2

Combinatorics

C1. Let S_n be the sum of reciprocal values of non-zero digits of all positive integers up to (and including) n. For instance, $S_{13} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{3}$. Find the least positive integer k making the number $k! \cdot S_{2016}$ an integer.

Solution.
We will first calculate S_{999}, then $S_{1999} - S_{999}$, and then $S_{2016} - S_{1999}$.

Writing the integers from 1 to 999 as 001 to 999, adding eventually also 000(since 0 digits actually do not matter), each digit appears exactly 100 times in each position(as unit, ten, or hundred). Hence

$$S_{999} = 300 \cdot \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}\right)$$

For the numbers in the interval $1000 \to 1999$, compared to $0 \to 999$, there are precisely 1000 more digits 1. We get

$$S_{1999} - S_{999} = 1000 + S_{999} \Longrightarrow S_{1999} = 1000 + 600 \cdot \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{9}\right)$$

Finally, in the interval $2000 \to 2016$, the digit 1 appears twice as unit and seven times as a ten, the digit 2 twice as a unit and 17 times as a thousand, the digits 3, 4, 5, and 6 each appear exactly twice as units, and the digits 7, 8, and 9 each appear exactly once as a unit. Hence

$$S_{2016} - S_{1999} = 9 \cdot 1 + 19 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) + 1 \cdot \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right).$$

In the end, we get

$$\begin{aligned} S_{2016} &= 1609 \cdot 1 + 619 \cdot \frac{1}{2} + 602 \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) + 601 \cdot \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) \\ &= m + \frac{1}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{6}{7} + \frac{1}{8} + \frac{7}{9} = n + \frac{p}{2^3 \cdot 3^2 \cdot 5 \cdot 7}, \end{aligned}$$

where m, n, and p are positive integers, p coprime to $2^3 \cdot 3^2 \cdot 5 \cdot 7$. Then $k! \cdot S_{2016}$ is an integer precisely when $k!$ is a multiple of $2^3 \cdot 3^2 \cdot 5 \cdot 7$. Since $7|k!$, it follows that $k \geq 7$. Also, $7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, implying that the least k satisfying $k! \cdot S_{2016} \in \mathbb{Z}$ is $k = 7$. $\square$

C2. The natural numbers from 1 to 50 are written down on the blackboard. At least how many of them should be deleted, in order that the sum of any two of the remaining numbers is not a prime?

Solution. Notice that if the odd, respectively even, numbers are all deleted, then the sum of any two remaining numbers is even and exceeds 2, so it is certainly not a prime. We prove that 25 is the minimal number of deleted numbers. To this end, we group the positive integers from 1 to 50 in 25 pairs, such that the sum of the numbers within each pair is a prime:

$$(1,2),(3,4),(5,6),(7,10),(8,9),(11,12),(13,16),(14,15),(17,20),$$

$$(18,19),(21,22),(23,24),(25,28),(26,27),(29,30),(31,36),(32,35),$$

$$(33,34),(37,42),(38,41),(39,40),(43,46),(44,45),(47,50),(48,49).$$

Since at least one number from each pair has to be deleted, the minimal number is 25.

□

C3. Consider any four pairwise distinct real numbers and write one of these numbers in each cell of a 5×5 array so that each number occurs exactly once in every 2×2 subarray. The sum over all entries of the array is called the *total sum* of that array. Determine the maximum number of distinct total sums that may be obtained in this way.

Solution. We will prove that the maximum number of total sums is 60.

The proof is based on the following claim.

Claim. Either each row contains exactly two of the numbers, or each column contains exactly two of the numbers.

Proof of the Claim. Indeed, let R be a row containing at least three of the numbers. Then, in row R we can find three of the numbers in consecutive positions, let x, y, z be the numbers in consecutive positions(where $\{x, y, s, z\} = \{a, b, c, d\}$). Due to our hypothesis that in every 2×2 subarray each number is used exactly once, in the row above R(if there is such a row), precisely above the numbers x, y, z will be the numbers z, t, x in this order. And above them will be the numbers x, y, z in this order. The same happens in the rows below R (see at the following figure).

$$\begin{pmatrix} \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \\ \bullet & z & t & x & \bullet \\ \bullet & x & y & z & \bullet \end{pmatrix}$$

Completing all the array, it easily follows that each column contains exactly two of the numbers and our claim has been proven.

Rotating the matrix (if it is necessary), we may assume that each row contains exactly two of the numbers. If we forget the first row and column from the array, we obtain a 4×4 array, that can be divided into four 2×2 subarrays, containing thus each number exactly four times, with a total sum of $4(a+b+c+d)$. It suffices to find how many different ways are there to put the numbers in the first row R_1 and the first column C_1.

Denoting by a_1, b_1, c_1, d_1 the number of appearances of a, b, c, and respectively d in R_1 and C_1, the total sum of the numbers in the entire 5×5 array will be

$$S = 4(a+b+c+d) + a_1 \cdot a + b_1 \cdot b + c_1 \cdot c + d_1 \cdot d.$$

If the first, the third and the fifth row contain the numbers x, y, with x denoting the number at the entry $(1,1)$, then the second and the fourth row will contain only the numbers z, t, with z denoting the number at the entry $(2,1)$. Then $x_1 + y_1 = 7$ and $x_1 \geqslant 3$, $y_1 \geqslant 2$, $z_1 + t_1 = 2$, and $z_1 \geqslant t_1$. Then $\{x_1, y_1\} = \{5, 2\}$ or $\{x_1, y_1\} = \{4, 3\}$, respectively $\{z_1, t_1\} = \{2, 0\}$ or $\{z_1, t_1\} = \{1, 1\}$. Then (a_1, b_1, c_1, d_1) is obtained by permuting one of the following quadruples:

$$(5,2,2,0), (5,2,1,1), (4,3,2,0), (4,3,1,1).$$

There are a total of $\frac{4!}{2!} = 12$ permutations of $(5,2,2,0)$, also 12 permutations of $(5,2,1,1)$, 24 permutations of $(4,3,2,0)$ and finally, there are 12 permutations of $(4,3,1,1)$. Hence, there are at most 60 different possible total sums.

We can obtain indeed each of these 60 combinations: take three rows *ababa* alternating with two rows *cdcdc* to get $(5,2,2,0)$; take three rows *ababa* alternating with one row *cdcdc* and a row (*dcdcd*) to get $(5,2,1,1)$; take three rows *ababc* alternating with two rows *cdcda* to get $(4,3,2,0)$; take three rows *abcda* alternating with two rows *cdabc* to get $(4,3,1,1)$. By choosing for example $a = 10^3$, $b = 10^2$, $c = 10$, $d = 1$, we can make all these sums different. Hence, 60 is indeed the maximum possible number of different sums.

□

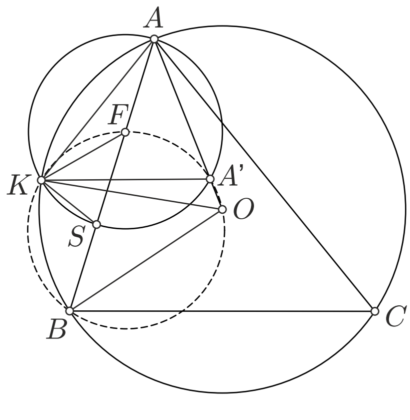
C4. A splitting of a planar polygon is a finite set of triangles whose interiors are pairwise disjoint, and whose union is the polygon in question. Given an integer $n \geq 3$, determine the largest integer m such that no planar n-gon splits into less than m triangles.

Solution. The required maximum is $\lceil n/3 \rceil$, the least integer greater than or equal to $n/3$. To describe a planar n-gon splitting into this many triangles, write $n = 3m - r$, where m is a positive integer and $r = 0, 1, 2$, and consider m coplanar equilateral triangles $A_{3i}A_{3i+1}A_{3i+2}$, $i = 0, \ldots, m-1$, where the A_i are pairwise distinct, the A's of rank congruent to 0 or 2 modulo 3 are all collinear, A_2, A_3, A_5, …, A_{3m-3}, A_{3m-1}, A_0, in order, and the line A_0A_2 separates A_1 from the remaining A's of rank congruent to 1 modulo 3. The polygon $A_0A_1A_2A_3A_4A_5 \ldots A_{3m-3}A_{3m-2}A_{3m-1}$ settles the case $r = 0$; removal of A_3 from the list settles the case $r = 1$; and removal of A_3 and A_{3m-1} settles the case $r = 2$.
Next, we prove that no planar n-gon splits into less than $n/3$ triangles. Alternatively, but equivalently, if a planar polygon splits into t triangles, then its boundary has (combinatorial) length at most $3t$. Proceed by induction on t. The base case $t = 1$ is clear, so let $t > 1$.
The vertices of the triangles in the splitting may subdivide the boundary of the polygon, making it into a possibly combinatorially longer simple loop Ω. Clearly, it is sufficient to prove that the length of Ω does not exceed $3t$.
To this end, consider a triangle in the splitting whose boundary ω meets Ω along at least one of its edges. Trace Ω counterclockwise and let α_1, …, α_k, in order, be the connected components of $\Omega - \omega$. Each α_i is a path along Ω with distinct end points, whose terminal point is joined to the starting point of α_{i+1} by a (possibly constant) path β_i along ω. Trace ω clockwise from the terminal point of α_i to its starting point to obtain a path α'_i of positive length, and notice that $\alpha_i + \alpha'_i$ is the boundary of a polygon split into $t_i < t$ triangles. By the induction hypothesis, the length of $\alpha_i + \alpha'_i$ does not exceed $3t_i$, and since α'_i has positive length, the length of α_i is at most $3t_i - 1$. Consequently, the length of $\Omega - \omega$ does not exceed $\sum_{i=1}^{k}(3t_i - 1) = 3t - 3 - k$.
Finally, we prove that the total length of the β_i does not exceed $k + 3$. Begin by noticing that no β_i has length greater than 4, at most one has length greater than 2, and at most three have length 2. If some β_i has length 4, then the remaining $k - 1$ are all of length at most 1, so the total length of the β's does not exceed $4 + (k-1) = k + 3$. Otherwise, either some β_i has length 3, in which case at most one other has length 2 and the remaining $k - 2$ all have length at most 1, or the β_i all have length less than 3, in which case there are at most three of length 2 and the remaining $k - 3$ all have length at most 1; in the former case, the total length of the β's does not exceed $3 + 2 + (k-2) = k + 3$, and in the latter, the total length of the β's does not exceed $3 \cdot 2 + (k-3) = k + 3$. The conclusion follows. □

Chapter 3

Geometry

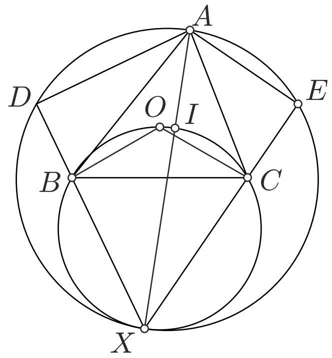
G1. Let ABC be an acute angled triangle, let O be its circumcentre, and let D, E, F be points on the sides BC, AC, AB, respectively. The circle (c_1) of radius FA, centred at F, crosses the segment (OA) at A' and the circumcircle (c) of the triangle ABC again at K. Similarly, the circle (c_2) of radius DB, centred at D, crosses the segment (OB) at B' and the circle (c) again at L. Finally, the circle (c_3) of radius EC, centred at E, crosses the segment (OC) at C' and the circle (c) again at M. Prove that the quadrilaterals $BKFA'$, $CLDB'$ and $AMEC'$ are all cyclic, and their circumcircles share a common point.



Solution. We will prove that the quadrilateral $BKFA'$ is cyclic and its circumcircle passes through the center O of the circle (c).
The triangle AFK is isosceles, so $m(\widehat{KFB}) = 2m(\widehat{KAB}) = m(\widehat{KOB})$. It follows that the quadrilateral $BKFO$ is cyclic. (1)
The triangles OFK and OFA are congruent (S.S.S.), hence $m(\widehat{OKF}) = m(\widehat{OAF})$. The triangle FAA' is isosceles, so $m(\widehat{FA'A}) = m(\widehat{OAF})$. Therefore $m(\widehat{FA'A}) = m(\widehat{OKF})$, so the quadrilateral $OKFA'$ is cyclic. (2)
(1) and (2) prove the initial claim.
Along the same lines, we can prove that the points C, D, L, B', O and A, M, E, C', O are concylic, respectively, so their circumcircles also pass through O.

□

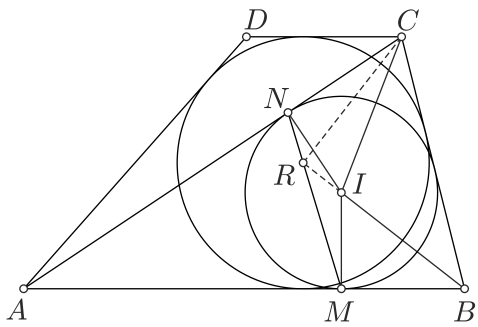
G2. Let ABC be a triangle with $m(\widehat{BAC}) = 60°$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors of $\widehat{ABC}$ and $\widehat{ACB}$, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles ΔADE and ΔBOC are tangent to each other.



Solution. Let X be the intersection of the lines BD and CE.
We will prove that X lies on the circumcircles of both triangles ΔADE and ΔBOC and then we will prove that the centers of these circles and the point X are collinear, which is enough for proving that the circles are tangent to each other.
In this proof we will use the notation (MNP) to denote the circumcircle of the triangle ΔMNP.
Obviously, the quadrilateral $ADXE$ is cyclic, and the circle (DAE) has $[AX]$ as diameter. (1)
Let I be the incenter of triangle ABC. So, the point I lies on the segment $[AX]$ (2), and the quadrilateral $XBIC$ is cyclic because $IC \perp XC$ and $IB \perp XB$. So, the circle (BIC) has $[IX]$ as diameter.
Finally, $m(\widehat{BIC}) = 90° + \frac{1}{2}m(\widehat{BAC}) = 120°$ and $m(\widehat{BOC}) = 2m(\widehat{BAC}) = 120°$.
So, the quadrilateral $BOIC$ is cyclic and the circle (BOC) has $[IX]$ as diameter. (3)
(1), (2), (3) imply the conclusion.

□

G3. A trapezoid $ABCD$ ($AB \parallel CD$, $AB > CD$) is circumscribed. The incircle of triangle ABC touches the lines AB and AC at M and N, respectively. Prove that the incenter of the trapezoid lies on the line MN.



Solution. Let I be the incenter of triangle ABC and R be the common point of the lines BI and MN. Since

$$m(\widehat{ANM}) = 90^\circ - \frac{1}{2}m(\widehat{MAN}) \quad \text{and} \quad m(\widehat{BIC}) = 90^\circ + \frac{1}{2}m(\widehat{MAN})$$

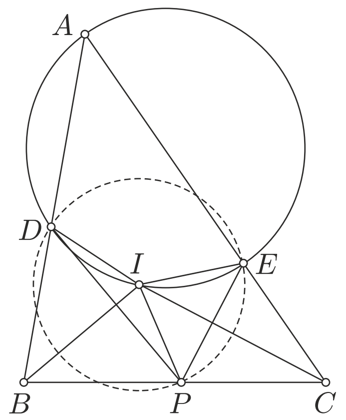
the quadrilateral $IRNC$ is cyclic.
It follows that $m(\widehat{BRC}) = 90^\circ$ and therefore

$$m(\widehat{BCR}) = 90^\circ - m(\widehat{CBR}) = 90^\circ - \frac{1}{2}\big(180^\circ - m(\widehat{BCD})\big) = \frac{1}{2}m(\widehat{BCD}).$$

So, $(CR$ is the angle bisector of $\widehat{DCB}$ and R is the incenter of the trapezoid.

□

G4. Let ABC be an acute angled triangle whose shortest side is $[BC]$. Consider a variable point P on the side $[BC]$, and let D and E be points on $(AB]$ and $(AC]$, respectively, such that $BD = BP$ and $CP = CE$. Prove that, as P traces $[BC]$, the circumcircle of the triangle ADE passes through a fixed point.



Solution. We claim that the fixed point is the center of the incircle of ABC.
Let I be the center of the incircle of ABC. Since $BD = BP$ and $[BI$ is the bisector of $\widehat{DBP}$, the line BI is the perpendicular bisector of $[DP]$. This yields $DI = PI$. Analogously we get $EI = PI$. So, the point I is the circumcenter of the triangle DEP.
This means $m(\widehat{DIE}) = 2m(\widehat{DPE})$.
On the other hand

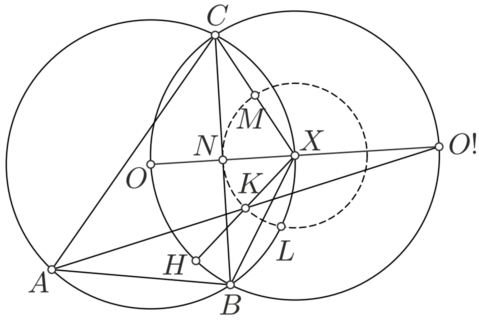
$$\begin{aligned} m(\widehat{DPE}) &= 180^\circ - m(\widehat{DPB}) - m(\widehat{EPC}) \\ &= 180^\circ - \left(90^\circ - \frac{1}{2}m(\widehat{DBP})\right) - \left(90^\circ - \frac{1}{2}m(\widehat{ECP})\right) \\ &= 90^\circ - \frac{1}{2}m(\widehat{BAC}). \end{aligned}$$

So, $m(\widehat{DIE}) = 2m(\widehat{DPE}) = 180^\circ - m(\widehat{DAE})$, which means that the points A, D, E and I are cocyclic.

□

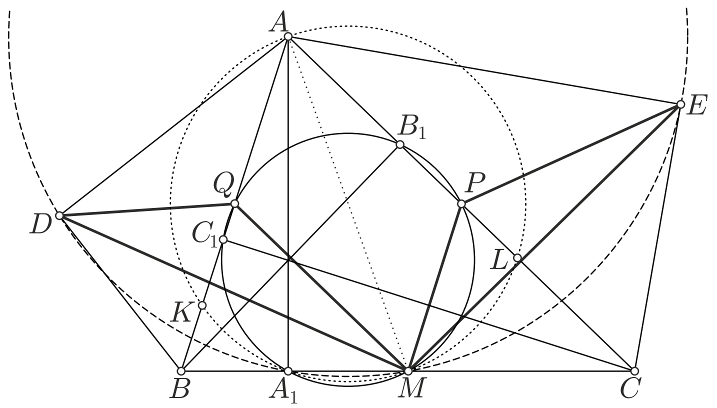
Remark. The fact that the incentre I of the triangle ABC is the required fixed point could be guessed by considering the two extremal positions of P. Thus, if $P = B$, then $D = D_B = B$ as well, and $CE = CE_B = BC$, so $m(\angle AEB) = m(\angle C) + m(\angle EBC) = m(\angle C) + \frac{180^\circ - m(\angle C)}{2} = 90^\circ + \frac{m(\angle C)}{2} = m(\angle AIB)$. Hence the points A, $E = E_B$, I, $D_B = B$ are cocyclic. Similarly, letting $P = C$, the points A, $D = D_C$, I, $E_C = C$ are cocyclic. Consequently, the circles AD_BE_B and AD_CE_C meet again at I.

G5. Let ABC be an acute angled triangle with orthocenter H and circumcenter O. Assume the circumcenter X of BHC lies on the circumcircle of ABC. Reflect O across X to obtain O', and let the lines XH and $O'A$ meet at K. Let L, M and N be the midpoints of $[XB]$, $[XC]$ and $[BC]$, respectively. Prove that the points K, L, M and N are cocyclic.



Solution. The circumcircles of ABC and BHC have the same radius. So, $XB = XC = XH = XO = r$ (where r is the radius of the circle ABC) and O' lies on $C(X, r)$. We conclude that OX is the perpendicular bisector for $[BC]$. So, BOX and COX are equilateral triangles.
It is known that $AH = 2ON = r$. So, $AHO'X$ is parallelogram, and $XK = KH = r/2$.
Finally, $XL = XK = XN = XM = r/2$. So, K, L, M and N lie on the circle $c(X, r/2)$.
□

G6. Given an acute triangle ABC, erect triangles ABD and ACE externally, so that $m(\widehat{ADB}) = m(\widehat{AEC}) = 90^\circ$ and $\widehat{BAD} \equiv \widehat{CAE}$. Let $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$ be the feet of the altitudes of the triangle ABC, and let K and L be the midpoints of $[BC_1]$ and $[CB_1]$, respectively. Prove that the circumcenters of the triangles AKL, $A_1B_1C_1$ and DEA_1 are collinear.



Solution. Let M, P and Q be the midpoints of $[BC]$, $[CA]$ and $[AB]$, respectively.
The circumcircle of the triangle $A_1B_1C_1$ is the Euler's circle. So, the point M lies on this circle.
It is enough to prove now that $[A_1M]$ is a common chord of the three circles $(A_1B_1C_1)$, (AKL) and (DEA_1).
The segments $[MK]$ and $[ML]$ are midlines for the triangles BCC_1 and BCB_1 respectively, hence $MK \parallel CC_1 \perp AB$ and $ML \parallel BB_1 \perp AC$. So, the circle (AKL) has diameter $[AM]$ and therefore passes through M.
Finally, we prove that the quadrilateral DA_1ME is cyclic.
From the cyclic quadrilaterals $ADBA_1$ and $AECA_1$, $\widehat{AA_1D} \equiv \widehat{ABD}$ and $\widehat{AA_1E} \equiv \widehat{ACE} \equiv \widehat{ABD}$, so $m(\widehat{DA_1E}) = 2m(\widehat{ABD}) = 180^\circ - 2m(\widehat{DAB})$.
We notice now that $DQ = AB/2 = MP$, $QM = AC/2 = PE$ and

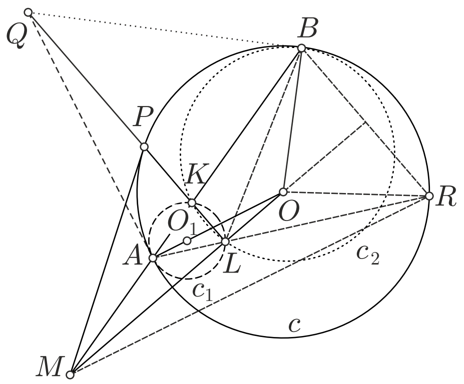
$$m(\widehat{DQM}) = m(\widehat{DQB}) + m(\widehat{BQM}) = 2m(\widehat{DAB}) + m(\widehat{BAC}),$$

$$m(\widehat{EPM}) = m(\widehat{EPC}) + m(\widehat{CPM}) = 2m(\widehat{EAC}) + m(\widehat{CAB}),$$

so $\Delta MPE \equiv \Delta DQM$ (S.A.S.). This leads to $m(\widehat{DME}) = m(\widehat{DMQ}) + m(QMP) + m(PME) = m(\widehat{DMQ}) + m(BQM) + m(QDM) = 180^\circ - m(\widehat{DQB}) = 180^\circ - 2m(\widehat{DAB})$. Since $m(\widehat{DA_1E}) = m(\widehat{DME})$, the quadrilateral DA_1ME is cyclic.

□

G7. Let $[AB]$ be a chord of a circle (c) centered at O, and let K be a point on the segment (AB) such that $AK < BK$. Two circles through K, internally tangent to (c) at A and B, respectively, meet again at L. Let P be one of the points of intersection of the line KL and the circle (c), and let the lines AB and LO meet at M. Prove that the line MP is tangent to the circle (c).



Solution. Let (c_1) and (c_2) be circles through K, internally tangent to (c) at A and B, respectively, and meeting again at L, and let the common tangent to (c_1) and (c) meet the common tangent to (c_2) and (c) at Q. Then the point Q is the radical center of the circles (c_1), (c_2) and (c), and the line KL passes through Q.

We have $m(\widehat{QLB}) = m(\widehat{QBK}) = m(\widehat{QBA}) = \frac{1}{2}m(\widehat{BA}) = m(\widehat{QOB})$. So, the quadrilateral $OBQL$ is cyclic. We conclude that $m(\widehat{QLO}) = 90°$ and the points O, B, Q, A and L are cocyclic on a circle (k).

In the sequel, we will denote $\mathcal{P}_\omega(X)$ the power of the point X with respect of the circle ω.

The first continuation.

From $MO^2 - OP^2 = \mathcal{P}_c(M) = MA \cdot MB = \mathcal{P}_k(M) = ML \cdot MO = (MO - OL) \cdot MO = MO^2 - OL \cdot MO$ follows that $OP^2 = OL \cdot OM$. Since $PL \perp OM$, this shows that the triangle MPO is right at point P. Thus, the line MP is tangent to the circle (c).

The second continuation.

Let $R \in (c)$ be so that $BR \perp MO$. The triangle LBR is isosceles with $LB = LR$, so $\widehat{OLR} \equiv \widehat{OLB} \equiv \widehat{OQB} \equiv \widehat{OQA} \equiv \widehat{MLA}$. We conclude that the points A, L and R are collinear.

Now $m(\widehat{AMR}) + m(\widehat{AOR}) = m(\widehat{AMR}) + 2m(\widehat{ABR}) = m(\widehat{AMR}) + m(\widehat{ABR}) + m(\widehat{MRB}) = 180°$, since the triangle MBR is isosceles. So, the quadrilateral $MAOR$ is cyclic.

This yields $LM \cdot LO = -\mathcal{P}_{(MAOR)}(L) = LA \cdot LR = -\mathcal{P}_c(L) = LP^2$, which as above, shows that $OP \perp PM$.

The third continuation.

$\widehat{KLA} \equiv \widehat{KAQ} \equiv \widehat{KLB}$ and $m(\widehat{MLK}) = 90°$ show that $[LK$ and $[LM$ are the internal and external bisectors af the angle $\widehat{ALB}$, so (M, K) and (A, B) are harmonic conjugates. So, LK is the polar line of M in the circle (c).

□

Chapter 4

Number Theory

N1. Determine the largest positive integer n that divides $p^6 - 1$ for all primes $p > 7$.

Solution. Note that

$$p^6 - 1 = (p-1)(p+1)(p^2 - p + 1)(p^2 + p + 1).$$

For $p = 11$ we have

$$p^6 - 1 = 1771560 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 37.$$

For $p = 13$ we have

$$p^6 - 1 = 2^3 \cdot 3^2 \cdot 7 \cdot 61 \cdot 157.$$

From the last two calculations we find evidence to try showing that $p^6 - 1$ is divisible by $2^3 \cdot 3^2 \cdot 7 = 504$ and this would be the largest positive integer that divides $p^6 - 1$ for all primes greater than 7.

By Fermat's theorem, $7 \mid p^6 - 1$.

Next, since p is odd, $8 \mid p^2 - 1 = (p-1)(p+1)$, hence $8 \mid p^6 - 1$.

It remains to show that $9 \mid p^6 - 1$.

Any prime number p, $p > 3$ is 1 or -1 modulo 3.

In the first case both $p - 1$ and $p^2 + p + 1$ are divisible by 3, and in the second case, both $p + 1$ and $p^2 - p + 1$ are divisible by 3.

Consequently, the required number is indeed 504.

Alternative solution

Let q be a (positive) prime factor of n. Then $q \leq 7$, as $q \nmid q^6 - 1$. Also, q is not 5, as the last digit of $13^6 - 1$ is 8.

Hence, the prime factors of n are among 2, 3, and 7.

Next, from $11^6 - 1 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 37$ it follows that the largest integer n such that $n \mid p^6 - 1$ for all primes p greater than 7 is at most $2^3 \cdot 3^2 \cdot 7$, and it remains to prove that 504 divides $p^6 - 1$ for all primes grater than 7.

□

N2. Find the maximum number of natural numbers $x_1, x_2, \ldots, x_m$ satisfying the conditions:
a) No $x_i - x_j$, $1 \leq i < j \leq m$ is divisible by 11; and
b) The sum $x_2x_3\ldots x_m + x_1x_3\ldots x_m + \cdots + x_1x_2\ldots x_{m-1}$ is divisible by 11.

Solution. The required maximum is 10.
According to a), the numbers $x_i, 1 \leq i \leq m$, are all different $\pmod{11}$ (1)
Hence, the number of natural numbers satisfying the conditions is at most 11.
If $x_j \equiv 0 \pmod{11}$ for some j, then

$$x_2x_3\ldots x_m + x_1x_3\ldots x_m + \cdots + x_1x_2\ldots x_{m-1} \equiv x_1\ldots x_{j-1}x_{j+1}\ldots x_m \pmod{11},$$

which would lead to $x_i \equiv 0 \pmod{11}$ for some $i \neq j$, contradicting (1).
We now prove that 10 is indeed the required maximum.
Consider $x_i = i$, for all $i \in \{1, 2, \ldots, 10\}$. The products $2 \cdot 3 \cdot \cdots \cdot 10$, $1 \cdot 3 \cdot \cdots \cdot 10$, $\ldots$, $1 \cdot 2 \cdot \cdots \cdot 9$ are all different $\pmod{11}$, and so

$$2 \cdot 3 \cdot \cdots \cdot 10 + 1 \cdot 3 \cdot \cdots \cdot 10 + \cdots + 1 \cdot 2 \cdot \cdots \cdot 9 \equiv 1 + 2 + \cdots + 10 \pmod{11},$$

and condition b) is satisfied, since $1 + 2 + \cdots + 10 = 55 = 5 \cdot 11$.

□

N3. Find all positive integers n such that the number $A_n = \frac{2^{4n+2}+1}{65}$ is
a) an integer;
b) a prime.

Solution. a) Note that $65 = 5 \cdot 13$.
Obviously, $5 = 2^2 + 1$ is a divisor of $(2^2)^{2n+1} + 1 = 2^{4n+2} + 1$ for any positive integer n. Since $2^{12} \equiv 1 \pmod{13}$, if $n \equiv r \pmod 3$, then $2^{4n+2} + 1 \equiv 2^{4r+2} + 1 \pmod{13}$. Now, $2^{4\cdot 0+2} + 1 = 5$, $2^{4\cdot 1+2} + 1 = 65$, and $2^{4\cdot 2+2} + 1 = 1025 = 13 \cdot 78 + 11$. Hence 13 is a divisor of $2^{4n+2} + 1$ precisely when $n \equiv 1 \pmod 3$. Hence, A_n is an integer iff $n \equiv 1 \pmod 3$.
b) Applying the identity $4x^4 + 1 = (2x^2 - 2x + 1)(2x^2 + 2x + 1)$, we have $2^{4n+2} + 1 = (2^{2n+1} - 2^{n+1} + 1)(2^{2n+1} + 2^{n+1} + 1)$. For $n = 1$, $A_1 = 1$, which is not a prime. According to a), if $n \neq 1$, then $n \geq 4$. But then $2^{2n+1} + 2^{n+1} + 1 > 2^{2n+1} - 2^{n+1} + 1 > 65$, and A_n has at least two factors. We conclude that A_n can never be a prime.
Alternative Solution to b): Knowing that $n = 3k + 1$ in order for A_n to be an integer, $2^{4n+2} + 1 = 2^{12k+6} + 1 = (2^{4k+2})^3 + 1 = (2^{4k+2} + 1)(2^{8k+4} - 2^{4k+2} + 1) \quad (*)$. As in the previous solution, if $k = 0$, then $A_1 = 1$, if $k = 1$, then $A_4 = 2^{12} - 2^6 + 1 = 4033 = 37 \cdot 109$, and for $k \geq 2$ both factors in $(*)$ are larger than 65, so A_{3k+1} is not a prime.

□

N4. Find all triples of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016.

Solution. Let z be a positive integer such that

$$(a-b)(b-c)(c-a) + 4 = 2 \cdot 2016^z.$$

We set $a - b = -x$, $b - c = -y$ and we rewrite the equation as

$$xy(x+y) + 4 = 2 \cdot 2016^z.$$

Note that the right hand side is divisible by 7, so we have that

$$xy(x+y) + 4 \equiv 0 \pmod 7$$

or

$$3xy(x+y) \equiv 2 \pmod 7$$

or

$$(x+y)^3 - x^3 - y^3 \equiv 2 \pmod 7. \tag{4.1}$$

Note that, by Fermat's Little Theorem, we have that for any integer k the cubic residues are $k^3 \equiv -1, 0, 1 \mod 7$. It follows that in (4.1) some of $(x+y)^3$, x^3 and y^3 should be divisible by 7, but in this case, $xy(x+y)$ is divisible by 7 and this is a contradiction. So, the only possibility is to have $z = 0$ and consequently, $xy(x+y) + 4 = 2$, or, equivalently, $xy(x+y) = -2$. The only solution of the latter is $(x, y) = (-1, -1)$, so the required triples are $(a, b, c) = (k+2, k+1, k)$, $k \in \mathbb{Z}$, and all their cyclic permutations.

□

N5. Determine all four-digit numbers $\overline{abcd}$ such that

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)=\overline{abcd}.$$

Solution. Depending on the parity of a,b,c,d, at least two of the factors $(a+b)$, $(a+c)$, $(a+d)$, $(b+c)$, $(b+d)$, $(c+d)$ are even, so that $4|\overline{abcd}$.
We claim that $3|\overline{abcd}$.
Assume $a+b+c+d\equiv 2 \pmod 3$. Then $x+y\equiv 1 \pmod 3$, for all distinct $x,y\in\{a,b,c,d\}$. But then the left hand side in the above equality is congruent to $1 \pmod 3$ and the right hand side congruent to $2 \pmod 3$, contradiction.
Assume $a+b+c+d\equiv 1 \pmod 3$. Then $x+y\equiv 2 \pmod 3$, for all distinct $x,y\in\{a,b,c,d\}$, and $x\equiv 1 \pmod 3$, for all $x,y\in\{a,b,c,d\}$. Hence, $a,b,c,d\in\{1,4,7\}$, and since $4|\overline{abcd}$, we have $c=d=4$. Therefore, $8|\overline{ab44}$, and since at least one more factor is even, it follows that $16\overline{ab44}$. Then $b\neq 4$, and the only possibilities are $b=1$, implying $a=4$, which is impossible because 4144 is not divisible by $5=1+4$, or $b=7$, implying $11|\overline{a744}$, hence $a=7$, which is also impossible because 7744 is not divisible by $14=7+7$.
We conclude that $3|\overline{abcd}$, hence also $3|a+b+c+d$. Then at least one factor $x+y$ of $(a+b),(a+c),(a+d),(b+c),(b+d),(c+d)$ is a multiple of 3, implying that also $3|a+b+c+d-x-y$, so $9|\overline{abcd}$. Then $9|a+b+c+d$, and $a+b+c+d\in\{9,18,27,36\}$.
Using the inequality $xy\geq x+y-1$, valid for all $x,y\in\mathbb{N}^*$, if $a+b+c+d\in\{27,36\}$, then

$$\overline{abcd}=(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)\geq 26^3>10^4,$$

which is impossible.
Using the inequality $xy\geq 2(x+y)-4$ for all $x,y\geq 2$, if $a+b+c+d=18$ and all two-digit sums are greater than 1, then $\overline{abcd}\geq 32^3>10^4$. Hence, if $a+b+c+d=18$, some two-digit sum must be 1, hence the complementary sum will be 17, and the digits are $\{a,b,c,d\}=\{0,1,8,9\}$. But then $\overline{abcd}=1\cdot 17\cdot 8\cdot 9^2\cdot 10>10^4$.
We conclude that $a+b+c+d=9$. Then among a,b,c,d there are either three odd or three even numbers, and $8|\overline{abcd}$.
If three of the digits are odd, then d is even and since c is odd, divisibility by 8 implies that $d\in\{2,6\}$. If $d=6$, then $a=b=c=1$. But 1116 is not divisible by 7, so this is not a solution. If $d=2$, then a,b,c are either $1,1,5$ or $1,3,3$ in some order. In the first case $2\cdot 6^2\cdot 3^2\cdot 7=4536\neq\overline{abcd}$. The second case cannot hold because the resulting number is not a multiple of 5.
Hence, there has to be one odd and three even digits. At least one of the two-digits sums of even digits is a multiple of 4, and since there cannot be two zero digits, we have either $x+y=4$ and $z+t=5$, or $x+y=8$ and $z+t=1$ for some ordering x,y,z,t of a,b,c,d. In the first case we have $d=0$ and the digits are $0,1,4,4$, or $0,2,3,4$, or $0,2,2,5$. None of these is a solution because $1\cdot 4^2\cdot 5^2\cdot 8=3200$, $2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7=5040$ and $2^2\cdot 5\cdot 4\cdot 7^2=3920$. In the second case two of the digits are 0 and 1, and the other two have to be either 4 and 4, or 2 and 6. We already know that the first possibility fails. For the second, we get

$$(0+1)\cdot(0+2)\cdot(0+6)\cdot(1+2)\cdot(1+6)\cdot(2+6)=2016$$

and $\overline{abcd}=2016$ is the only solution.

□