2011 APMO PROBLEMS

Time allowed: 4 hours *Each problem is worth 7 points*

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c,\ b^2 + c + a,\ c^2 + a + b$ to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_i A_j A_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC = 30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

Problem 4. Let n be a fixed positive odd integer. Take $m + 2$ **distinct** points $P_0, P_1, \cdots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:
(1) $P_0 = (0, 1),\ P_{m+1} = (n + 1, n)$, and for each integer i, $1 \le i \le m$, both x- and y- coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).
(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.
(3) For each pair i, j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Determine all functions $f : \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers, satisfying the following 2 conditions:
(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
(2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

SOLUTIONS FOR 2011 APMO PROBLEMS

Problem 1.

Solution: Suppose all of the 3 numbers $a^2 + b + c$, $b^2 + c + a$ and $c^2 + a + b$ are perfect squares. Then from the fact that $a^2 + b + c$ is a perfect square bigger than a^2 it follows that $a^2 + b + c \geq (a + 1)^2$, and therefore, $b + c \geq 2a + 1$. Similarly we obtain $c + a \geq 2b + 1$ and $a + b \geq 2c + 1$.

Adding the corresponding sides of the preceding 3 inequalities, we obtain $2(a + b + c) \geq 2(a + b + c) + 3$, a contradiction. This proves that it is impossible to have all the 3 given numbers to be perfect squares.

Alternate Solution: Since the given conditions of the problem are symmetric in a, b, c, we may assume that $a \geq b \geq c$ holds. From the assumption that a^2+b+c is a perfect square, we can deduce as in the solution above the inequality $b+c \geq 2a+1$. But then we have

$$2a \geq b + c \geq 2a + 1,$$

a contradiction, which proves the assertion of the problem.

Problem 2.

Solution: We will show that $36°$ is the desired answer for the problem.

First, we observe that if the given 5 points form a regular pentagon, then the minimum of the angles formed by any triple among the five vertices is $36°$, and therefore, the answer we seek must be bigger than or equal to $36°$.

Next, we show that for any configuration of 5 points satisfying the condition of the problem, there must exist an angle smaller than or equal to $36°$ formed by a triple chosen from the given 5 points. For this purpose, let us start with any 5 points, say A_1, A_2, A_3, A_4, A_5, on the plane satisfying the condition of the problem, and consider the smallest convex subset, call it Γ, in the plane containing all of the 5 points. Since this convex subset Γ must be either a triangle or a quadrilateral or a pentagon, it must have an interior angle with $108°$ or less. We may assume without loss of generality that this angle is $\angle A_1A_2A_3$. By the definition of Γ it is clear that the remaining 2 points A_4 and A_5 lie in the interior of the angular region determined by $\angle A_1A_2A_3$, and therefore, there must be an angle smaller than or equal to $\frac{1}{3} \cdot 108° = 36°$, which is formed by a triple chosen from the given 5 points, and this proves that $36°$ is the desired maximum.

Problem 3.

Solution: Since $\angle B_1BB_2 = 90^\circ$, the circle having B_1B_2 as its diameter goes through the points B, B_1, B_2. From $B_1A : B_1C = B_2A : B_2C = BA : BC$, it follows that this circle is the Apolonius circle with the ratio of the distances from the points A and C being $BA : BC$. Since the point P lies on this circle, we have

$$PA : PC = BA : BC = \sin C : \sin A,$$

from which it follows that $PA \sin A = PC \sin C$. Similarly, we have $PA \sin A = PB \sin B$, and therefore, $PA \sin A = PB \sin B = PC \sin C$.

Let us denote by D, E, F the foot of the perpendicular line drawn from P to the line segment BC, CA and AB, respectively. Since the points E, F lie on a circle having PA as its diameter, we have by the law of sines $EF = PA \sin A$. Similarly, we have $FD = PB \sin B$ and $DE = PC \sin C$. Consequently, we conclude that DEF is an equilateral triangle. Furthermore, we have $\angle CPE = \angle CDE$, since the quadrilateral $CDPE$ is cyclic. Similarly, we have $\angle FPB = \angle FDB$. Putting these together, we get

$$\begin{aligned}\angle BPC &= 360^\circ - (\angle CPE + \angle FPB + \angle EPF) \\ &= 360^\circ - \{(\angle CDE + \angle FDB) + (180^\circ - \angle FAE)\} \\ &= 360^\circ - (120^\circ + 150^\circ) = 90^\circ,\end{aligned}$$

which proves the assertion of the problem.

Alternate Solution: Let O be the midpoint of the line segment B_1B_2. Then the points B and P lie on the circle with center at O and going through the point B_1. From

$$\angle OBC = \angle OBB_1 - \angle CBB_1 = \angle OB_1B - \angle B_1BA = \angle BAC$$

it follows that the triangles OCD and OBA are similar, and therefore we have that $OC \cdot OA = OB^2 = OP^2$. Thus we conclude that the triangles OCP and OPA are similar, and therefore, we have $\angle OPC = \angle PAC$. Using this fact, we obtain

$$\begin{aligned}\angle PBC - \angle PBA &= (\angle B_1BC + \angle PBB_1) - (\angle ABB_1 - \angle PBB_1) \\ &= 2\angle PBB_1 = \angle POB_1 = \angle PCA - \angle OPC \\ &= \angle PCA - \angle PAC,\end{aligned}$$

from which we conclude that $\angle PAC + \angle PBC = \angle PBA + \angle PCA$. Similarly, we get $\angle PAB + \angle PCB = \angle PBA + \angle PCA$. Putting these facts together and taking into account the fact that

$$(\angle PAC + \angle PBC) + (\angle PAB + \angle PCB) + (\angle PBA + \angle PCA) = 180^\circ,$$

we conclude that $\angle PBA + \angle PCA = 60^\circ$, and finally that

$$\angle BPC = (\angle PBA + \angle PAB) + (\angle PCA + \angle PAC) = \angle BAC + (\angle PBA + \angle PCA) = 90^\circ,$$

proving the assertion of the problem.

Problem 4.

Solution: We will show that the desired maximum value for m is $n(n-1)$.

First, let us show that $m \le n(n-1)$ always holds for any sequence $P_0, P_1, \cdots, P_{m+1}$ satisfying the conditions of the problem.

Call a point a **turning point** if it coincides with P_i for some i with $1 \leq i \leq m$. Let us say also that 2 points $\{P, Q\}$ are **adjacent** if $\{P, Q\} = \{P_{i-1}, P_i\}$ for some i with $1 \leq i \leq m$, and **vertically adjacent** if, in addition, PQ is parallel to the y-axis.

Any turning point is vertically adjacent to exactly one other turning point. Therefore, the set of all turning points is partitioned into a set of pairs of points using the relation of "vertical adjacency". Thus we can conclude that if we fix $k \in \{1, 2, \cdots, n\}$, the number of turning points having the x-coordinate k must be even, and hence it is less than or equal to $n - 1$. Therefore, altogether there are less than or equal to $n(n-1)$ turning points, and this shows that $m \leq n(n-1)$ must be satisfied.

It remains now to show that for any positive odd number n one can choose a sequence for which $m = n(n-1)$. We will show this by using the mathematical induction on n. For $n = 1$, this is clear. For $n = 3$, choose

$$P_0 = (0, 1), \qquad P_1 = (1, 1), \qquad P_2 = (1, 2), \qquad P_3 = (2, 2),$$

$$P_4 = (2, 1), \qquad P_5 = (3, 1), \qquad P_6 = (3, 3), \qquad P_7 = (4, 3).$$

It is easy to see that these points satisfy the requirements (See fig. 1 below).

figure 1

Let n be an odd integer ≥ 5, and suppose there exists a sequence satisfying the desired conditions for $n-4$. Then, it is possible to construct a sequence which gives a configuration indicated in the following diagram (fig. 2), where the configuration inside of the dotted square is given by the induction hypothesis:

figure 2

By the induction hypothesis, there are exactly $(n-4)(n-5)$ turning points for the configuration inside of the dotted square in the figure 2 above, and all of the lattice points in the figure 2 lying outside of the dotted square except for the 4 points $(n, 2)$, $(n-1, n-2)$, $(2, 3)$, $(1, n-1)$ are turning points. Therefore, the total number of turning points in this configuration is

$$(n-4)(n-5)+(n^2-(n-4)^2-4)=n(n-1),$$

showing that for this n there exists a sequence satisfying the desired properties, and thus completing the induction process.

Problem 5.

Solution: By substituting $x=1$ and $y=1$ into the given identity we obtain $f(f(1))=f(1)$. Next, by substituting $x=1$ and $y=f(1)$ into the given identity and using $f(f(1))=f(1)$, we get $f(1)^2=f(1)$, from which we conclude that either $f(1)=0$ or $f(1)=1$. But if $f(1)=1$, then substituting $y=1$ into the given identity, we get $f(x)=x$ for all x, which contradicts the condition (1). Therefore, we must have $f(1)=0$.

By substituting $x=1$ into the given identity and using the fact $f(1)=0$, we then obtain $f(f(y))=2f(y)$ for all y. This means that if a number t belongs to the range of the function f, then so does $2t$, and by induction we can conclude that for any non-negative integer n, $2^n t$ belongs to the range of f if t does. Now suppose that there exists a real number a for which $f(a)>0$, then for any non-negative integer n $2^n f(a)$ must belong to the range of f, which leads to a contradiction to the condition (1). Thus we conclude that $f(x)\le 0$ for any real number x.

By substituting $\frac{x}{2}$ for x and $f(y)$ for y in the given identity and using the fact that $f(f(y))=2f(y)$, we obtain

$$f(xf(y))+f(y)f\left(\frac{x}{2}\right)=xf(y)+f\left(\frac{x}{2}f(y)\right),$$

from which it follows that $xf(y)-f(xf(y))=f(y)f\left(\frac{x}{2}\right)-f\left(\frac{x}{2}f(y)\right)\ge 0$, since the values of f are non-positive. Combining this with the given identity, we conclude that $yf(x)\ge f(xy)$. When $x>0$, by letting y to be $\dfrac{1}{x}$ and using the fact that $f(1)=0$, we get $f(x)\ge 0$. Since $f(x)\le 0$ for any real number x, we conclude that $f(x)=0$ for any positive real number x. We also have $f(0)=f(f(1))=2f(1)=0$.

If f is identically 0, i.e., $f(x)=0$ for all x, then clearly, this f satisfies the given identity. If f satisfies the given identity but not identically 0, then there exists a $b<0$ for which $f(b)<0$. If we set $c=f(b)$, then we have $f(c)=f(f(b))=2f(b)=2c$. For any negative real number x, we have $cx>0$ so that $f(cx)=f(2cx)=0$, and by substituting $y=c$ into the given identity, we get

$$f(2cx)+cf(x)=2cx+f(cx),$$

from which it follows that $f(x)=2x$ for any negative real x.

We therefore conclude that if f satisfies the given identity and is not identically 0, then f is of the form $f(x)=\begin{cases}0 & \text{if } x\ge 0\\ 2x & \text{if } x<0.\end{cases}$ Finally, let us show that the function f of the form shown above does satisfy the conditions of the problem. Clearly, it satisfies the condition (1). We can check that f satisfies the condition (2) as well by separating into the following 4 cases depending on whether x,y are non-negative or negative.

- when both x and y are non-negative, both sides of the given identity are 0.
- when x is non-negative and y is negative, we have $xy\le 0$ and both sides of the given identity are $4xy$.

- when x is negative and y is non-negative, we have $xy \leq 0$ and both sides of the given identity are $2xy$.
- when both x and y are negative, we have $xy > 0$ and both sides of the given identity are $2xy$.

Summarizing the arguments above, we conclude that the functions f satisfying the conditions of the problem are

$$f(x) = 0 \qquad \text{and} \qquad f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0. \end{cases}$$