Balkan MO Shortlist 2015

– Algebra

A1 If a, b and c are positive real numbers, prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc\left(a^3b^3 + b^3c^3 + c^3a^3\right) + a^2b^2c^2\left(a^3 + b^3 + c^3\right).$$

(Montenegro).

A2 Let a, b, c be sidelengths of a triangle and r, R, s be the inradius, the circumradius and the semiperimeter respectively of the same triangle. Prove that:

$$\frac{1}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} \leq \frac{r}{16Rs} + \frac{s}{16Rr} + \frac{11}{8s}$$

(Albania)

A3 Let a, b, c be sidelengths of a triangle and m_a, m_b, m_c the medians at the corresponding sides. Prove that

$$m_a\left(\frac{b}{a} - 1\right)\left(\frac{c}{a} - 1\right) + m_b\left(\frac{a}{b} - 1\right)\left(\frac{c}{b} - 1\right) + m_c\left(\frac{a}{c} - 1\right)\left(\frac{b}{c} - 1\right) \geq 0.$$

(FYROM)

A4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(x+y)f(2yf(x) + f(y)) = x^3 f(yf(x)), \quad \forall x, y \in \mathbb{R}^+.$$

(Albania)

A5 Let m, n be positive integers and a, b positive real numbers different from 1 such thath $m > n$ and

$$\frac{a^{m+1} - 1}{a^m - 1} = \frac{b^{n+1} - 1}{b^n - 1} = c$$

. Prove that $a^m c^n > b^n c^m$

(Turkey)

A6 For a polynomials $P \in \mathbb{R}[x]$, denote $f(P) = n$ if n is the smallest positive integer for which is valid

$$(\forall x \in \mathbb{R})(\underbrace{P(P(\ldots P}_{n}(x))\ldots) > 0),$$

and $f(P) = 0$ if such n doeas not exist. Exists polyomial $P \in \mathbb{R}[x]$ of degree 2014^{2015} such that $f(P) = 2015$?

(Serbia)

–	Combinatorics
C1	A committee of 3366 film critics are voting for the Oscars. Every critic voted just an actor and just one actress. After the voting, it was found that for every positive integer $n \in \{1, 2, \ldots, 100\}$, there is some actor or some actress who was voted exactly n times. Prove that there are two critics who voted the same actor and the same actress. *(Cyprus)*
C2	Isaak and Jeremy play the following game. Isaak says to Jeremy that he thinks a few 2^n integers $k_1, .., k_{2^n}$. Jeremy asks questions of the form: "Is it true that $k_i < k_j$?" in which Isaak answers by always telling the truth. After $n2^{n-1}$ questions, Jeramy must decide whether numbers of Isaak are all distinct each other or not. Prove that Jeremy has bo way to be "sure" for his final decision. (UK)
C3	A chessboard 1000×1000 is covered by dominoes 1×10 that can be rotated. We don't know which is the cover, but we are looking for it. For this reason, we choose a few N cells of the chessboard, for which we know the position of the dominoes that cover them. Which is the minimum N such that after the choice of N and knowing the dominoed that cover them, we can be sure and for the rest of the cover? (Bulgaria)

G1 In an acute angled triangle ABC , let BB' and CC' be the altitudes. Ray $C'B'$ intersects the circumcircle at B'' andl let α_A be the angle $\widehat{ABB''}$. Similarly are defined the angles α_B and α_C. Prove that

$$\sin\alpha_A \sin\alpha_B \sin\alpha_C \leq \frac{3\sqrt{6}}{32}$$

(Romania)

G2 Let ABC be a triangle with circumcircle ω . Point D lies on the arc BC ω and is different than B, C and the midpoint of arc BC. Tangent of Γ on D intersects lines BC, CA, AB at A', B', C', respectively. Lines BB' and CC' intersect at E. Line AA' intersects again the circle ω at F. Prove that points D, E, F are collinear.

(Saudi Arabia)

G3 A set of points of the plane is called *obtuse-angled* if every three of it's points are not collinear and every triangle with vertices inside the set has one angle $> 91^o$. Is it correct that every finite *acute-angled* set can be extended to an infinite *obtuse-angled* set?

(UK)

G4 Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear.
(Cyprus)

G5 Quadrilateral $ABCD$ is given with $AD \nparallel BC$. The midpoints of AD and BC are denoted by M and N, respectively. The line MN intersects the diagonals AC and BD in points K and L, respectively. Prove that the circumcircles of the triangles AKM and BNL have common point on the line AB.(Proposed by Emil Stoyanov)
`http://estoyanov.net/wp-content/uploads/2015/09/est.png`

G6 Let AB be a diameter of a circle (ω) with centre O. From an arbitrary point M on AB such that $MA < MB$ we draw the circles (ω_1) and (ω_2) with diameters AM and BM respectively. Let CD be an exterior common tangent of $(\omega_1), (\omega_2)$ such that C belongs to (ω_1) and D belongs to (ω_2). The point E is diametrically opposite to C with respect to (ω_1) and the tangent to (ω_1) at the point E intersects (ω_2) at the points F, G. If the line of the common chord of the circumcircles of the triangles CED and CFG intersects the circle (ω) at the points K, L and the circle (ω_2) at the point N (with N closer to L), then prove that $KC = NL$.

G7 Let scalene triangle ABC have orthocentre H and circumcircle Γ. AH meets Γ at D distinct from A. BH and CH meet CA and AB at E and F respectively, and EF meets BC at P. The tangents to Γ at B and C meet at T. Show that AP and DT are concurrent on the circumcircle of AFE.

– Number Theory

N1 Let d be an even positive integer.
John writes the numbers $1^2, 3^2, \ldots, (2n-1)^2$ on the blackboard and then chooses three of them, let them be a_1, a_2, a_3, erases them and writes the number $1 + \sum |a_i - a_j|$

He continues until two numbers remain written on on the blackboard.
Prove that the sum of squares of those two numbers is different than the numbers $1^2, 3^2, \ldots, (2n$ $1)^2$.

(Albania)

N2 Sequence $(a_n)_{n\geq 0}$ is defined as $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 6$,
and $a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n, n \geq 0$.
Prove that n^2 divides a_n for infinite n.

(Romania)

N3 Let a be a positive integer. For all positive integer n, we define $a_n = 1 + a + a^2 + \ldots + a^{n-1}$.
Let s, t be two different positive integers with the following property:
If p is prime divisor of $s - t$, then p divides $a - 1$.
Prove that number $\frac{a_s - a_t}{s-t}$ is an integer.

(FYROM)

N4 Find all pairs of positive integers (x, y) with the following property:
If a, b are relative prime and positive divisors of $x^3 + y^3$, then $a + b - 1$ is divisor of $x^3 + y^3$.

(Cyprus)

N5 For a positive integer s, denote with $v_2(s)$ the maximum power of 2 that divides s.
Prove that for any positive integer m that:

$$v_2\left(\prod_{n=1}^{2^m} \binom{2n}{n}\right) = m2^{m-1} + 1.$$

(FYROM)

N6 Prove that among 20 consecutive positive integers there is an integer d such that for every positive integer n the following inequality holds

$$n\sqrt{d}\left\{n\sqrt{d}\right\} > \frac{5}{2}$$

where by $\{x\}$ denotes the fractional part of the real number x. The fractional part of the real number x is defined as the difference between the largest integer that is less than or equal to x to the actual number x.

(Serbia)

N7 Positive integer m shall be called *anagram* of positive n if every digit a appears as many times in the decimal representation of m as it appears in the decimal representation of n also. Is it possible to find 4 different positive integers such that each of the four to be *anagram* of the sum of the other 3?

(Bulgaria)