The 13[th] Romanian Master of Mathematics Competition

Day 1 — Solutions

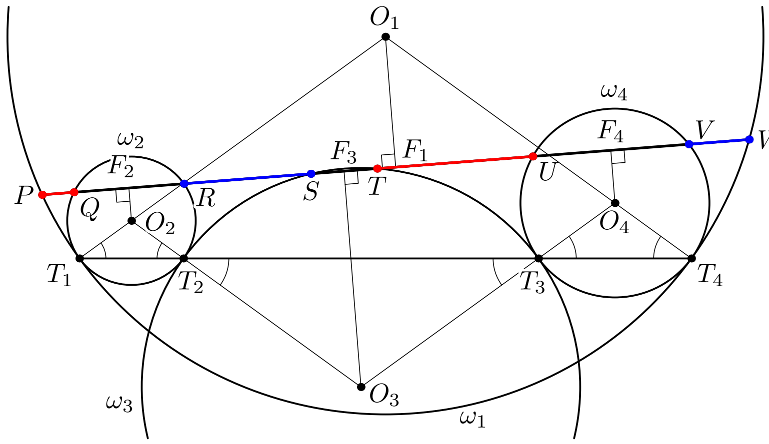
Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Hungary, Geza Kos

Solution. Let O_i be the centre of ω_i, $i = 1, 2, 3, 4$. Notice that the isosceles triangles $O_iT_iT_{i-1}$ are similar (indices are reduced modulo 4), to infer that ω_4 is internally tangent to ω_1 at T_4, and $O_1O_2O_3O_4$ is a (possibly degenerate) parallelogram.

Let F_i be the foot of the perpendicular from O_i to PW. The F_i clearly bisect the segments PW, QR, ST and UV, respectively.

The proof can now be concluded in two similar ways.



First Approach. Since $O_1O_2O_3O_4$ is a parallelogram, $\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4} = \mathbf{0}$ and $\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1} = \mathbf{0}$; this still holds in the degenerate case, for if the O_i are collinear, then they all lie on the line T_1T_4, and each O_i is the midpoint of the segment T_iT_{i+1}. Consequently,

$$\begin{aligned}\overrightarrow{PQ} - \overrightarrow{RS} + \overrightarrow{TU} - \overrightarrow{VW} &= \left(\overrightarrow{PF_1} + \overrightarrow{F_1F_2} + \overrightarrow{F_2Q}\right) - \left(\overrightarrow{RF_2} + \overrightarrow{F_2F_3} + \overrightarrow{F_3S}\right) \\ &\quad + \left(\overrightarrow{TF_3} + \overrightarrow{F_3F_4} + \overrightarrow{F_4U}\right) - \left(\overrightarrow{VF_4} + \overrightarrow{F_4F_1} + \overrightarrow{F_1W}\right) \\ &= \left(\overrightarrow{PF_1} - \overrightarrow{F_1W}\right) - \left(\overrightarrow{RF_2} - \overrightarrow{F_2Q}\right) + \left(\overrightarrow{TF_3} - \overrightarrow{F_3S}\right) - \left(\overrightarrow{VF_4} - \overrightarrow{F_4U}\right) \\ &\quad + \left(\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4}\right) - \left(\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1}\right) = \mathbf{0}.\end{aligned}$$

Alternatively, but equivalently, $\overrightarrow{PQ} + \overrightarrow{TU} = \overrightarrow{RS} + \overrightarrow{VW}$, as required.

Second Approach. This is merely another way of reading the previous argument. Fix an orientation of the line PW, say, from P towards W, and use a lower case letter to denote the coordinate of a point labelled by the corresponding upper case letter.

Since the diagonals of a parallelogram bisect one another, $f_1 + f_3 = f_2 + f_4$, the common value being twice the coordinate of the projection to PW of the point where O_1O_3 and O_2O_4 cross; the relation clearly holds in the degenerate case as well.

Plug $f_1 = \frac{1}{2}(p+w)$, $f_2 = \frac{1}{2}(q+r)$, $f_3 = \frac{1}{2}(s+t)$ and $f_4 = \frac{1}{2}(u+v)$ into the above equality to get $p+w+s+t = q+r+u+v$. Alternatively, but equivalently, $(q-p)+(u-t) = (s-r)+(w-v)$, that is, $PQ + TU = RQ + VW$, as required.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers $a_1, a_2, \dots, a_{20}$ such that, for each $k = 1, 2, \dots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k : k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Russia, Sergey Kudrya

Solution. Sergey can determine Xenia's number in 2 but not fewer moves.

We first show that 2 moves are sufficient. Let Sergey provide the set $\{17, 18\}$ on his first move, and the set $\{18, 19\}$ on the second move. In Xenia's two responses, exactly one number occurs twice, namely, a_{18}. Thus, Sergey is able to identify a_{17}, a_{18}, and a_{19}, and thence the residue of N modulo $17 \cdot 18 \cdot 19 = 5814 > 5000$, by the Chinese Remainder Theorem. This means that the given range contains a single number satisfying all congruences, and Sergey achieves his goal.

To show that 1 move is not sufficient, let $M = \operatorname{lcm}(1, 2, \dots, 10) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. Notice that M is divisible by the greatest common divisor of every pair of distinct positive integers not exceeding 20. Let Sergey provide the set $S = \{s_1, s_2, \dots, s_k\}$. We show that there exist pairwise distinct positive integers $b_1, b_2, \dots, b_k$ such that $1 \equiv b_i \pmod{s_i}$ and $M + 1 \equiv b_{i-1} \pmod{s_i}$ (indices are reduced modulo k). Thus, if in response Xenia provides the set $\{b_1, b_2, \dots, b_k\}$, then Sergey will be unable to distinguish 1 from $M + 1$, as desired.

To this end, notice that, for each i, the numbers of the form $1 + ms_i$, $m \in \mathbb{Z}$, cover all residues modulo s_{i+1} which are congruent to $1\,(\equiv M + 1)$ modulo $\gcd(s_i, s_{i+1}) \mid M$. Xenia can therefore choose a positive integer b_i such that $b_i \equiv 1 \pmod{s_i}$ and $b_i \equiv M + 1 \pmod{s_{i+1}}$. Clearly, such choices can be performed so as to make the b_i pairwise distinct, as required.

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Russia, Mikhail Antipov

Solution. Assume that every single worker also forms a brigade (with a unique possible leader). In this modified setting, we are interested in the number N of ways to assign leadership so that each worker's number of assignments is congruent to 1 modulo 4.

Consider the variables x_1, x_2, $\dots$, x_{17} corresponding to the workers. Assign each brigade (from the i-th through the j-th worker) the polynomial $f_{ij} = x_i + x_{i+1} + \cdots + x_j$, and form the product $f = \prod_{1 \le i \le j \le 17} f_{ij}$. The number N is the sum $\Sigma(f)$ of the coefficients of all monomials $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{17}^{\alpha_{17}}$ in the expansion of f, where the α_i are all congruent to 1 modulo 4. For any polynomial P, let $\Sigma(P)$ denote the corresponding sum. From now on, all polynomials are considered with coefficients in the finite field $\mathbb{F}_{17}$.

Recall that for any positive integer n, and any integers a_1, a_2, $\dots$, a_n, there exist indices $i \le j$ such that $a_i + a_{i+1} + \cdots + a_j$ is divisible by n. Consequently, $f(a_1, a_2, \dots, a_{17}) = 0$ for all a_1, a_2, $\dots$, a_{17} in $\mathbb{F}_{17}$.

Now, if some monomial in the expansion of f is divisible by x_i^{17}, replace that x_i^{17} by x_i; this does not alter the above overall vanishing property (by Fermat's Little Theorem), and preserves $\Sigma(f)$. After several such changes, f transforms into a polynomial g whose degree in each variable does not exceed 16, and $g(a_1, a_2, \dots, a_{17}) = 0$ for all a_1, a_2, $\dots$, a_{17} in $\mathbb{F}_{17}$. For such a polynomial, an easy induction on the number of variables shows that it is identically zero. Consequently, $\Sigma(g) = 0$, so $\Sigma(f) = 0$ as well, as desired.

The 13$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Consider an integer $n \geq 2$ and write the numbers 1, 2, ..., n down on a board. A move consists in erasing any two numbers a and b, and, for each c in $\{a+b, |a-b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

China

Solution. The answer is in the affirmative for all $n \geq 2$. Induct on n. Leaving aside the trivial case $n = 2$, deal first with particular cases $n = 5$ and $n = 6$.

If $n = 5$, remove first the pair $(2, 5)$, notice that $3 = |2-5|$ is already on the board, so $7 = 2+5$ alone is written down. Removal of the pair $(3, 4)$ then leaves exactly two numbers on the board, 1 and 7, since $|3 \pm 4|$ are both already there.

If $n = 6$, remove first the pair $(1, 6)$, notice that $5 = |1-6|$ is already on the board, so $7 = 1+6$ alone is written down. Next, remove the pair $(2, 5)$ and notice that $|2 \pm 5|$ are both already on the board, so no new number is written down. Finally, removal of the pair $(3, 4)$ provides a single number to be written down, $1 = |3-4|$, since $7 = 3+4$ is already on the board. At this stage, the process comes to an end: 1 and 7 are the two numbers left.

In the remaining cases, the problem for n is brought down to the corresponding problem for $\lceil n/2 \rceil < n$ by a finite number of moves. The conclusion then follows by induction.

Let $n = 4k$ or $4k-1$, where k is a positive integer. Remove the pairs $(1, 4k-1)$, $(3, 4k-3)$, ..., $(2k-1, 2k+1)$ in turn. Each time, two odd numbers are removed, and the corresponding $c = |a \pm b|$ are even numbers in the range 2 through $4k$, of which one is always $4k$. These even numbers are already on the board at each stage, so no c is to be written down, unless $n = 4k-1$ in which case $4k$ is written down during the first move. The outcome of this k-move round is the string of even numbers 2 through $4k$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, ..., $2k = \lceil n/2 \rceil$, as desired.

Finally, let $n = 4k+1$ or $4k+2$, where $k \geq 2$. Remove first the pair $(4, 2k+1)$ and notice that no new number is to be written down on the board, since $4 + (2k+1) = 2k+5 \leq 4k+1 \leq n$. Next, remove the pairs $(1, 4k+1)$, $(3, 4k-1)$, ..., $(2k-1, 2k+3)$ in turn. As before, at each of these stages, two odd numbers are removed; the corresponding $c = |a \pm b|$ are even numbers, this time in the range 4 through $4k+2$, of which one is always $4k+2$; and no new numbers are to be written down on the board, except $4 = |(2k-1) - (2k+3)|$ during the last move, and, possibly, $4k+2 = 1 + (4k+1)$ during the first move if $n = 4k+1$. Notice that 2 has not yet been involved in the process, to conclude that the outcome of this $(k+1)$-move round is the string of even numbers 2 through $4k+2$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, ..., $2k+1 = \lceil n/2 \rceil$, as desired.

Solution 2. We will prove the following, more general statement:

Claim. Write down a finite number (at least two) of pairwise distinct positive integers on a board. A *move* consists in erasing any two numbers a and b, and, for each c in $\{a+b, |a-b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. Then it is possible to be left with exactly two numbers on the board after a finite number of moves.

Notice that, if we divide all numbers on the board by some common factor, the resulting process goes on equally well. Such a reduction can therefore be performed after any move.

Notice that we cannot be left with less than two numbers. So it suffices to show that, given k positive integers on the board, $k \geq 3$, we can always decrease their number by at least 1. Arguing indirectly, choose a set of $k \geq 3$ positive integers $S = \{a_1, \dots, a_k\}$ which cannot be reduced in size by a sequence of moves, having a minimal possible sum σ. So, in any sequence of moves applied to S, two numbers are erased and exactly two numbers appear on each move. Moreover, the sum of any resulting set of k numbers is at least σ.

Notice that, given two numbers $a > b$ on the board, we can replace them by $a + b$ and $a - b$, and then, performing a move on the two new numbers, by $(a + b) + (a - b) = 2a$ and $(a + b) - (a - b) = 2b$. So we can double any two numbers on the board.

We now show that, if the board contains two even numbers a and b, we can divide them both by 2, while keeping the other numbers unchanged. If k is even, split the other numbers into pairs to multiply each pair by 2; then clear out the common factor 2. If k is odd, split all numbers but a into pairs to multiply each by 2; then do the same for all numbers but b; finally, clear out the common factor 4.

Back to the problem, if two of the numbers $a_1, \dots, a_k$ are even, reduce them both by 2 to get a set with a smaller sum, which is impossible. Otherwise, two numbers, say, $a_1 < a_2$, are odd, and we may replace them by the two even numbers $a_1 + a_2$ and $a_2 - a_1$, and then by $\frac{1}{2}(a_1 + a_2)$ and $\frac{1}{2}(a_2 - a_1)$, to get a set with a smaller sum, which is again impossible.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and 60° rotational symmetry (that is, there is a point O such that a 60° rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2+3n+1$ citizens at $3n^2+3n+1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

USA, Ankan Bhattacharya

Solution. Let P denote the given polygon, i.e., the kingdom of Zoomtopia. Throughout the solution, we interpret polygons with integer sides and perimeter $6k$ as $6k$-gons with unit sides (some of their angles may equal 180°). The argument hinges on the claim below:

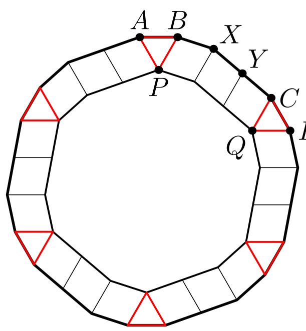
Claim. Let P be a convex polygon satisfying the problem conditions — i.e., it has integer sides, perimeter $6n$, and 60° rotational symmetry. Then P can be tiled with unit equilateral triangles and unit lozenges with angles at least 60°, with tiles meeting completely along edges, so that the tile configuration has a total of exactly $3n^2+3n+1$ distinct vertices.

Proof. Induct on n. The base case, $n=1$, is clear.

Now take a polygon P of perimeter $6n \geq 12$. Place six equilateral triangles inwards on six edges corresponding to each other upon rotation at 60°. It is possible to stick a lozenge to each other edge, as shown in the Figure below.

We show that all angles of the lozenges are at least 60°. Let an edge XY of the polygon bearing some lozenge lie along a boundary segment between edges AB and CD bearing equilateral triangles ABP and CDQ. Then the angle formed by $\overrightarrow{XY}$ and $\overrightarrow{BP}$ is between those formed by $\overrightarrow{AB}$, $\overrightarrow{BP}$ and $\overrightarrow{CD}$, $\overrightarrow{CQ}$, i.e., between 60° and 120°, as desired.

Removing all obtained tiles, we get a 60°-symmetric convex $6(n-1)$-gon with unit sides which can be tiled by the inductive hypothesis. Finally, the number of vertices in the tiling of P is $6n+3(n-1)^2+3(n-1)+1 = 3n^2+3n+1$, as desired.



Using the Claim above, we now show that the citizens may be placed at the $3n^2+3n+1$ tile vertices.

Consider any tile T_1; its vertices are at least 1 apart from each other. Moreover, let BAC be a part of the boundary of some tile T, and let X be any point of the boundary of T, lying outside the half-open intervals $[A, B)$ and $[A, C)$ (in this case, we say that X is not adjacent to A). Then $AX \geq \sqrt{3}/2$.

Now consider any two tile vertices A and B. If they are vertices of the same tile we already know $AB \geq 1$; otherwise, the segment AB crosses the boundaries of some tiles containing A and B at some points X and Y not adjacent to A and B, respectively. Hence $AB \geq AX+YB \geq \sqrt{3} > 1$.

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, a polynomial $f(x)$ with real coefficients is (A, B)-*nice* if $f(A) = B$, where $f(A) = \{f(a_i) \colon i = 1, 2, \ldots, n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A| = |B|$, one can produce an (A, B)-nice polynomial in a finite number of steps.

Iran, Navid Safaei

Solution. The required polynomials are all polynomials of an even degree $d \geq 2$, and all polynomials of odd degree $d \geq 3$ with negative leading coefficient.

Part I. We begin by showing that any (non-constant) polynomial $S(x)$ **not** listed above is not (A, B)-nice for some pair (A, B) with either $|A| = |B| = 2$, or $|A| = |B| = 3$.

If $S(x)$ is linear, then so are all the polynomials appearing on the board. Therefore, none of them will be (A, B)-nice, say, for $A = \{1, 2, 3\}$ and $B = \{1, 2, 4\}$, as desired.

Otherwise, $\deg S = d \geq 3$ is odd, and the leading coefficient is positive. In this case, we make use of the following technical fact, whose proof is presented at the end of the solution.

Claim. There exists a positive constant T such that $S(x)$ satisfies the following condition:

$$S(b) - S(a) \geq b - a \qquad \text{whenever} \quad b - a \geq T. \tag{$*$}$$

Fix a constant T provided by the Claim. Then, an immediate check shows that all newly appearing polynomials on the board also satisfy $(*)$ (with the same value of T). Therefore, none of them will be (A, B)-nice, say, for $A = \{0, T\}$ and $B = \{0, T/2\}$, as desired.

Part II. We show that the polynomials listed in the Answer satisfy the requirements. We will show that for any $a_1 < a_2 < \cdots < a_n$ and any $b_1 \leq b_2 \leq \cdots \leq b_n$ there exists a polynomial $f(x)$ satisfying $f(a_i) = b_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$, where σ is some permutation.

The proof goes by induction on $n \geq 2$. It is based on the following two lemmas, first of which is merely the base case $n = 2$; the proofs of the lemmas are also at the end of the solution.

Lemma 1. For any $a_1 < a_2$ and any b_1, b_2 one can write down on the board a polynomial $F(x)$ satisfying $F(a_i) = b_i$, $i = 1, 2$.

Lemma 2. For any distinct numbers $a_1 < a_2 < \cdots < a_n$ one can produce a polynomial $F(x)$ on the board such that the list $F(a_1)$, $F(a_2)$, $\ldots$, $F(a_n)$ contains exactly $n-1$ distinct numbers, and $F(a_1) = F(a_2)$.

Now, in order to perform the inductive step, we may replace the polynomial $S(x)$ with its shifted copy $S(C + x)$ so that the values $S(a_i)$ are pairwise distinct. Applying Lemma 2, we get a polynomial $f(x)$ such that only two among the numbers $c_i = f(a_i)$ coincide, namely c_1 and c_2. Now apply Lemma 1 to get a polynomial $g(x)$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$. Apply the inductive hypothesis in order to obtain a polynomial $h(x)$ satisfying $h(c_i) = b_i - g(a_i)$ for all $i = 2, 3, \ldots, n$. Then the polynomial $h(f(x)) + g(x)$ is a desired one; indeed, we have $h(f(a_i)) + g(a_i) = h(c_i) + g(a_i) = b_i$ for all $i = 2, 3, \ldots, n$, and finally $h(f(a_1)) + g(a_1) = h(c_1) + g(a_1) = b_2 - g(a_2) + g(a_1) = b_1$.

It remains to prove the Claim and the two Lemmas.

Proof of the Claim. There exists some segment $\Delta = [\alpha', \beta']$ such that $S(x)$ is monotone increasing outside that segment. Now one can choose $\alpha \leq \alpha'$ and $\beta \geq \beta'$ such that $S(\alpha) < \min_{x \in \Delta} S(x)$ and

$S(\beta) > \max_{x\in\Delta} S(x)$. Therefore, for any x, y, z with $x \le \alpha \le y \le \beta \le z$ we get $S(x) \le S(\alpha) \le S(y) \le S(\beta) \le S(z)$.

We may decrease α and increase β (preserving the condition above) so that, in addition, $S'(x) > 3$ for all $x \notin [\alpha, \beta]$. Now we claim that the number $T = 3(\beta - \alpha)$ fits the bill.

Indeed, take any a and b with $b - a \ge T$. Even if the segment $[a, b]$ crosses $[\alpha, \beta]$, there still is a segment $[a', b'] \subseteq [a, b] \setminus (\alpha, \beta)$ of length $b' - a' \ge (b - a)/3$. Then

$$S(b) - S(a) \ge S(b') - S(a') = (b' - a') \cdot S'(\xi) \ge 3(b' - a') \ge b - a$$

for some $\xi \in (a', b')$.

Proof of Lemma 1. If $S(x)$ has an even degree, then the polynomial $T(x) = S(x+a_2) - S(x+a_1)$ has an odd degree, hence there exists x_0 with $T(x_0) = S(x_0+a_2) - S(x_0+a_1) = b_2 - b_1$. Setting $G(x) = S(x+x_0)$, we see that $G(a_2) - G(a_1) = b_2 - b_1$, so a suitable shift $F(x) = G(x) + \big(b_1 - G(a_1)\big)$ fits the bill.

Assume now that $S(x)$ has odd degree and a negative leading coefficient. Notice that the polynomial $S^2(x) := S(S(x))$ has an odd degree and a positive leading coefficient. So, the polynomial $S^2(x+a_2) - S^2(x+a_1)$ attains all sufficiently large positive values, while $S(x+a_2) - S(x+a_1)$ attains all sufficiently large negative values. Therefore, the two-variable polynomial $S^2(x+a_2) - S^2(x+a_1) + S(y+a_2) - S(y+a_1)$ attains all real values; in particular, there exist x_0 and y_0 with $S^2(x_0+a_2) + S(y_0+a_2) - S^2(x_0+a_1) - S(y_0+a_1) = b_2 - b_1$. Setting $G(x) = S^2(x+x_0) + S(x+y_0)$, we see that $G(a_2) - G(a_1) = b_2 - b_1$, so a suitable shift of G fits the bill.

Proof of Lemma 2. Let Δ denote the segment $[a_1; a_n]$. We modify the proof of Lemma 1 in order to obtain a polynomial F convex (or concave) on Δ such that $F(a_1) = F(a_2)$; then F is a desired polynomial. Say that a polynomial $H(x)$ is *good* if H is convex on Δ.

If $\deg S$ is even, and its leading coefficient is positive, then $S(x+c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S(x+c)$ is good for all sufficiently large positive c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x+c_1) + S(x+c_2)$ is a desired polynomial. If the leading coefficient of H is negative, we similarly find a desired polynomial which is concave on Δ.

If $\deg S \ge 3$ is odd (and the leading coefficient is negative), then $S(x+c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S^2(x+c)$ is good for all sufficiently large positive c, and $S^2(a_2+c) - S^2(a_1+c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x+c_1) + S^2(x+c_2)$ is a desired polynomial.

Comment. Both parts above allow some variations.

In Part I, the same scheme of the proof works for many conditions similar to $(*)$, e.g.,

$$S(b) - S(a) > T \qquad \text{whenever} \quad b - a > T.$$

Let us sketch an alternative approach for Part II. It suffices to construct, for each i, a polynomial $f_i(x)$ such that $f_i(a_i) = b_i$ and $f_i(a_j) = 0$, $j \ne i$. The construction of such polynomials may be reduced to the construction of those for $n = 3$ similarly to what happens in the proof of Lemma 2. However, this approach (as well as any in this part) needs some care in order to work properly.