Chapter 1

2008 Shortlist JBMO - Problems

1.1 Algebra

A1 If for the real numbers x, y, z, k the following conditions are valid, $x \neq y \neq z \neq x$ and $x^3 + y^3 + k(x^2 + y^2) = y^3 + z^3 + k(y^2 + z^2) = z^3 + x^3 + k(z^2 + x^2) = 2008$, find the product xyz.
A2 Find all real numbers a, b, c, d such that $a+b+c+d = 20$ and $ab+ac+ad+bc+bd+cd = 150$.
A3 Let the real parameter p be such that the system

$$\begin{cases} p(x^2 - y^2) = (p^2 - 1)xy \\ |x - 1| + |y| = 1 \end{cases}$$

has at least three different real solutions. Find p and solve the system for that p.
A4 Find all triples (x, y, z) of real numbers that satisfy the system

$$\begin{cases} x + y + z = 2008 \\ x^2 + y^2 + z^2 = 6024^2 \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{2008}\,. \end{cases}$$

A5 Find all triples (x, y, z) of real positive numbers, which satisfy the system

$$\begin{cases} \dfrac{1}{x} + \dfrac{4}{y} + \dfrac{9}{z} = 3 \\ x + y + z \leq 12\,. \end{cases}$$

A6 If the real numbers a, b, c, d are such that $0 < a, b, c, d < 1$, show that

$$1 + ab + bc + cd + da + ac + bd > a + b + c + d.$$

A7 Let a, b and c be positive real numbers such that $abc = 1$. Prove the inequality

$$\left(ab + bc + \frac{1}{ca}\right)\left(bc + ca + \frac{1}{ab}\right)\left(ca + ab + \frac{1}{bc}\right) \geq (1 + 2a)(1 + 2b)(1 + 2c).$$

A8 Show that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 4\left(\frac{x}{xy+1}+\frac{y}{yz+1}+\frac{z}{zx+1}\right)^2,$$

for all real positive numbers x, y and z.

A9 Consider an integer $n \geq 4$ and a sequence of real numbers $x_1, x_2, x_3, \ldots, x_n$. An operation consists in eliminating all numbers not having the rank of the form $4k+3$, thus leaving only the numbers $x_3, x_7, x_{11}, \ldots$ (for example, the sequence $4, 5, 9, 3, 6, 6, 1, 8$ produces the sequence $9, 1$). Upon the sequence $1, 2, 3, \ldots, 1024$ the operation is performed successively for 5 times. Show that at the end only one number remains and find this number.

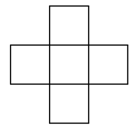
1.2 Combinatorics

C1 On a 5×5 board, n white markers are positioned, each marker in a distinct 1×1 square. A smart child got an assignment to recolor in black as many markers as possible, in the following manner: a white marker is taken from the board; it is colored in black, and then put back on the board on an empty square such that none of the neighboring squares contains a white marker (two squares are called neighboring if they share a common side). If it is possible for the child to succeed in coloring all the markers black, we say that the initial positioning of the markers was *good*.
a) Prove that if $n = 20$, then a good initial positioning exists.
b) Prove that if $n = 21$, then a good initial positioning does not exist.

C2 Kostas and Helene have the following dialogue:
Kostas: I have in my mind three positive real numbers with product 1 and sum equal to the sum of all their pairwise products.
Helene: I think that I know the numbers you have in mind. They are all equal to 1.
Kostas: In fact, the numbers you mentioned satisfy my conditions, but I did not think of these numbers. The numbers you mentioned have the minimal sum between all possible solutions of the problem.
Can you decide if Kostas is right? (Explain your answer).

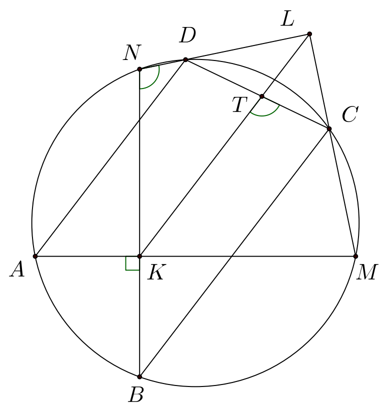
C3 Integers $1, 2, \ldots, 2n$ are arbitrarily assigned to boxes labeled with numbers $1, 2, \ldots, 2n$. Now, we add the number assigned to the box to the number on the box label. Show that two such sums give the same remainder modulo $2n$.

C4 Every cell of table 4×4 is colored into white. It is permitted to place the cross (pictured below) on the table such that its center lies on the table (the whole figure does not need to lie on the table) and change colors of every cell which is covered into opposite (white and black). Find all n such that after n steps it is possible to get the table with every cell colored black.



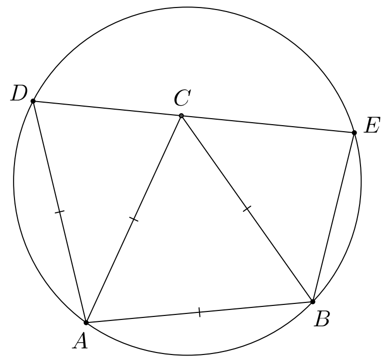
1.3 Geometry

G1 Two perpendicular chords of a circle, AM, BN, which intersect at point K, define on the circle four arcs with pairwise different length, with AB being the smallest of them. We draw the chords AD, BC with $AD \parallel BC$ and C, D different from N, M. If L is the point of intersection of DN, MC and T the point of intersection of DC, KL, prove that $\angle KTC = \angle KNL$.

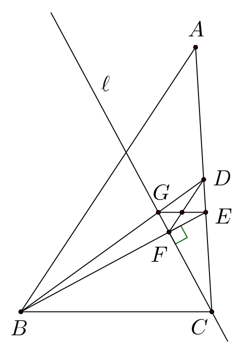


G2 For a fixed triangle ABC we choose a point M on the ray CA (after A), a point N on the ray AB (after B) and a point P on the ray BC (after C) in a way such that $AM - BC = BN - AC = CP - AB$. Prove that the angles of triangle MNP do not depend on the choice of M, N, P.

G3 The vertices A and B of an equilateral $\triangle ABC$ lie on a circle k of radius 1, and the vertex C is inside k. The point $D \neq B$ lies on k, $AD = AB$ and the line DC intersects k for the second time in point E. Find the length of the segment CE.

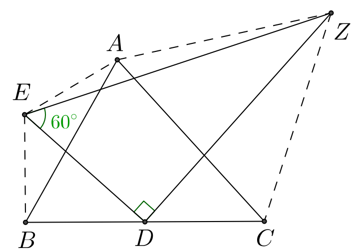


G4 Let ABC be a triangle, $(BC < AB)$. The line ℓ passing trough the vertices C and orthogonal to the angle bisector BE of $\angle B$, meets BE and the median BD of the side AC at points F and G, respectively. Prove that segment DF bisects the segment EG.



G5 Is it possible to cover a given square with a few congruent right-angled triangles with acute angle equal to 30°? (The triangles may not overlap and may not exceed the margins of the square.)

G6 Let ABC be a triangle with $A < 90^\circ$. Outside of a triangle we consider isosceles triangles ABE and ACZ with bases AB and AC, respectively. If the midpoint D of the side BC is such that $DE \perp DZ$ and $EZ = 2 \cdot ED$, prove that $\widehat{AEB} = 2 \cdot \widehat{AZC}$.



G7 Let ABC be an isosceles triangle with $AC = BC$. The point D lies on the side AB such that the semicircle with diameter BD and center O is tangent to the side AC in the point P and intersects the side BC at the point Q. The radius OP intersects the chord DQ at the point E such that $5 \cdot PE = 3 \cdot DE$. Find the ratio $\dfrac{AB}{BC}$.

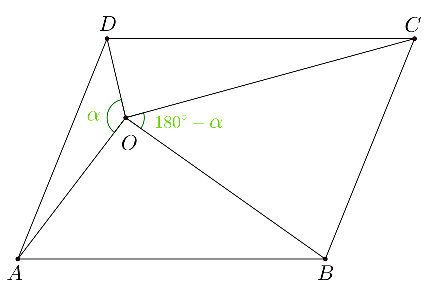
G8 The side lengths of a parallelogram are a, b and diagonals have lengths x and y, Knowing that $ab = \dfrac{xy}{2}$, show that

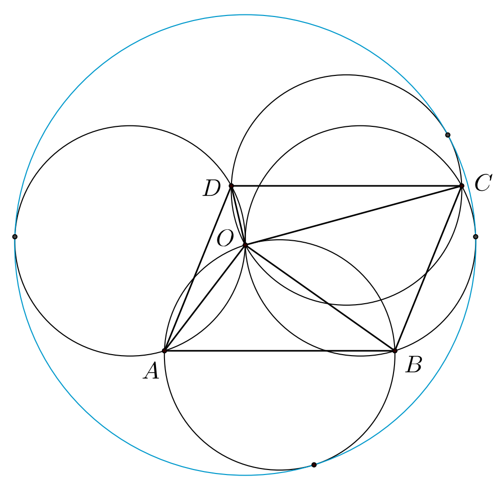
$$a = \frac{x}{\sqrt{2}}, \ b = \frac{y}{\sqrt{2}} \ \text{ or } \ a = \frac{y}{\sqrt{2}}, \ b = \frac{x}{\sqrt{2}}.$$

G9 Let O be a point inside the parallelogram $ABCD$ such that

$$\angle AOB + \angle COD = \angle BOC + \angle COD.$$

Prove that there exists a circle k tangent to the circumscribed circles of the triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$ and $\triangle DOA$.





G10 Let Γ be a circle of center O, and δ be a line in the plane of Γ, not intersecting it. Denote by A the foot of the perpendicular from O onto δ, and let M be a (variable) point on Γ. Denote by γ the circle of diameter AM, by X the (other than M) intersection point of γ and Γ, and by Y the (other than A) intersection point of γ and δ. Prove that the line XY passes through a fixed point.

G11 Consider ABC an acute-angled triangle with $AB \neq AC$. Denote by M the midpoint of BC, by D, E the feet of the altitudes from B, C respectively and let P be the intersection point of the lines DE and BC. The perpendicular from M to AC meets the perpendicular from C to BC at point R. Prove that lines PR and AM are perpendicular.

1.4 Number Theory

NT1 Find all the positive integers x and y that satisfy the equation

$$x(x-y) = 8y - 7.$$

NT2 Let $n \geq 2$ be a fixed positive integer. An integer will be called "n-free" if it is not a multiple of an n-th power of a prime. Let M be an infinite set of rational numbers, such that the product of every n elements of M is an n-free integer. Prove that M contains only integers.

NT3 Let $s(a)$ denote the sum of digits of a given positive integer a. The sequence $a_1, a_2, \ldots a_n, \ldots$ of positive integers is such that $a_{n+1} = a_n + s(a_n)$ for each positive integer n. Find the greatest possible n for which it is possible to have $a_n = 2008$.

NT4 Find all integers n such that $n^4 + 8n + 11$ is a product of two or more consecutive integers.

NT5 Is it possible to arrange the numbers $1^1, 2^2, \ldots, 2008^{2008}$ one after the other, in such a way that the obtained number is a perfect square? (Explain your answer.)

NT6 Let $f : \mathbb{N} \to \mathbb{R}$ be a function, satisfying the following condition:

for every integer $n > 1$, there exists a prime divisor p of n such that $f(n) = f\left(\frac{n}{p}\right) - f(p)$.

If

$$f(2^{2007}) + f(3^{2008}) + f(5^{2009}) = 2006,$$

determine the value of

$$f(2007^2) + f(2008^3) + f(2009^5).$$

NT7 Determine the minimal prime number $p > 3$ for which no natural number n satisfies

$$2^n + 3^n \equiv 0 \pmod{p}.$$

NT8 Let a, b, c, d, e, f are nonzero digits such that the natural numbers $\overline{abc}$, $\overline{def}$ and $\overline{abcdef}$ are squares.

a) Prove that $\overline{abcdef}$ can be represented in two different ways as a sum of three squares of natural numbers.

b) Give an example of such a number.

NT9 Let p be a prime number. Find all positive integers a and b such that:

$$\frac{4a+p}{b}+\frac{4b+p}{a}$$

and

$$\frac{a^2}{b}+\frac{b^2}{a}$$

are integers.

NT10 Prove that 2^n+3^n is not a perfect cube for any positive integer n.

NT11 Determine the greatest number with n digits in the decimal representation which is divisible by 429 and has the sum of all digits less than or equal to 11.

NT12 Solve the equation $\frac{p}{q}-\frac{4}{r+1}=1$ in prime numbers.

Chapter 2

2008 Shortlist JBMO - Solutions

2.1 Algebra

A1 If for the real numbers x, y, z, k the following conditions are valid, $x \neq y \neq z \neq x$ and $x^3 + y^3 + k(x^2 + y^2) = y^3 + z^3 + k(y^2 + z^2) = z^3 + x^3 + k(z^2 + x^2) = 2008$, find the product xyz.

Solution

$x^3 + y^3 + k(x^2 + y^2) = y^3 + z^3 + k(y^2 + z^2) \Rightarrow x^2 + xz + z^2 = -k(x + z) : (1)$ and
$y^3 + z^3 + k(y^2 + z^2) = z^3 + x^3 + k(z^2 + x^2) \Rightarrow y^2 + yx + x^2 = -k(y + x) : (2)$

- From $(1) - (2) \Rightarrow x + y + z = -k : (*)$
- If $x + z = 0$, then from $(1) \Rightarrow x^2 + xz + z^2 = 0 \Rightarrow (x + z)^2 = xz \Rightarrow xz = 0$

So $x = z = 0$, contradiction since $x \neq z$ and therefore $(1) \Rightarrow -k = \dfrac{x^2 + xz + z^2}{x + z}$

Similarly we have: $-k = \dfrac{y^2 + yx + x^2}{y + x}$.

So $\dfrac{x^2 + xz + z^2}{x + z} = \dfrac{y^2 + xy + x^2}{x + y}$ from which $xy + yz + zx = 0 : (**)$.

We substitute k in $x^3 + y^3 + k(x^2 + y^2) = 2008$ from the relation $(*)$ and using the $(**)$, we finally obtain that $2xyz = 2008$ and therefore $xyz = 1004$.

Remark: x, y, z must be the distinct real solutions of the equation $t^3 + kt^2 - 1004 = 0$. Such solutions exist if (and only if) $k > 3\sqrt[3]{251}$.

A2 Find all real numbers a, b, c, d such that $a+b+c+d = 20$ and $ab+ac+ad+bc+bd+cd = 150$.

Solution

$400 = (a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2 \cdot 150$, so $a^2 + b^2 + c^2 + d^2 = 100$. Now $(a-b)^2 + (a-c)^2 + (a-d)^2 + (b-c)^2 + (b-d)^2 + (c-d)^2 = 3(a^2 + b^2 + c^2 + d^2) - 2(ab + ac + ad + bc + bd + cd) = 300 - 300 = 0$. Thus $a = b = c = d = 5$.

A3 Let the real parameter p be such that the system

$$\begin{cases} p(x^2 - y^2) = (p^2 - 1)xy \\ |x - 1| + |y| = 1 \end{cases}$$

has at least three different real solutions. Find p and solve the system for that p.

Solution

The second equation is invariant when y is replaced by $-y$, so let us assume $y \geq 0$. It is also invariant when $x - 1$ is replaced by $-(x-1)$, so let us assume $x \geq 1$. Under these conditions the equation becomes $x + y = 2$, which defines a line on the coordinate plane. The set of points on it that satisfy the inequalities is a segment with endpoints $(1,1)$ and $(2,0)$. Now taking into account the invariance under the mentioned replacements, we conclude that the set of points satisfying the second equation is the square $\diamond$ with vertices $(1,1)$, $(2,0)$, $(1,-1)$ and $(0,0)$.

The first equation is equivalent to

$px^2 - p^2xy + xy - py^2 = 0$

$px(x - py) + y(x - py) = 0$

$(px + y)(x - py) = 0.$

Thus $y = -px$ or $x = py$. These are equations of two perpendicular lines passing through the origin, which is also a vertex of $\diamond$. If one of them passes through an interior point of the square, the other cannot have any common points with $\diamond$ other than $(0,0)$, so the system has two solutions. Since we have at least three different real solutions, the lines must contain some sides of $\diamond$, i.e. the slopes of the lines have to be 1 and -1. This happens if $p = 1$ or $p = -1$. In either case $x^2 = y^2$, $|x| = |y|$, so the second equation becomes $|1 - x| + |x| = 1$. It is true exactly when $0 \leq x \leq 1$ and $y = \pm x$.

A4 Find all triples (x, y, z) of real numbers that satisfy the system

$$\begin{cases} x + y + z = 2008 \\ x^2 + y^2 + z^2 = 6024^2 \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{2008}\,. \end{cases}$$

Solution

The last equation implies $xyz = 2008(xy + yz + zx)$, therefore $xyz - 2008(xy + yz + zx) + 2008^2(x + y + z) - 2008^3 = 0$.

$(x - 2008)(y - 2008)(z - 2008) = 0.$

Thus one of the variable equals 2008. Let this be x. Then the first equation implies $y = -z$. From the second one it now follows that $2y^2 = 6024^2 - 2008^2 = 2008^2(9 - 1) = 2 \cdot 4016^2$. Thus (x, y, z) is the triple $(2008, 4016, -4016)$ or any of its rearrangements.

A5 Find all triples (x, y, z) of real positive numbers, which satisfy the system

$$\begin{cases} \dfrac{1}{x} + \dfrac{4}{y} + \dfrac{9}{z} = 3 \\ x + y + z \leq 12\,. \end{cases}$$

Solution
If we multiply the given equation and inequality ($x > 0, y > 0, z > 0$), we have

$$\left(\frac{4x}{y}+\frac{y}{x}\right)+\left(\frac{z}{x}+\frac{9x}{z}\right)+\left(\frac{4z}{y}+\frac{9y}{z}\right)\le 22. \tag{1}$$

From AM-GM we have

$$\frac{4x}{y}+\frac{y}{x}\ge 4, \quad \frac{z}{x}+\frac{9x}{z}\ge 6, \quad \frac{4z}{y}+\frac{9y}{z}\ge 12. \tag{2}$$

Therefore

$$22\le\left(\frac{4x}{y}+\frac{y}{x}\right)+\left(\frac{z}{x}+\frac{9x}{z}\right)+\left(\frac{4z}{y}+\frac{9y}{z}\right). \tag{3}$$

Now from (1) and (3) we get

$$\left(\frac{4x}{y}+\frac{y}{x}\right)+\left(\frac{z}{x}+\frac{9x}{z}\right)+\left(\frac{4z}{y}+\frac{9y}{z}\right)=22,$$

which means that in (2), everywhere equality holds i.e. we have equality between means, also $x+y+z=12$.
Therefore $\frac{4x}{y}=\frac{y}{x}$, $\frac{z}{x}=\frac{9x}{z}$ and, as $x>0$, $y>0$, $z>0$, we get $y=2x$, $z=3x$. Finally if we substitute for y and z, in $x+y+z=12$, we get $x=2$, therefore $y=2\cdot 2=4$ and $z=3\cdot 2=6$.
Thus the unique solution is $(x,y,z)=(2,4,6)$.

A6 If the real numbers a,b,c,d are such that $0<a,b,c,d<1$, show that

$$1+ab+bc+cd+da+ac+bd>a+b+c+d.$$

Solution
If $1\ge a+b+c$ then we write the given inequality equivalently as

$$1-(a+b+c)+d[(a+b+c)-1]+ab+bc+ca>0$$

$$\Leftrightarrow [1-(a+b+c)](1-d)+ab+bc+ca>0$$

which is of course true.
If instead $a+b+c>1$, then $d(a+b+c)>d$ i.e.

$$da+db+dc>d. \tag{1}$$

We are going to prove that also

$$1+ab+bc+ca>a+b+c \tag{2}$$

thus adding (1) and (2) together we'll get the desired result in this case too.

For the truth of (2):
If $1 \geq a+b$, then we rewrite (2) equivalently as

$$1-(a+b)+c[(a+b)-1]+ab>0$$

$$\Leftrightarrow [1-(a+b)](1-c)+ab>0$$

which is of course true.
If instead $a+b>1$, then $c(a+b)>c$, i.e.

$$ca+cb>c \tag{3}$$

But it is also true that

$$1+ab>a+b \tag{4}$$

because this is equivalent to $(1-a)+b(a-1)>0$, i.e. to $(1-a)(1-b)>0$ which holds. Adding (3) and (4) together we get the truth of (2) in this case too and we are done. You can instead consider the following generalization:
Exercise. If for the real numbers $x_1, x_2, \ldots, x_n$ it is $0<x_i<1$, for any i, show that

$$1+\sum_{1\leq i<j\leq n} x_i x_j > \sum_{i=1}^{n} x_i.$$

Solution
We'll prove it by induction.
For $n=1$ the desired result becomes $1>x_1$ which is true.
Let the result be true for some natural number $n \geq 1$.
We'll prove it to be true for $n+1$ as well, and we'll be done.
So let $x_1, x_2, \ldots, x_n, x_{n+1}$ be $n+1$ given real numbers with $0<x_i<1$, for any i. We wish to show that

$$1+\sum_{1\leq i<j\leq n+1} x_i x_j > x_1+x_2+\ldots+x_n+x_{n+1}. \tag{5}$$

If $1 \geq x_1+x_2+\ldots+x_n$ then we rewrite (5) equivalently as

$$1-(x_1+x_2+\ldots+x_n)+x_{n+1}(x_1+x_2+\ldots+x_n-1)+\sum_{1\leq i<j\leq n} x_i x_j > 0.$$

This is also written as

$$(1-x_{n+1})[1-(x_1+x_2+\ldots+x_n)]+\sum_{1\leq i<j\leq n} x_i x_j > 0$$

which is clearly true.
If instead $x_1+x_2+\ldots+x_n>1$ then $x_{n+1}(x_1+x_2+\ldots+x_n)>x_{n+1}$, i.e.

$$x_{n+1}x_1+x_{n+1}x_2+\ldots+x_{n+1}x_n > x_{n+1}. \tag{6}$$

By the induction hypothesis applied to the n real numbers $x_1, x_2, \dots, x_n$ we also know that

$$1 + \sum_{1 \le i < j \le n} x_i x_j > \sum_{i=1}^{n} x_i. \tag{7}$$

Adding (6) and (7) together we get the validity of (5) in this case too, and we are done.

You can even consider the following variation:

Exercise. If the real numbers $x_1, x_2, \dots, x_{2008}$ are such that $0 < x_i < 1$, for any i, show that

$$1 + \sum_{1 \le i < j \le 2008} x_i x_j > \sum_{i=1}^{2008} x_i.$$

Remark: Inequality (2) follows directly from $(1-a)(1-b)(1-c) > 0 \Leftrightarrow 1 - a - b - c + ab + bc + ca > abc > 0$.

A7 Let a, b and c be a positive real numbers such that $abc = 1$. Prove the inequality

$$\left(ab + bc + \frac{1}{ca}\right)\left(bc + ca + \frac{1}{ab}\right)\left(ca + ab + \frac{1}{bc}\right) \ge (1+2a)(1+2b)(1+2c).$$

Solution 1
By Cauchy-Schwarz inequality and $abc = 1$ we get

$$\sqrt{\left(bc + ca + \frac{1}{ab}\right)\left(ab + bc + \frac{1}{ca}\right)} = \sqrt{\left(bc + ca + \frac{1}{ab}\right)\left(\frac{1}{ca} + ab + bc\right)} \ge$$

$$\left(\sqrt{ab} \cdot \sqrt{\frac{1}{ab}} + \sqrt{bc} \cdot \sqrt{bc} + \sqrt{\frac{1}{ca}} \cdot \sqrt{ca}\right) = (2 + bc) = (2abc + bc) = bc(1+2a)$$

Analogously we get $\sqrt{\left(bc + ca + \frac{1}{ab}\right)\left(ca + ab + \frac{1}{bc}\right)} \ge ca(1+2b)$ and
$\sqrt{\left(ca + ab + \frac{1}{bc}\right)\left(ab + bc + \frac{1}{ca}\right)} \ge ab(1+2a)$.
Multiplying these three inequalities we get:

$$\left(ab + bc + \frac{1}{ca}\right)\left(bc + ca + \frac{1}{ab}\right)\left(ca + ab + \frac{1}{bc}\right) \ge a^2b^2c^2(1+2a)(1+2b)(1+2c) =$$

$(1+2a)(1+2b)(1+2c)$ because $abc = 1$.
Equality holds if and only if $a = b = c = 1$.

Solution 2
Using $abc=1$ we get

$$\left(ab+bc+\frac{1}{ca}\right)\left(bc+ca+\frac{1}{ab}\right)\left(ca+ab+\frac{1}{bc}\right)=$$

$$=\left(\frac{1}{c}+\frac{1}{a}+b\right)\left(\frac{1}{a}+\frac{1}{b}+c\right)\left(\frac{1}{b}+\frac{1}{c}+a\right)=$$

$$=\frac{(a+c+abc)}{ac}\cdot\frac{(b+a+abc)}{ab}\cdot\frac{(b+c+abc)}{bc}=(a+b+1)(b+c+1)(c+a+1).$$

Thus, we need to prove

$$(a+b+1)(b+c+1)(c+a+1)\geq(1+2a)(1+2b)(1+2c).$$

After multiplication and using the fact $abc=1$ we have to prove

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+3(ab+bc+ca)+2(a+b+c)+a^2+b^2+c^2+3\geq$$

$$\geq 4(ab+bc+ca)+2(a+b+c)+9.$$

So we need to prove

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+a^2+b^2+c^2\geq ab+bc+ca+6$$

This follows from the well-known (AM-GM inequality) inequalities

$$a^2+b^2+c^2\geq ab+bc+ca$$

and

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b\geq 6abc=6.$$

A8 Show that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 4\left(\frac{x}{xy+1}+\frac{y}{yz+1}+\frac{z}{zx+1}\right)^2,$$

for any real positive numbers x, y and z.
Solution
The idea is to split the inequality in two, showing that

$$\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{x}}\right)^2$$

can be intercalated between the left-hand side and the right-hand side.
Indeed, using the Cauchy-Schwarz inequality one has

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{x}}\right)^2.$$

On the other hand, as

$$\sqrt{\frac{x}{y}} \geq \frac{2x}{xy+1} \Leftrightarrow (\sqrt{xy}-1)^2 \geq 0$$

by summation one has

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{z}} + \sqrt{\frac{z}{x}} \geq \frac{2x}{xy+1} + \frac{2y}{yz+1} + \frac{2z}{zx+1}.$$

The rest is obvious.

A9 Consider an integer $n \geq 4$ and a sequence of real numbers $x_1, x_2, x_3, \ldots, x_n$. An operation consists in eliminating all numbers not having the rank of the form $4k+3$, thus leaving only the numbers $x_3, x_7, x_{11}, \ldots$ (for example, the sequence $4, 5, 9, 3, 6, 6, 1, 8$ produces the sequence $9, 1$). Upon the sequence $1, 2, 3, \ldots, 1024$ the operation is performed successively for 5 times. Show that at the end only 1 number remains and find this number.
Solution
After the first operation 256 number remain; after the second one, 64 are left, then 16, next 4 and ultimately only one number.
Notice that the 256 numbers left after the first operation are 3, 7, ..., 1023, hence they are in arithmetical progression of common difference 4. Successively, the 64 numbers left after the second operation are in arithmetical progression of ratio 16 and so on.
Let a_1, a_2, a_3, a_4, a_5 be the first term in the 5 sequences obtained after each of the 5 operations. Thus $a_1 = 3$ and a_5 is the requested number. The sequence before the fifth operation has 4 numbers, namely

$$a_4, a_4+256, a_4+512, a_4+768$$

and $a_5 = a_4 + 512$. Similarly, $a_4 = a_3 + 128$, $a_3 = a_2 + 32$, $a_2 = a_1 + 8$.
Summing up yields $a_5 = a_1 + 8 + 32 + 128 + 512 = 3 + 680 = 683$.

2.2 Combinatorics

C1 On a 5×5 board, n white markers are positioned, each marker in a distinct 1×1 square. A smart child got an assignment to recolor in black as many markers as possible, in the following manner: a white marker is taken from the board; it is colored in black, and then put back on the board on an empty square such that none of the neighboring squares contains a white marker (two squares are called neighboring if they contain a common side). If it is possible for the child to succeed in coloring all the markers black, we say that the initial positioning of the markers was *good*.
a) Prove that if $n = 20$, then a good initial positioning exists.
b) Prove that if $n = 21$, then a good initial positioning does not exist.
Solution
a) Position 20 white markers on the board such that the left-most column is empty. This

positioning is good because the coloring can be realized column by column, starting with the second (from left), then the third, and so on, so that the white marker on position (i,j) after the coloring is put on position $(i,j-1)$.
b) Suppose there exists a good positioning with 21 white markers on the board i.e. there exists a re-coloring of them all, one by one. In any moment when there are 21 markers on the board, there must be at least one column completely filled with markers, and there must be at least one row completely filled with markers. So, there exists a "cross" of markers on the board. At the initial position, each such cross is completely white, at the final position each such cross is completely black, and at every moment when there are 21 markers on the board, each such cross is monochromatic. But this cannot be, since every two crosses have at least two common squares and therefore it is not possible for a white cross to vanish and for a black cross to appear by re-coloring of only one marker. Contradiction!

C2 Kostas and Helene have the following dialogue:
Kostas: I have in my mind three positive real numbers with product 1 and sum equal to the sum of all their pairwise products.
Helene: I think that I know the numbers you have in mind. They are all equal to 1.
Kostas: In fact, the numbers you mentioned satisfy my conditions, but I did not think of these numbers. The numbers you mentioned have the minimal sum between all possible solutions of the problem.
Can you decide if Kostas is right? (Explain your answer).
Solution
Kostas is right according to the following analysis:
If x, y, z are the three positive real numbers Kostas thought about, then they satisfy the following equations:

$$xy + yz + zx = x + y + z \tag{1}$$

$$xyz = 1. \tag{2}$$

Subtracting (1) from (2) by parts we obtain

$$xyz - (xy + yz + zx) = 1 - (x + y + z)$$

$$\Leftrightarrow xyz - xy - yz - zx + x + y + z - 1 = 0$$

$$\Leftrightarrow xy(z-1) - x(z-1) - y(z-1) + (z-1) = 0$$

$$\Leftrightarrow (z-1)(xy - x - y + 1) = 0$$

$$(z-1)(x-1)(y-1) = 0$$

$$\Leftrightarrow x = 1 \text{ or } y = 1 \text{ or } z = 1.$$

For $x = 1$, from (1) and (2) we have the equation $yz = 1$, which has the solutions

$$(y, z) = \left(a, \frac{1}{a}\right), a > 0,$$

And therefore the solutions of the problem are the triples

$$(x, y, z) = \left(1, a, \frac{1}{a}\right), \ a > 0.$$

Similarly, considering $y = 1$ or $z = 1$ we get the solutions

$$(x, y, z) = \left(a, 1, \frac{1}{a}\right) \text{ or } (x, y, z) = \left(a, \frac{1}{a}, 1\right), \ a > 0.$$

Since for each $a > 0$ we have

$$x + y + z = 1 + a + \frac{1}{a} \geq 1 + 2 = 3$$

and equality is valid only for $a = 1$, we conclude that among the solutions of the problem, the triple $(x, y, z) = (1, 1, 1)$ is the one whose sum $x + y + z$ is minimal.

C3 Integers $1, 2, \ldots, 2n$ are arbitrarily assigned to boxes labeled with numbers $1, 2, \ldots, 2n$. Now, we add the number assigned to the box to the number on the box label. Show that two such sums give the same remainder modulo $2n$.
Solution
Let us assume that all sums give different remainder modulo $2n$, and let S denote the value of their sum.
For our assumption,

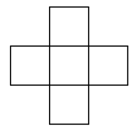
$$S \equiv 0 + 1 + \ldots + 2n - 1 = \frac{(2n-1)2n}{2} = (2n-1)n \equiv n \pmod{2n}.$$

But, if we sum, breaking all sums into its components, we derive

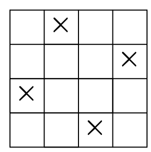
$$S \equiv 2(1 + \ldots + 2n) = 2 \cdot \frac{2n(2n+1)}{2} = 2n(2n+1) \equiv 0 \pmod{2n}.$$

From the last two conclusions we derive $n \equiv 0 \pmod{2n}$. Contradiction.
Therefore, there are two sums with the same remainder modulo $2n$.
Remark: The result is no longer true if one replaces $2n$ by $2n + 1$. Indeed, one could assign the number k to the box labeled k, thus obtaining the sums $2k$, $k = \overline{1, 2n+1}$. Two such numbers give different remainders when divided by $2n + 1$.

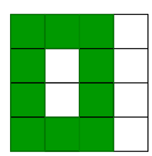
C4 Every cell of table 4×4 is colored into white. It is permitted to place the cross (pictured below) on the table such that its center lies on the table (the whole figure does not need to lie on the table) and change colors of every cell which is covered into opposite (white and black). Find all n such that after n steps it is possible to get the table with every cell colored black.



Solution
The cross covers at most five cells so we need at least 4 steps to change the color of every cell. If we place the cross 4 times such that its center lies in the cells marked below, we see that we can turn the whole square black in $n = 4$ moves.



Furthermore, applying the same operation twice (,,do and undo"), we get that is possible to turn all the cells black in n steps for every even $n \geq 4$.
We shall prove that for odd n it is not possible to do that. Look at the picture below.



Let k be a difference between white and black cells in the green area in picture. Every figure placed on the table covers an odd number of green cells, so after every step k is changed by a number $\equiv 2 \pmod 4$. At the beginning $k = 10$, at the end $k = -10$. From this it is clear that we need an even number of steps.
Solution for n is: every even number except 2.

2.3 Geometry

G1 Two perpendicular chords of a circle, AM, BN, which intersect at point K, define on the circle four arcs with pairwise different length, with AB being the smallest of them. We draw the chords AD, BC with $AD \parallel BC$ and C, D different from N, M. If L is the point of intersection of DN, MC and T the point of intersection of DC, KL, prove that $\angle KTC = \angle KNL$.

Solution
First we prove that $NL \perp MC$. The arguments depend slightly on the position of D. The other cases are similar.
From the cyclic quadrilaterals $ADCM$ and $DNBC$ we have:

$$\sphericalangle DCL = \sphericalangle DAM \text{ and } \sphericalangle CDL = \sphericalangle CBN.$$

So we obtain
$$\sphericalangle DCL + \sphericalangle CDL = \sphericalangle DAM + \sphericalangle CBN.$$

And because $AD \parallel BC$, if Z the point of intersection of AM, BC then $\sphericalangle DAM = \sphericalangle BZA$, and we have
$$\sphericalangle DCL + \sphericalangle CDL = \sphericalangle BZA + \sphericalangle CBN = 90^\circ.$$

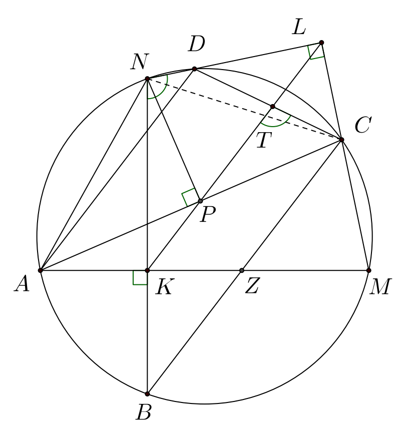
Let P the point of intersection of KL, AC, then $NP \perp AC$, because the line KPL is a Simson line of the point N with respect to the triangle ACM.
From the cyclic quadrilaterals $NPCL$ and $ANDC$ we obtain:

$$\sphericalangle CPL = \sphericalangle CNL \text{ and } \sphericalangle CNL = \sphericalangle CAD,$$

so $\sphericalangle CPL = \sphericalangle CAD$, that is $KL \parallel AD \parallel BC$ therefore $\sphericalangle KTC = \sphericalangle ADC$ (1).
But $\sphericalangle ADC = \sphericalangle ANC = \sphericalangle ANK + \sphericalangle KNC = \sphericalangle CNL + \sphericalangle KNC$, so

$$\sphericalangle ADC = \sphericalangle KNL \qquad (2).$$

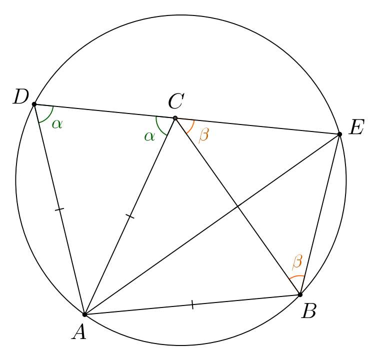
From (1) and (2) we obtain the result.



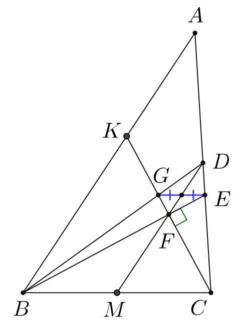
G2 For a fixed triangle ABC we choose a point M on the ray CA (after A), a point N on the ray AB (after B) and a point P on the ray BC (after C) in a way such that $AM - BC = BN - AC = CP - AB$. Prove that the angles of triangle MNP do not depend on the choice of M, N, P.

Solution
Consider the points M' on the ray BA (after A), N' on the ray CB (after B) and P' on the ray AC (after C), so that $AM = AM'$, $BN = BN'$, $CP = CP'$. Since $AM - BC = BN - AC = BN' - AC$, we get $CM = AC + AM = BC + BN' = CN'$. Thus triangle MCN' is isosceles, so the perpendicular bisector of $[MN']$ bisects angle ACB and hence passes through the incenter I of triangle ABC. Arguing similarly, we may conclude that I lies also on the perpendicular bisectors of $[NP']$ and $[PM']$. On the other side, I clearly lies on the perpendicular bisectors of $[MM']$, $[NN']$ and $[PP']$. Thus the hexagon $M'MN'NP'P$ is cyclic. Then angle PMN equals angle $PN'N$, which measures $90^\circ - \dfrac{\beta}{2}$ (the angles of triangle ABC are α, β, γ). In the same way angle MNP measures $90^\circ - \dfrac{\gamma}{2}$ and angle MPN measures $90^\circ - \dfrac{\alpha}{2}$.

G3 The vertices A and B of an equilateral $\triangle ABC$ lie on a circle k of radius 1, and the vertex C is inside k. The point $D \neq B$ lies on k, $AD = AB$ and the line DC intersects k for the second time in point E. Find the length of the segment CE.
Solution
As $AD = AC$, $\triangle CDA$ is isosceles. If $\sphericalangle ADC = \sphericalangle ACD = \alpha$ and $\sphericalangle BCE = \beta$, then $\beta = 120^\circ - \alpha$. The quadrilateral $ABED$ is cyclic, so $\sphericalangle ABE = 180^\circ - \alpha$. Then $\sphericalangle CBE = 120^\circ - \alpha$ so $\sphericalangle CBE = \beta$. Thus $\triangle CBE$ is isosceles, so AE is the perpendicular bisector of BC, so it bisects $\sphericalangle BAC$. Now the arc BE is intercepted by a 30° inscribed angle, so it measures 60°. Then BE equals the radius of k, namely 1. Hence $CE = BE = 1$.



G4 Let ABC be a triangle, ($BC < AB$). The line ℓ passing trough the vertices C and orthogonal to the angle bisector BE of $\angle B$, meets BE and the median BD of the side AC at points F and G, respectively. Prove that segment DF bisect the segment EG.



Solution

Let $CF \cap AB = \{K\}$ and $DF \cap BC = \{M\}$. Since $BF \perp KC$ and BF is angle bisector of $\sphericalangle KBC$, we have that $\triangle KBC$ is isosceles i.e. $BK = BC$, also F is midpoint of KC. Hence DF is midline for $\triangle ACK$ i.e. $DF \parallel AK$, from where it is clear that M is a midpoint of BC.

We will prove that $GE \parallel BC$. It is sufficient to show $\dfrac{BG}{GD} = \dfrac{CE}{ED}$. From $DF \parallel AK$ and $DF = \dfrac{AK}{2}$ we have

$$\frac{BG}{GD} = \frac{BK}{DF} = \frac{2BK}{AK} \tag{1}$$

Also

$$\frac{CE}{DE} = \frac{CD-DE}{DE} = \frac{CD}{DE} - 1 = \frac{AD}{DE} - 1 = \frac{AE-DE}{DE} - 1 = \frac{AE}{DE} - 2 =$$

$$= \frac{AB}{DF} - 2 = \frac{AK+BK}{\frac{AK}{2}} - 2 = 2 + 2\frac{BK}{AK} - 2 = \frac{2BK}{AK}. \tag{2}$$

From (1) and (2) we have $\dfrac{BG}{GD} = \dfrac{CE}{ED}$, so $GE \parallel BC$, as M is the midpoint of BC, it follows that the segment DF, bisects the segment GE.

G5 Is it possible to cover a given square with a few congruent right-angled triangles with acute angle equal to $30°$? (The triangles may not overlap and may not exceed the margins of the square.)

Solution

We will prove that desired covering is impossible.

Let assume the opposite i.e. a square with side length a, can be tiled with k congruent right angled triangles, whose sides are of lengths b, $b\sqrt{3}$ and $2b$.

Then the area of such a triangle is $\dfrac{b^2\sqrt{3}}{2}$.

And the area of the square is

$$S_{sq} = kb^2\frac{\sqrt{3}}{2}. \tag{1}$$

Furthermore, the length of the side of the square, a, is obtained by the contribution of an integer number of length b, $2b$ and $b\sqrt{3}$, hence

$$a = mb\sqrt{3} + nb,$$

where $m, n \in \mathbb{N} \cup \{0\}$, and at least one of the numbers m and n is different from zero.
So the area of the square is

$$S_{sq} = a^2 = (mb\sqrt{3} + nb)^2 = b^2(3m^2 + n^2 + 2\sqrt{3}mn). \tag{2}$$

Now because of (1) and (2) it follows $3m^2 + n^2 + 2\sqrt{3}mn = k\frac{\sqrt{3}}{2}$ i.e.

$$6m^2 + 2n^2 = (k - 4mn)\sqrt{3} \tag{3}$$

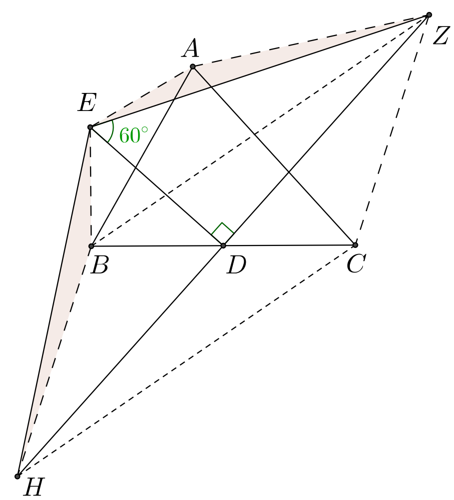
Because of $3m^2 + n^2 \neq 0$ and from the equality (3) it follows $4mn \neq k$.
Using once more (3), we get

$$\sqrt{3} = \frac{6m^2 + 2n^2}{k - 4mn},$$

which contradicts at the fact that $\sqrt{3}$ is irrational, because $\dfrac{6m^2 + 2n^2}{k - 4mn}$ is a rational number.
Finally, we have obtained a contradiction, which proves that the desired covering is impossible.
Remark.
This problem has been given in Russian Mathematical Olympiad 1993 - 1995 for 9-th Grade.

G6 Let ABC be a triangle with $A < 90^\circ$. Outside of a triangle we consider isosceles triangles ABE and ACZ with bases AB and AC, respectively. If the midpoint D of the side BC is such that $DE \perp DZ$ and $EZ = 2 \cdot ED$, prove that $\widehat{AEB} = 2 \cdot \widehat{AZC}$.
Solution
Since D is the midpoint of the side BC, in the extension of the line segment ZD we take a point H such that $ZD = DH$. Then the quadrilateral $BHCZ$ is parallelogram and therefore we have

$$BH = ZC = ZA. \tag{1}$$



Also from the isosceles triangle ABE we get

$$BE = AE. \tag{2}$$

Since $DE \perp DZ$, ED is altitude and median of the triangle EZH and so this triangle is isosceles with

$$EH = EZ. \tag{3}$$

From (1), (2) and (3) we conclude that the triangles BEH and AEZ are equal. Therefore they have also

$$\widehat{BEH} = \widehat{AEZ},\ \widehat{EBH} = \widehat{EAZ} \text{ and } \widehat{EHB} = \widehat{AZE}. \tag{4}$$

Putting $\widehat{EBA} = \widehat{EAB} = \omega$, $\widehat{ZAC} = \widehat{ZCA} = \varphi$, then we have $\widehat{CBH} = \widehat{BCZ} = \widehat{C} + \varphi$, and therefore from the equality $\widehat{EBH} = \widehat{EAZ}$ we receive:

$$360^\circ - \widehat{EBA} - \widehat{B} - \widehat{CBH} = \widehat{EAB} + \widehat{A} + \widehat{ZAC}$$

$$\Rightarrow 360^\circ - \widehat{B} - \omega - \varphi - \widehat{C} = \omega + \widehat{A} + \varphi$$

$$\Rightarrow 2(\omega + \varphi) = 360^\circ - \left(\widehat{A} + \widehat{B} + \widehat{C}\right)$$

$$\Rightarrow \omega + \varphi = 90^\circ$$

$$\Rightarrow \frac{180^\circ - \widehat{AEB}}{2} + \frac{180^\circ - \widehat{AZC}}{2} = 90^\circ$$

$$\Rightarrow \widehat{AEB} + \widehat{AZC} = 180^\circ. \tag{5}$$

From the supposition $EZ = 2 \cdot ED$, we get that the right triangle ZEH has $\widehat{EZD} = 30°$ and $\widehat{ZED} = 60°$. Thus we have $\widehat{ZEH} = 120°$.
However, since we have proved that $\widehat{BEH} = \widehat{AEZ}$, we get that

$$\widehat{AEB} = \widehat{AEZ} + \widehat{ZEB} = \widehat{ZEB} + \widehat{BEH} = \widehat{ZEH} = 120°. \qquad (6)$$

From (5) and (6) we obtain that $\widehat{AZC} = 60°$ and thus $\widehat{AEB} = 2 \cdot \widehat{AZC}$.

G7 Let ABC be an isosceles triangle with $AC = BC$. The point D lies on the side AB such that the semicircle with diameter $[BD]$ and center O is tangent to the side AC in the point P and intersects the side BC at the point Q. The radius OP intersects the chord DQ at the point E such that $5 \cdot PE = 3 \cdot DE$. Find the ratio $\frac{AB}{BC}$.

Solution

We denote $OP = OD = OB = R$, $AC = BC = b$ and $AB = 2a$. Because $OP \perp AC$ and $DQ \perp BC$, then the right triangles APO and BQD are similar and $\sphericalangle BDQ = \sphericalangle AOP$. So, the triangle DEO is isosceles with $DE = OE$. It follows that

$$\frac{PE}{DE} = \frac{PE}{OE} = \frac{3}{5}.$$

Let F and G are the orthogonal projections of the points E and P respectively on the side AB and M is the midpoint of the side $[AB]$. The triangles OFE, OGP, OPA and CMA are similar. We obtain the following relations

$$\frac{OF}{OE} = \frac{OG}{OP} = \frac{CM}{AC} = \frac{OP}{OA}.$$

But $CM = \sqrt{b^2 - a^2}$ and we have $OG = \frac{R}{b} \cdot \sqrt{b^2 - a^2}$. In isosceles triangle DEO the point F is the midpoint of the radius DO. So, $OF = R/2$. By using Thales' theorem we obtain

$$\frac{3}{5} = \frac{PE}{OE} = \frac{GF}{OF} = \frac{OG - OF}{OF} = \frac{OG}{OF} - 1 = 2 \cdot \sqrt{1 - \left(\frac{a}{b}\right)^2} - 1.$$

From the last relations it is easy to obtain that $\frac{a}{b} = \frac{3}{5}$ and $\frac{AB}{BC} = \frac{6}{5}$.
The problem is solved.

G8 The side lengths of a parallelogram are a, b and diagonals have lengths x and y, Knowing that $ab = \frac{xy}{2}$, show that

$$a = \frac{x}{\sqrt{2}}, \ b = \frac{y}{\sqrt{2}} \ \text{ or } \ a = \frac{y}{\sqrt{2}}, \ b = \frac{x}{\sqrt{2}}.$$

Solution 1.

Let us consider a parallelogram $ABCD$, with $AB = a$, $BC = b$, $AC = x$, $BD = y$, $\widehat{AOD} = \theta$.
For the area of $ABCD$ we know $(ABCD) = ab \sin A$.
But it is also true that $(ABCD) = 4(AOD) = 4 \cdot \dfrac{OA \cdot OD}{2} \sin\theta = 2OA \cdot OD \sin\theta =$
$= 2 \cdot \dfrac{x}{2} \cdot \dfrac{y}{2} \sin\theta = \dfrac{xy}{2} \sin\theta$. So $ab \sin A = \dfrac{xy}{2} \sin\theta$ and since $ab = \dfrac{xy}{2}$ by hypothesis, we get

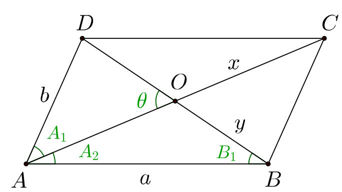
$$\sin A = \sin\theta.$$

Thus

$$\theta = \widehat{A} \text{ or } \theta = 180^\circ - \widehat{A} = \widehat{B}$$

If $\theta = A$ then (see Figure below) $A_2 + B_1 = A_1 + A_2$, so $B_1 = A_1$ which implies that AD is tangent to the circumcircle of triangle OAB. So

$$DA^2 = DO \cdot DB \Rightarrow b^2 = \frac{y}{2} \cdot y \Rightarrow b = \frac{y}{\sqrt{2}}.$$



Then by $ab = \dfrac{xy}{2}$ we get $a = \dfrac{x}{\sqrt{2}}$.
If $\theta = B$ we similarly get $a = \dfrac{x}{\sqrt{2}}$, $b = \dfrac{y}{\sqrt{2}}$.

Solution 2.

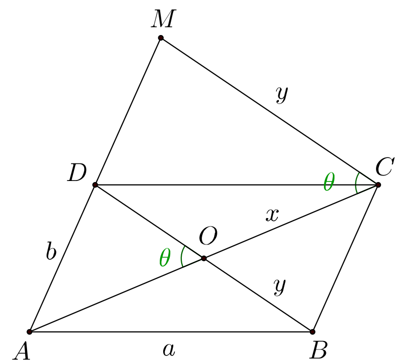
Let us consider a parallelogram $ABCD$, with $AB = a$, $BC = b$, $AC = x$, $BD = y$, $\widehat{BOC} = \theta$, and let us produce the line AD towards D and consider $M \in (AD$ so that $AD = DM$. Then $BCMD$ is a parallelogram, so $CM = BD = y$.
Observe also that $(ABCD) = 2(ACD) = (ACM)$ which is written equivalently as

$$CB \cdot CD \cdot \sin C = \frac{AC \cdot CM \cdot \sin\theta}{2} \text{ i.e. } ab \sin C = \frac{xy \sin\theta}{2}.$$

Because of the given relation $ab = \dfrac{xy}{2}$ the last relation becomes $\sin C = \sin\theta$, i.e.

$$\theta = \widehat{C} \text{ or } \theta = 180^\circ - \widehat{C} = \widehat{B}.$$

If $\theta = \widehat{C}$, then the triangles ACM and BCD are similar because their angles at C are equal, as well as their angles at B, M (remember $BCMD$ is a parallelogram).



Then

$$\frac{b}{y} = \frac{a}{x} = \frac{y}{2b} \Rightarrow \left(b = \frac{y}{2},\ a = \frac{x}{2} \right).$$

If $\theta = \widehat{B}$, then similarly we prove that the triangles ACM and ACD are similar, which then implies

$$\frac{a}{y} = \frac{b}{x} = \frac{x}{2b} \Rightarrow \left(a = \frac{y}{2},\ b = \frac{x}{2} \right).$$

Solution 3.

The *Parallelogram Law* states that, in any parallelogram, the sum of the squares of its diagonals is equal to the sum of the squares of its sides.

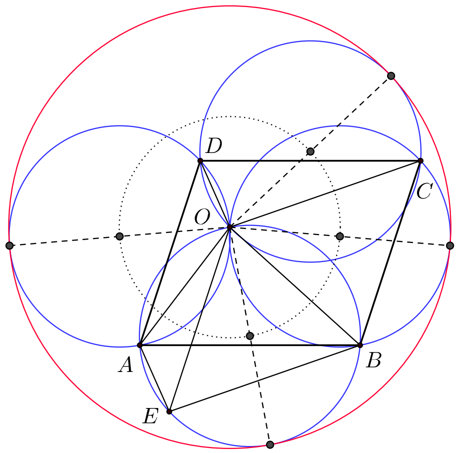
In our case, this translates to $x^2+y^2 = 2(a^2+b^2)$. First adding $2xy = 4ab$, then subtracting the same equality, yields $(x + y)^2 = 2(a + b)^2$ and $(x - y)^2 = 2(a - b)^2$. It follows that $x + y = a\sqrt{2} + b\sqrt{2}$ and either $x - y = a\sqrt{2} - b\sqrt{2}$, or $x - y = b\sqrt{2} - a\sqrt{2}$. In the first case one obtains $x = a\sqrt{2}$, $y = b\sqrt{2}$, in the latter case, $x = b\sqrt{2}$, $y = a\sqrt{2}$.

For the proof of the Parallelogram Law, simply apply the Law of cosines in triangles ABC and ABD and use the fact that $\cos(\sphericalangle ABC) = -\cos(\sphericalangle BAD)$. Adding the two relations gives the desired condition.

G9 Let O be a point inside the parallelogram $ABCD$ such that

$$\angle AOB + \angle COD = \angle BOC + \angle COD.$$

Prove that there exists a circle k tangent to the circumscribed circles of the triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$ and $\triangle DOA$.



Solution

From given condition it is clear that $\sphericalangle AOB + \sphericalangle COD = \sphericalangle BOC + \sphericalangle AOD = 180^\circ$.

Let E be a point such that $AE = DO$ and $BE = CE$. Clearly, $\triangle AEB \equiv \triangle DOC$ and from that $AE \parallel DO$ and $BE \parallel CO$. Also, $\sphericalangle AEB = \sphericalangle COD$ so $\sphericalangle AOB + \sphericalangle AEB = \sphericalangle AOB + \sphericalangle COD = 180^\circ$. Thus, the quadrilateral $AOBE$ is cyclic.

So $\triangle AOB$ and $\triangle AEB$ the same circumcircle, therefor the circumcircles of the triangles $\triangle AOB$ and $\triangle COD$ have the same radius.

Also, $AE \parallel DO$ and $AE = DO$ gives $AEOD$ is parallelogram and $\triangle AOD \equiv \triangle OAE$. So $\triangle AOB$, $\triangle COD$ and $\triangle DOA$ has the same radius of their circumcircle (the radius of the cyclic quadrilateral $AEBO$). Analogously, triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$ and $\triangle DOA$ has same radius R.

Obviously, the circle with center O and radius $2R$ is externally tangent to each of these circles, so this will be the circle k.

G10 Let Γ be a circle of center O, and δ be a line in the plane of Γ, not intersecting it. Denote by A the foot of the perpendicular from O onto δ, and let M be a (variable) point on Γ. Denote by γ the circle of diameter AM, by X the (other than M) intersection point of γ and Γ, and by Y the (other than A) intersection point of γ and δ. Prove that the line XY passes through a fixed point.

Solution

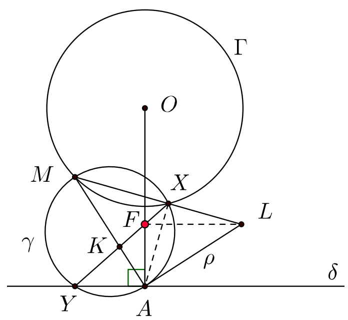
Consider the line ρ tangent to γ at A, and take the points $\{K\} = AM \cap XY$, $\{L\} = \rho \cap XM$, and $\{F\} = OA \cap XY$.

(*Remark:* Moving M into its reflection with respect to the line OA will move XY into its reflection with respect to OA. These old and the new XY meet on OA, hence it should be clear that the fixed point mult be F.)

Since $\sphericalangle LMA = \sphericalangle FYA$ and $\sphericalangle YAF = \sphericalangle LAM = 90^\circ$, it follows that triangles FAY and LAM are similar, therefore $\sphericalangle AFY = \sphericalangle ALM$, hence the quadrilateral $ALXF$ is cyclic.

But then $\sphericalangle AFL = \sphericalangle AXL = 90^\circ$, so $LF \perp AF$, hence $LF \parallel \delta$.

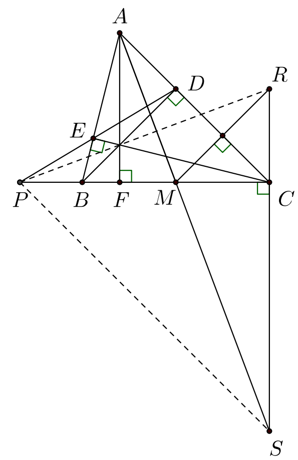
Now, ρ is the radical axis of circles γ and A (consider A as a circle of center A and radius 0), while XM is the radical axis of circles γ and Γ, so L is the radical center of the three circle, which means that L lies on the radical axis of circles Γ and A. From $LF \perp OA$, where OA is the line of the centers of the circles A and Γ, and $F \in XY$, it follows that F is (the) fixed point of XY.
(The degenerate two cases when $M \in OA$, where $X \equiv M$ and $Y \equiv A$, also trivially satisfy the conclusion, as then $F \in AM$).



G11 Consider ABC an acute-angled triangle with $AB \neq AC$. Denote by M the midpoint of BC, by D, E the feet of the altitudes from B, C respectively and let P be the intersection point of the lines DE and BC. The perpendicular from M to AC meets the perpendicular from C to BC at point R. Prove that lines PR and AM are perpendicular.
Solution
Let F be the foot of the altitude from A and let S be the intersection point of AM and RC. As PC is an altitude of the triangle PRS, the claim is equivalent to $RM \perp PS$, since the latter implies that M is the orthocenter of PRS. Due to $RM \perp AC$, we need to prove that $AC \parallel PS$, in other words

$$\frac{MC}{MP} = \frac{MA}{MS}.$$

Notice that $AF \parallel CS$, so $\frac{MA}{MS} = \frac{MF}{MC}$. Now the claim is reduced to proving $MC^2 = MF \cdot MP$, a well-known result considering that AF is the polar line of P with respect to circle of radius MC centered at M.
The "elementary proof" on the latter result may be obtained as follows: $\frac{PB}{PC} = \frac{FB}{FC}$, using, for instance, Menelaus and Ceva theorems with respect to ABC. Cross-multiplying one gets $(PM - x)(FM + x) = (x - FM)(PM + x)$
- x stands for the length of MC - and then $PM \cdot FM = x^2$.



Comment. The proof above holds for both cases $AB < AC$ and $AB > AC$; it is for the committee to decide if a contestant is supposed to (even) mention this.

2.4 Number Theory

NT1 Find all the positive integers x and y that satisfy the equation

$$x(x-y) = 8y - 7.$$

Solution 1:
The given equation can be written as:

$$x(x-y) = 8y - 7$$

$$x^2 + 7 = y(x+8)$$

Let $x + 8 = m$, $m \in \mathbb{N}$. Then we have: $x^2 + 7 \equiv 0 \pmod{m}$, and $x^2 + 8x \equiv 0 \pmod{m}$.
So we obtain that $8x - 7 \equiv 0 \pmod{m}$ (1).
Also we obtain $8x + 8^2 = 8(x+8) \equiv 0 \pmod{m}$ (2).
From (1) and (2) we obtain $(8x + 64) - (8x - 7) = 71 \equiv 0 \pmod{m}$, therefore $m \mid 71$, since 71 is a prime number, we have:
$x + 8 = 1$ or $x + 8 = 71$. The only accepted solution is $x = 63$, and from the initial equation we obtain $y = 56$.
Therefore the equation has a unique solution, namely $(x, y) = (63, 56)$.
Solution 2:
The given equation is $x^2 - xy + 7 - 8y = 0$.

Discriminant is $\Delta = y^2 + 32y - 28 = (y+16)^2 - 284$ and must be perfect square. So $(y+16)^2 - 284 = m^2$, and its follow $(y+16)^2 - m^2 = 284$, and after some casework, $y + 16 - m = 2$ and $y + 16 + m = 142$, hence $y = 56$, $x = 63$.

NT2 Let $n \geq 2$ be a fixed positive integer. An integer will be called "n-free" if it is not a multiple of an n-th power of a prime. Let M be an infinite set of rational numbers, such that the product of every n elements of M is an n-free integer. Prove that M contains only integers.

Solution

We first prove that M can contain only a finite number of non-integers. Suppose that there are infinitely many of them: $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_k}{q_k}, \ldots$, with $(p_k, q_k) = 1$ and $q_k > 1$ for each k. Let $\frac{p}{q} = \frac{p_1 p_2 \ldots p_{n-1}}{q_1 q_2 \ldots q_{n-1}}$, where $(p, q) = 1$. For each $i \geq n$, the number $\frac{p}{q} \cdot \frac{p_i}{q_i}$ is an integer, so q_i is a divisor of p (as q_i and p_i are coprime). But p has a finite set of divisors, so there are n numbers of M with equal denominators. Their product cannot be an integer, a contradiction.

Now suppose that M contains a fraction $\frac{a}{b}$ in lowest terms with $b > 1$. Take a prime divisor p of b. If we take any $n - 1$ integers from M, their product with $\frac{a}{b}$ is an integer, so some of them is a multiple of p. Therefore there are infinitely many multiples of p in M, and the product of n of them is not n-free, a contradiction.

NT3 Let $s(a)$ denote the sum of digits of a given positive integer a. The sequence $a_1, a_2, \ldots a_n, \ldots$ of positive integers is such that $a_{n+1} = a_n + s(a_n)$ for each positive integer n. Find the greatest possible n for which it is possible to have $a_n = 2008$.

Solution

Since $a_{n-1} \equiv s(a_{n-1})$ (all congruences are modulo 9), we have $2a_{n-1} \equiv a_n \equiv 2008 \equiv 10$, so $a_{n-1} \equiv 5$. But $a_{n-1} < 2008$, so $s(a_{n-1}) \leq 28$ and thus $s(a_{n-1})$ can equal 5, 14 or 23. We check $s(2008 - 5) = s(2003) = 5$, $s(2008 - 14) = s(1994) = 23$, $s(2008 - 23) = s(1985) = 23$. Thus a_{n-1} can equal 1985 or 2003. As above $2a_{n-2} \equiv a_{n-1} \equiv 5 \equiv 14$, so $a_{n-2} \equiv 7$. But $a_{n-2} < 2003$, so $s(a_{n-2}) \leq 28$ and thus $s(a_{n-2})$ can equal 16 or 25. Checking as above we see that the only possibility is $s(2003 - 25) = s(1978) = 25$. Thus a_{n-2} can be only 1978. Now $2a_{n-3} \equiv a_{n-2} \equiv 7 \equiv 16$ and $a_{n-3} \equiv 8$. But $s(a_{n-3}) \leq 27$ and thus $s(a_{n-3})$ can equal 17 or 26. The check works only for $s(1978 - 17) = s(1961) = 17$. Thus $a_{n-3} = 1961$ and similarly $a_{n-4} = 1939 \equiv 4$, $a_{n-5} = 1919 \equiv 2$ (if they exist). The search for a_{n-6} requires a residue of 1. But $a_{n-6} < 1919$, so $s(a_{n-6}) \leq 27$ and thus $s(a_{n-6})$ can be equal only to 10 or 19. The check fails for both $s(1919 - 10) = s(1909) = 19$ and $s(1919 - 19) = s(1900) = 10$. Thus $n \leq 6$ and the case $n = 6$ is constructed above (1919, 1939, 1961, 1978, 2003, 2008).

NT4 Find all integers n such that $n^4 + 8n + 11$ is a product of two or more consecutive integers.

Solution

We will prove that $n^4 + 8n + 11$ is never a multiple of 3. This is clear if n is a multiple of 3. If

n is not a multiple of 3, then $n^4+8n+11=(n^4-1)+12+8n=(n-1)(n+1)(n^2+1)+12+8n$, where $8n$ is the only term not divisible by 3. Thus $n^4+8n+11$ is never the product of three or more integers.
It remains to discuss the case when $n^4+8n+11=y(y+1)$ for some integer y. We write this as $4(n^4+8n+11)=4y(y+1)$ or $4n^4+32n+45=(2y+1)^2$. A check shows that among $n=\pm1$ and $n=0$ only $n=1$ satisfies the requirement, as $1^4+8\cdot1+11=20=4\cdot5$. Now let $|n|\geq 2$. The identities $4n^2+32n+45=(2n^2-2)^2+8(n+2)^2+9$ and $4n^4+32n+45=(2n^2+8)^2-32n(n-1)-19$ indicate that for $|n|\geq 2$, $2n^2-2<2y+1<2n^2+8$. But $2y+1$ is odd, so it can equal $2n^2\pm1$; $2n^2+3$; $2n^2+5$ or $2n^2+7$. We investigate them one by one.
If $4n^4+32n+45=(2n^2-1)^2 \Rightarrow n^2+8n+11=0 \Rightarrow (n+4)^2=5$, which is impossible, as 5 is not a perfect square.
If $4n^4+32n+45=(2n^2+1)^2 \Rightarrow n^2-8n-11=0 \Rightarrow (n-4)^2=27$ which also fails.
Also $4n^4+32n+45=(2n^2+3)^2 \Rightarrow 3n^2-8n-9=0 \Rightarrow 9n^2-24n-27=0 \Rightarrow (3n-4)^2=43$ fails.
If $4n^4+32n+45=(2n^2+5)^2 \Rightarrow 5n^2-8n=5 \Rightarrow 25n^2-40n=25 \Rightarrow (5n-4)^2=41$ which also fails.
Finally, if $4n^4+32n+45=(2n^2+7)^2$, then $28n^2-32n+4=0 \Rightarrow 4(n-1)(7n-1)=0$, whence $n=1$ that we already found. Thus the only solution is $n=1$.

NT5 Is it possible to arrange the numbers $1^1, 2^2, \ldots, 2008^{2008}$ one after the other, in such a way that the obtained number is a perfect square? (Explain your answer.)
Solution
We will use the following lemmas.
Lemma 1. If $x\in\mathbb{N}$, then $x^2\equiv 0$ or $1 \pmod 3$.
Proof: Let $x\in\mathbb{N}$, then $x=3k$, $x=3k+1$ or $x=3k+2$, hence

$$\begin{aligned} x^2 &= 9k^2\equiv 0 \pmod 3,\\ x^2 &= 9k^2+6k+1\equiv 1 \pmod 3,\\ x^2 &= 9k^2+12k+4\equiv 1 \pmod 3, \quad \text{respectively.}\end{aligned}$$

Hence $x^2\equiv 0$ or $1 \pmod 3$, for every positive integer x.

Without proof we will give the following lemma.

Lemma 2. If a is a positive integer then $a\equiv S(a) \pmod 3$, where $S(a)$ is the sum of the digits of the number a.

Further we have

$$\begin{aligned} (6k+1)^{6k+1} &= [(6k+1)^k]^6\cdot(6k+1)\equiv 1 \pmod 3\\ (6k+2)^{6k+2} &= [(6k+2)^{3k+1}]^2\equiv 1 \pmod 3\\ (6k+3)^{6k+3} &\equiv 0 \pmod 3\\ (6k+4)^{6k+4} &= [(6k+1)^{3k+2}]^2\equiv 1 \pmod 3\\ (6k+5)^{6k+5} &= [(6k+5)^{3k+2}]^2\cdot(6k+5)\equiv 2 \pmod 3\\ (6k+6)^{6k+6} &\equiv 0 \pmod 3\end{aligned} \tag{3}$$

for every $k = 1, 2, 3, \ldots$.
Let us separate the numbers $1^1, 2^2, \ldots, 2008^{2008}$ into the following six classes: $(6k+1)^{6k+1}$, $(6k+2)^{6k+2}$, $(6k+3)^{6k+3}$, $(6k+4)^{6k+4}$, $(6k+5)^{6k+5}$, $(6k+6)^{6k+6}$, $k = 1, 2, \ldots,$.
For $k = 1, 2, 3, \ldots$ let us denote by

$$s_k = (6k+1)^{6k+1} + (6k+2)^{6k+2} + (6k+3)^{6k+3} + (6k+4)^{6k+4} + (6k+5)^{6k+5} + (6k+6)^{6k+6}.$$

From (3) we have

$$s_k \equiv 1 + 1 + 0 + 1 + 2 + 0 \equiv 2 \pmod 3 \tag{4}$$

for every $k = 1, 2, 3, \ldots$.
Let A be the number obtained by writing one after the other (in some order) the numbers $1^1, 2^2, \ldots, 2008^{2008}$.
The sum of the digits, $S(A)$, of the number A is equal to the sum of the sums of digits, $S(i^i)$, of the numbers i^i, $i = 1, 2, \ldots, 2008$, and so, from Lemma 2, it follows that

$$A \equiv S(A) = S(1^1) + S(2^2) + \ldots + S(2008^{2008}) \equiv 1^1 + 2^2 + \ldots + 2008^{2008} \pmod 3.$$

Further on $2008 = 334 \cdot 6 + 4$ and if we use (3) and (4) we get

$$\begin{aligned} A &\equiv 1^1 + 2^2 + \ldots + 2008^{2008} \\ &\equiv s_1 + s_2 + \ldots + s_{334} + 2005^{2005} + 2006^{2006} + 2007^{2007} + 2008^{2008} \pmod 3 \\ &\equiv 334 \cdot 2 + 1 + 1 + 0 + 1 = 671 \equiv 2 \pmod 3. \end{aligned}$$

Finally, from Lemma 1, it follows that A can not be a perfect square.

NT6 Let $f : \mathbb{N} \to \mathbb{R}$ be a function, satisfying the following condition:
for every integer $n > 1$, there exists a prime divisor p of n such that $f(n) = f\left(\frac{n}{p}\right) - f(p)$.
If

$$f(2^{2007}) + f(3^{2008}) + f(5^{2009}) = 2006,$$

determine the value of

$$f(2007^2) + f(2008^3) + f(2009^5).$$

Solution
If $n = p$ is prime number, we have

$$f(p) = f\left(\frac{p}{p}\right) - f(p) = f(1) - f(p)$$

i.e.

$$f(p) = \frac{f(1)}{2}. \tag{1}$$

If $n = pq$, where p and q are prime numbers, then

$$f(n) = f\left(\frac{n}{p}\right) - f(p) = f(q) - f(p) = \frac{f(1)}{2} - \frac{f(1)}{2} = 0.$$

If n is a product of three prime numbers, we have

$$f(n) = f\left(\frac{n}{p}\right) - f(p) = 0 - f(p) = -f(p) = -\frac{f(1)}{2}.$$

With mathematical induction by a number of prime multipliers we shell prove that:
if n is a product of k prime numbers then

$$f(n) = (2-k)\frac{f(1)}{2}. \tag{2}$$

For $k = 1$, clearly the statement (2), holds.
Let statement (2) holds for all integers n, where n is a product of k prime numbers.
Now let n be a product of $k+1$ prime numbers. Then we have $n = n_1 p$, where n_1 is a product of k prime numbers.
So

$$f(n) = f\left(\frac{n}{p}\right) - f(p) = f(n_1) - f(p) = (2-k)\frac{f(1)}{2} - \frac{f(1)}{2} = (2-(k+1))\frac{f(1)}{2}.$$

So (2) holds for every integer $n > 1$.
Now from $f(2^{2007}) + f(3^{2008}) + f(5^{2009}) = 2006$ and because of (2) we have

$$\begin{aligned} 2006 &= f(2^{2007}) + f(3^{2008}) + f(5^{2009}) \\ &= \frac{2-2007}{2}f(1) + \frac{2-2008}{2}f(1) + \frac{2-2009}{2}f(1) = -\frac{3\cdot 2006}{2}f(1), \end{aligned}$$

i.e.

$$f(1) = -\frac{2}{3}.$$

Since

$$2007 = 3^2\cdot 223,\ 2008 = 2^3\cdot 251,\ 2009 = 7^2\cdot 41,$$

and because of (2) and (3), we get

$$\begin{aligned} f(2007^2) + f(2008^3) + f(2009^5) &= \frac{2-6}{2}f(1) + \frac{2-12}{2}f(1) + \frac{2-15}{2}f(1) \\ &= -\frac{27}{2}f(1) = -\frac{27}{2}\cdot\left(-\frac{2}{3}\right) = 9. \end{aligned}$$

NT7 Determine the minimal prime number $p > 3$ for which no natural number n satisfies

$$2^n + 3^n \equiv 0 \pmod{p}.$$

Solution
We put $A(n) = 2^n + 3^n$. From Fermat's little theorem, we have $2^{p-1} \equiv 1 \pmod{p}$ and $3^{p-1} \equiv 1 \pmod{p}$ from which we conclude $A(n) \equiv 2 \pmod{p}$. Therefore, after $p-1$ steps

at most, we will have repetition of the power. It means that in order to determine the minimal prime number p we seek, it is enough to determine a complete set of remainders $S(p) = \{0, 1, \ldots, p-1\}$ such that $2^n + 3^n \not\equiv 0 \pmod{p}$, for every $n \in S(p)$.
For $p = 5$ and $n = 1$ we have $A(1) \equiv 0 \pmod{5}$.
For $p = 7$ and $n = 3$ we have $A(3) \equiv 0 \pmod{7}$.
For $p = 11$ and $n = 5$ we have $A(5) \equiv 0 \pmod{11}$.
For $p = 13$ and $n = 2$ we have $A(2) \equiv 0 \pmod{13}$.
For $p = 17$ and $n = 8$ we have $A(8) \equiv 0 \pmod{17}$.
For $p = 19$ we have $A(n) \not\equiv 0 \pmod{19}$, for all $n \in S(19)$.
Hence the minimal value of p is 19.

NT8 Let a, b, c, d, e, f are nonzero digits such that the natural numbers $\overline{abc}$, $\overline{def}$ and $\overline{abcdef}$ are squares.
a) Prove that $\overline{abcdef}$ can be represented in two different ways as a sum of three squares of natural numbers.
b) Give an example of such a number.
Solution
a) Let $\overline{abc} = m^2$, $\overline{def} = n^2$ and $\overline{abcdef} = p^2$, where $11 \leq m \leq 31$, $11 \leq n \leq 31$ are natural numbers. So, $p^2 = 1000 \cdot m^2 + n^2$. But $1000 = 30^2 + 10^2 = 18^2 + 26^2$. We obtain the following relations

$$p^2 = (30^2 + 10^2) \cdot m^2 + n^2 = (18^2 + 26^2) \cdot m^2 + n^2 =$$

$$= (30m)^2 + (10m)^2 + n^2 = (18m)^2 + (26m)^2 + n^2.$$

The assertion a) is proved.
b) We write the equality $p^2 = 1000 \cdot m^2 + n^2$ in the equivalent form $(p+n)(p-n) = 1000 \cdot m^2$, where $349 \leq p \leq 979$. If $1000 \cdot m^2 = p_1 \cdot p_2$, such that $p + n = p_1$ and $p - n = p_2$, then p_1 and p_2 are even natural numbers with $p_1 > p_2 \geq 318$ and $22 \leq p_1 - p_2 \leq 62$.
For $m = 15$ we obtain $p_1 = 500$, $p_2 = 450$. So, $n = 25$ and $p = 475$. We have

$$225625 = 475^2 = 450^2 + 150^2 + 25^2 = 270^2 + 390^2 + 25^2.$$

The problem is solved.

NT9 Let p be a prime number. Find all positive integers a and b such that:

$$\frac{4a+p}{b} + \frac{4b+p}{a}$$

and

$$\frac{a^2}{b} + \frac{b^2}{a}$$

are integers.

Solution
Since a and b are symmetric we can assume that $a \le b$. Let $d = (a,b)$, $a = du$, $b = dv$ and $(u,v) = 1$. Then we have:

$$\frac{a^2}{b} + \frac{b^2}{a} = \frac{d(u^3+v^3)}{uv}$$

Since,

$$(u^3+v^3, u) = (u^3+v^3, v) = 1$$

we deduce that $u \mid d$ and $v \mid d$. But as $(u,v) = 1$, it follows that $uv \mid d$.
Now, let $d = uvt$. Furthermore,

$$\frac{4a+p}{b} + \frac{4b+p}{a} = \frac{4(a^2+b^2)+p(a+b)}{ab} = \frac{4uvt(u^2+v^2)+p(u+v)}{u^2v^2t}.$$

This implies,

$$uv \mid p(u+v).$$

But from our assumption $1 = (u,v) = (u,u+v) = (v,u+v)$ we conclude $uv \mid p$.
Therefore, we have three cases $\{u=v=1\}$, $\{u=1, v=p\}$, $\{u=p, v=1\}$.
We assumed that $a \le b$, and this implies $u \le v$.
If $a = b$, we need $\dfrac{4a+p}{a} + \dfrac{4a+p}{a} \in \mathbb{N}$, i.e. $a \mid 2p$. Then $a \in \{1,2,p,2p\}$. The other condition being fulfilled, we obtain the solutions $(1,1)$, $(2,2)$, (p,p) and $(2p,2p)$.
Now, we have only one case to investigate, $u = 1$, $v = p$. The last equation is transformed into:

$$\frac{4a+p}{b} + \frac{4b+p}{a} = \frac{4pt(1+p^2)+p(p+1)}{p^2t} = \frac{4t+1+p(1+4pt)}{pt}.$$

From the last equation we derive

$$p \,|\, (4t+1).$$

Let $4t+1 = pq$. From here we derive

$$\frac{4t+1+p(1+4pt)}{pt} = \frac{q+1+4pt}{t}.$$

Now, we have

$$t \,|\, (q+1)$$

or

$$q+1 = st.$$

Therefore,

$$p = \frac{4t+1}{q} = \frac{4t+1}{st-1}.$$

Since p is a prime number, we deduce

$$\frac{4t+1}{st-1} \ge 2$$

or

$$s \leq \frac{4t+3}{2t} = 2 + \frac{3}{2t} < 4.$$

Case 1: $s = 1$, $p = \dfrac{4t+1}{t-1} = 4 + \dfrac{5}{t-1}$. We conclude $t = 2$ or $t = 6$. But when $t = 2$, we have $p = 9$, not a prime. When $t = 6$, $p = 5$, $a = 30$, $b = 150$.
Case 2: $s = 2$, $p = \dfrac{4t+1}{2t-1} = 2 + \dfrac{3}{2t-1}$. We conclude $t = 1$, $p = 5$, $a = 5$, $b = 25$ or $t = 2$, $p = 3$, $a = 6$, $b = 18$.
Case 3: $s = 3$, $p = \dfrac{4t+1}{3t-1}$ or $3p = 4 + \dfrac{7}{3t-1}$. As 7 does not have any divisors of the form $3t-1$, in this case we have no solutions.
So, the solutions are

$$(a,b) = \{(1,1),(2,2),(p,p),(2p,2p),(5,25),(6,18),(18,6),(25,5),(30,150),(150,30)\}.$$

NT10 Prove that $2^n + 3^n$ is not a perfect cube for any positive integer n.
Solution
If $n = 1$ then $2^1 + 3^1 = 5$ is not perfect cube.
Perfect cube gives residues -1, 0 and 1 modulo 9. If $2^n + 3^n$ is a perfect cube, then n must be divisible with 3 (congruence $2^n + 3^n = x^3$ modulo 9).
If $n = 3k$ then $2^{3k} + 3^{2k} > (3^k)^3$. Also, $(3^k+1)^3 = 3^{3k} + 3 \cdot 3^{2k} + 3 \cdot 3^k + 1 > 3^{3k} + 3^{2k} = 3^{3k} + 9^k > 3^{3k} + 8^k = 3^{3k} + 2^{3k}$. But, 3^k and $3^k + 1$ are two consecutive integers so $2^{3k} + 3^{3k}$ is not a perfect cube.

NT11 Determine the greatest number with n digits in the decimal representation which is divisible by 429 and has the sum of all digits less than or equal to 11.
Solution
Let $A = \overline{a_n a_{n-1} \dots a_1}$ and notice that $429 = 3 \cdot 11 \cdot 13$.
Since the sum of the digits $\sum a_i \leq 11$ and $\sum a_i$ is divisible by 3, we get $\sum a_i = 3$, 6 or 9. As 11 divides A, we have

$$11 \mid a_n - a_{n-1} + a_{n-2} - a_{n-3} + \dots,$$

in other words $11 | \sum\limits_{i \text{ odd}} a_i - \sum\limits_{i \text{ even}} a_i$. But

$$-9 \leq -\sum a_i \leq \sum_{i \text{ odd}} a_i - \sum_{i \text{ even}} a_i \leq \sum a_i \leq 9,$$

so $\sum\limits_{i \text{ odd}} a_i - \sum\limits_{i \text{ even}} a_i = 0$. It follows that $\sum a_i$ is even, so $\sum a_i = 6$ and $\sum\limits_{i \text{ odd}} a_i = \sum\limits_{i \text{ even}} a_i = 3$.
The number 13 is a divisor of 1001, hence

$$13 \mid \overline{a_3 a_2 a_1} - \overline{a_6 a_5 a_4} + \overline{a_9 a_8 a_7} - \overline{a_{12} a_{11} a_{10}} + \dots \tag{1}$$

For each $k = 1,2,3,4,5,6$, let s_k be the sum of the digits a_{k+6m}, $m \geq 0$; that is

$$s_1 = a_1 + a_7 + a_{13} + \ldots \text{ and so on.}$$

With this notation, (1) rewrites as

$$13 \,|\, 100(s_3 - s_6) + 10(s_2 - s_5) + (s_1 - s_4), \text{ or } 13|4(s_6 - s_3) + 3(s_5 - s_2) + (s_1 - s_4).$$

Let $S_3 = s_3 - s_6$, $S_2 = s_2 - s_5$, and $S_1 = s_1 - s_4$. Recall that $\sum_{i \text{ odd}} a_i = \sum_{i \text{ even}} a_i$, which implies $S_2 = S_1 + S_3$. Then

$$13 \,|\, 4S_3 + 3S_2 - S_1 = 7S_3 + 2S_1 \Rightarrow 13 \,|\, 49S_3 + 14S_1 \Rightarrow 13 \,|\, S_1 - 3S_3.$$

Observe that $|S_1| \leq s_1 = \sum_{i \text{ odd}} a_i = 3$ and likewise $|S_2|$, $|S_3| \leq 3$. Then $-13 < S_1 - 3S_3 < 13$ and consequently $S_1 = 3S_3$. Thus $S_2 = 4S_3$ and $|S_2| \leq 3$ yields $S_2 = 0$ and then $S_1 = S_3 = 0$. We have $s_1 = s_4$, $s_2 = s_5$, $s_3 = s_6$ and $s_1 + s_2 + s_3 = 3$, so the greatest number A is 30030000 ... .

NT12 Solve the equation $\dfrac{p}{q} - \dfrac{4}{r+1} = 1$ in prime numbers.

Solution

We can rewrite the equation in the form

$$\frac{pr + p - 4q}{q(r+1)} = 1 \Rightarrow pr + p - 4q = qr + q$$

$$pr - qr = 5q - p \Rightarrow r(p - q) = 5q - p.$$

It follows that $p \neq q$ and

$$r = \frac{5q - p}{p - q} = \frac{4q + q - p}{p - q}$$

$$r = \frac{4q}{p - q} - 1$$

As p is prime, $p - q \neq q$, $p - q \neq 2q$, $p - q \neq 4q$.

We have $p - q = 1$ or $p - q = 2$ or $p - q = 4$

i) If $p - q = 1$ then

$$q = 2, \; p = 3, \; r = 7$$

ii) If $p - q = 2$ then $p = q + 2$, $r = 2q - 1$

If $q = 1 \pmod 3$ then $q + 2 \equiv 0 \pmod 3$

$$q + 2 = 3 \Rightarrow q = 1$$

contradiction.

If $q \equiv -1 \pmod 3$ then $r \equiv -2 - 1 \equiv 0 \pmod 3$

$$r = 3$$

$$r = 2q - 1 = 3$$

$$q = 2$$

$$p = 4$$

contradiction.
Hence $q = 3$, $p = 5$, $r = 5$.
iii) If $p - q = 4$ then $p = q + 4$.
$r = q - 1$
Hence $q = 3$, $p = 7$, $r = 2$.