JBMO ShortLists 2001

1 Find the positive integers n that are not divisible by 3 if the number $2^{n^2-10}+2133$ is a perfect cube.

The wording of this problem is perhaps not the best English. As far as I am aware, just solve the diophantine equation $x^3=2^{n^2-10}+2133$ where $x,n\in\mathbb{N}$ and $3\nmid n$.

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2 Let P_n $(n=3,4,5,6,7)$ be the set of positive integers $n^k+n^l+n^m$, where k,l,m are positive integers. Find n such that:

i) In the set P_n there are infinitely many squares.

ii) In the set P_n there are no squares.

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3 Find all the three-digit numbers $\overline{abc}$ such that the 6003-digit number $\overline{abcabc\dots abc}$ is divisible by 91.

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4 The discriminant of the equation $x^2-ax+b=0$ is the square of a rational number and a and b are integers. Prove that the roots of the equation are integers.

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5 Let $x_k=\frac{k(k+1)}{2}$ for all integers $k\ge 1$. Prove that for any integer $n\ge 10$, between the numbers $A=x_1+x_2+\ldots+x_{n-1}$ and $B=A+x_n$ there is at least one square.

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6 Find all integers x and y such that $x^3\pm y^3=2001p$, where p is prime.

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7 Prove that there are are no positive integers x and y such that $x^5+y^5+1=(x+2)^5+(y-3)^5$.

The restriction x,y are positive isn't necessary.

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8 Prove that no three points with integer coordinates can be the vertices of an equilateral triangle.

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9 Consider a convex quadrilateral $ABCD$ with $AB=CD$ and $\angle BAC=30^\circ$. If $\angle ADC=150^\circ$, prove that $\angle BCA=\angle ACD$.

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10 A triangle ABC is inscribed in the circle $\mathcal{C}(O,R)$. Let $\alpha<1$ be the ratio of the radii of the circles tangent to $\mathcal{C}$, and both of the rays $(AB$ and $(AC$. The numbers $\beta<1$ and $\gamma<1$ are defined analogously. Prove that $\alpha+\beta+\gamma=1$.

11 Consider a triangle ABC with $AB = AC$, and D the foot of the altitude from the vertex A. The point E lies on the side AB such that $\angle ACE = \angle ECB = 18^\circ$.

If $AD = 3$, find the length of the segment CE.

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12 Consider the triangle ABC with $\angle A = 90^\circ$ and $\angle B \neq \angle C$. A circle $\mathcal{C}(O, R)$ passes through B and C and intersects the sides AB and AC at D and E, respectively. Let S be the foot of the perpendicular from A to BC and let K be the intersection point of AS with the segment DE. If M is the midpoint of BC, prove that $AKOM$ is a parallelogram.

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13 At a conference there are n mathematicians. Each of them knows exactly k fellow mathematicians. Find the smallest value of k such that there are at least three mathematicians that are acquainted each with the other two.

Rewording of the last line for clarification:
Find the smallest value of k such that there (always) exists 3 mathematicians X, Y, Z such that X and Y know each other, X and Z know each other and Y and Z know each other.