12^{th} Iranian Geometry Olympiad

October 17, 2025

Contest problems with solutions

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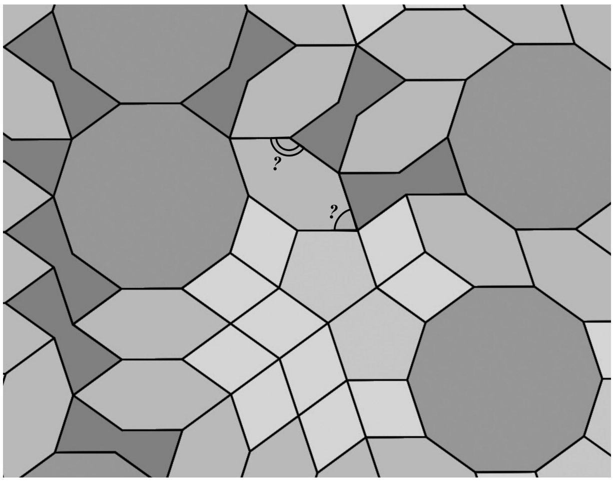
With special thanks to Mahdi Etesamifard and Hesam Rajabzadeh.

Contents

Elementary Level

Problems

Problem 1. A piece of tilework from a mosque in the city of Isfahan is shown below. The figure is made up of five different types of polygons. It consists of a regular pentagon and a regular decagon, with all sides having equal length. The hexagons in the figure have four equal angles, the two other angles are equal as well. Find the measures of the marked angles.



(→ p.5)

Problem 2. An equilateral triangle ABC is given. Points O_1, O_2 lie on the sides AB, AC respectively. It is known that the circle centered at O_1 and passing through B is externally tangent to the circle centered at O_2 and passing through C at a point P inside the triangle. Find the angle $\angle BPC$.

(→ p.6)

Problem 3. Arash is given a paper isosceles right triangle. A folding of this paper is called *good* if the polygon obtained after this fold has all angles less than $180°$. Arash performs a *good* fold. Babak takes the paper and performs two *good* folds, so that the paper is folded exactly three times at the end. Arash wants the final polygon to have the most possible number of sides but Babak wants the opposite. Assuming they do their best how many sides does the final polygon have?

(→ p.7)

Problem 4. Given that every diagonal of a convex quadrilateral is longer than each of its sides, prove that the length of each diagonal of this quadrilateral is less than $\sqrt{3}$ times the length of one of its sides.

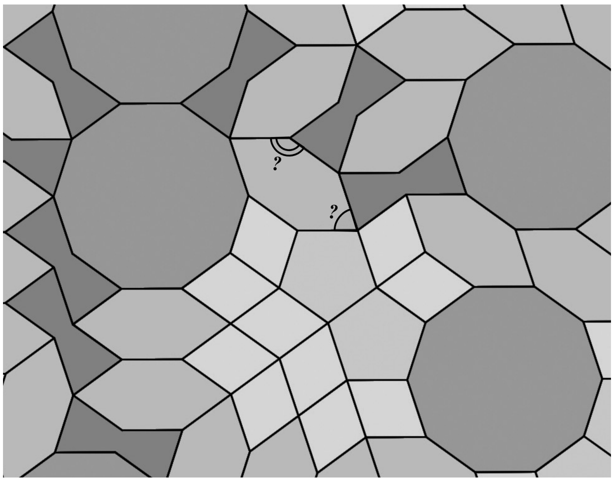
(→ p.8)

Problem 5. In triangle ABC with $\angle CAB = 15°$ and $\angle CBA = 30°$, points X and Y lie inside the angle $\angle BCA$ such that $\angle BCX = \angle ACY = 45°$ and $BC = CY, AC = CX$. Let the line XY meet AB at point Z. Prove that $AZ = BC$.

(→ p.9)

Solutions

Problem 1. A piece of tilework from a mosque in the city of Isfahan is shown below. The figure is made up of five different types of polygons. It consists of a regular pentagon and a regular decagon, with all sides having equal length. The hexagons in the figure have four equal angles, the two other angles are equal as well. Find the measures of the marked angles.



Proposed by Mahdi Norouzi - Iran

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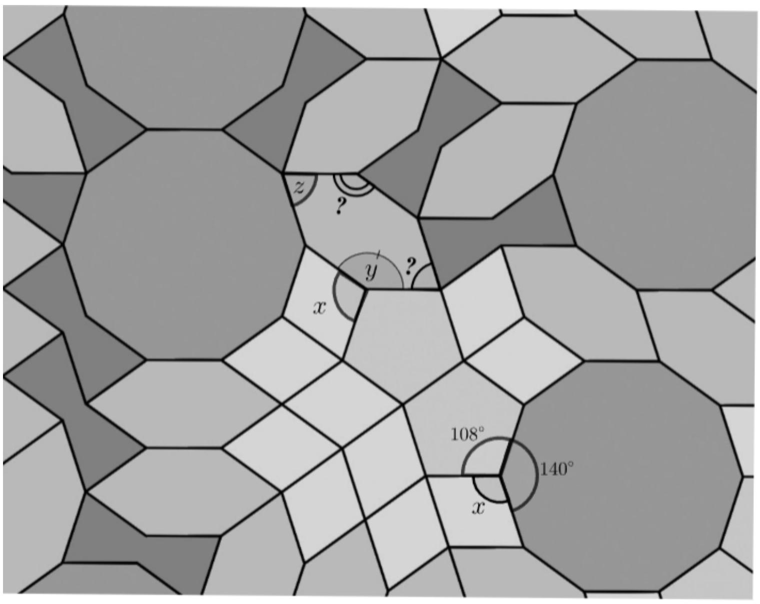
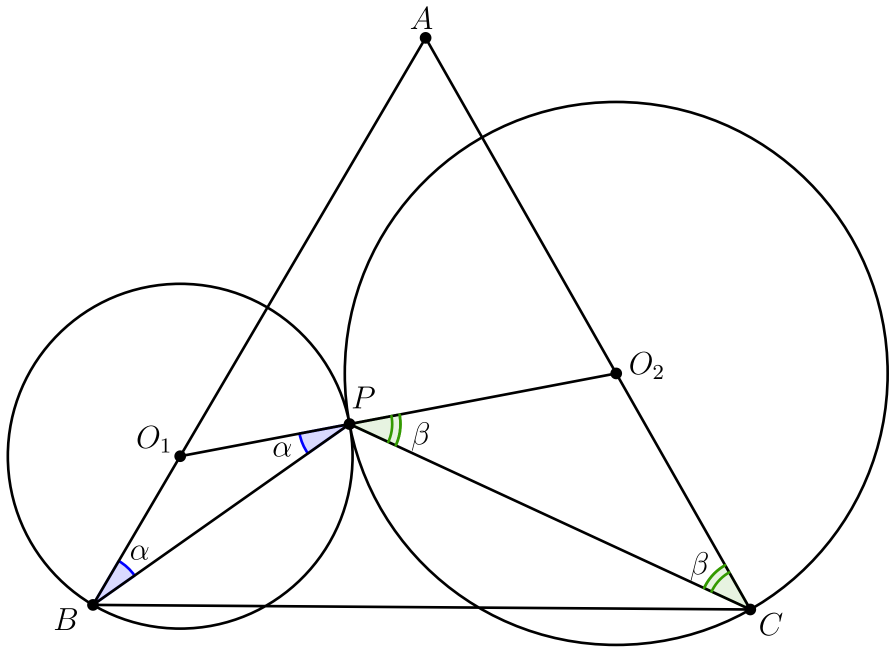
Solution. Note that the angle of a vertice of a regular polygon with n vertices is given by the formula $\frac{(n-2).180}{n}$. Hence the angles of vertices of the pentagons and decagons in the figure are $108°$ and $144°$ respectively. Considering the following figure the measure of angle x is $360-(108+144) = 108$. Similarly the measure of angle y in the figure is $360 - (108 + 108) = 144$. Considering the hexagon in the figure we know that the sum of all its angles is 720. Therefore we have the equation $2z + 4y = 720$ and $z = 72$.



Problem 2. An equilateral triangle ABC is given. Points O_1, O_2 lie on the sides AB, AC respectively. It is known that the circle centered at O_1 and passing through B is externally tangent to the circle centered at O_2 and passing through C at a point P inside the triangle. Find the angle $\angle BPC$.

Proposed by Davood Vakili - Iran

Solution. Let $\angle PBO_1 = \angle BPO_1 = \alpha$ and $\angle PCO_2 = \angle CPO_2 = \beta$. Then $\angle BPC = 180° - \alpha - \beta$. Also since $\angle O_1BC = \angle O_2CB = 60°$, therefore $\angle PBC = 60 - \alpha$ and $\angle PCB = 60 - \beta$. Hence $\angle BPC = 180° - \angle PBC - \angle PCB = 180° - (60° - \alpha) - (60° - \beta) = 60° + \alpha + \beta$. So $\angle BPC = 180° - \alpha - \beta = 60 + \alpha + \beta$. Therefore $\alpha + \beta = 60°$, and $\angle BPC = 120°$



Problem 3. Arash is given a paper isosceles right triangle. A folding of this paper is called *good* if the polygon obtained after this fold has all angles less than 180°. Arash performs a *good* fold. Babak takes the paper and performs two *good* folds, so that the paper is folded exactly three times at the end. Arash wants the final polygon to have the most possible number of sides but Babak wants the opposite. Assuming they do their best how many sides does the final polygon have?

Proposed by Arvin Taheri - Iran

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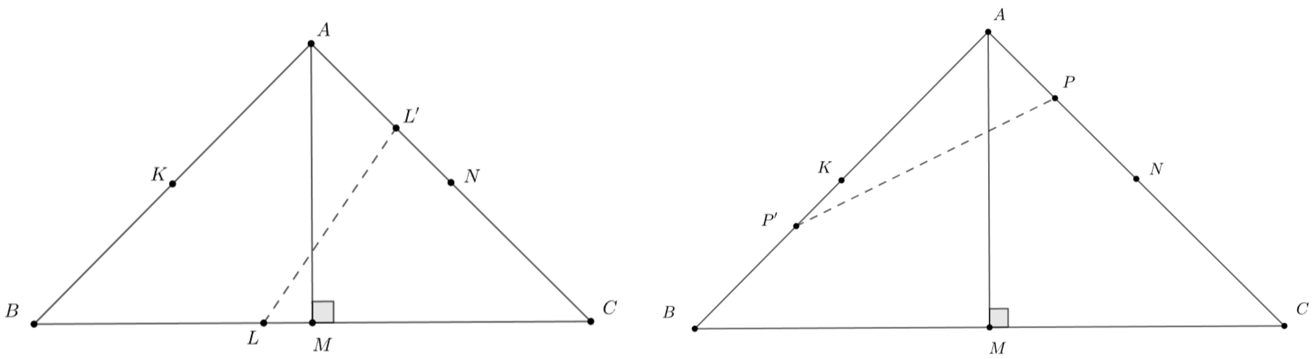
Solution. we claim that the answer is 3. Suppose the initial paper is the right isosceles triangle ABC shown in the figure. Let M, N, K be the midpoints of sides BC, CA, AB respectively.
Draw the line which will be formed after Arash folds the paper. We have two cases :
Case 1. *The line intersects BC and one of the other sides.*
Without loss of generality suppose that this line intersects BC and the side AC at points L, L' respectively. We claim that LL' doesn't intersect the perpendicular bisector of BC. Assuming the contrary, point L must lie on the segment BM(Figure 1.a). If $\angle CL'L > 90$ then $\angle CL'L > \angle LL'A$ and this will be a contradiction because the folding will not be *good*. Hence L' lies on the segment AN. But knowing that L lies on the segment BM, the folding of the paper will not be *good* which contradicts the assumption. Thus, LL' doesn't intersect the perpendicular bisector AM of this triangle. After Arash's fold, Babak simply folds the paper with respect to the line AM which leads to another right isosceles triangle. Then he will fold this triangle with respect to the perpendicular bisector of its hypotenuse. The number of sides in this case will be three !



(a) Case 1 (b) Case 2

Figure 1: The two cases after Arash's *good* fold

Case 2. *The line intersects the sides AC and AB.*
Assume that P, P' are the intersections of this line with sides AC, AB respectively.(Figure1.b) Note that P and P' cannot lie on segments CN and BK at the same time since PP' needs to lead to a *good* fold.Without loss of generality suppose that P lies on segment AN. In this case Babak folds the paper with respect to the perpendicular bisector of BC. Now folding the paper with respect to the line MN will lead to a right isosceles triangle.

Problem 4. Given that every diagonal of a convex quadrilateral is longer than each of its sides, prove that the length of each diagonal of this quadrilateral is less than $\sqrt{3}$ times the length of one of its sides.

Proposed by Morteza Saghafian - Iran

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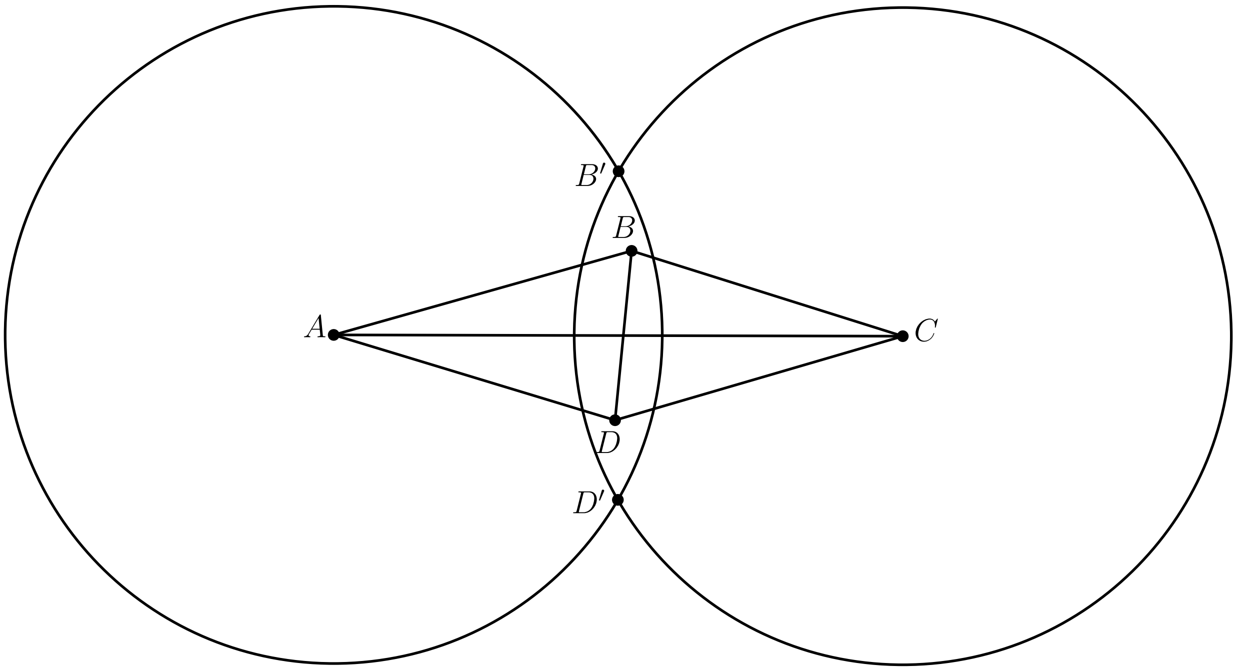
Solution. Assume the contrary. Let AC be the bigger diameter. Draw two circles with centers A, C with radius $\frac{AC}{\sqrt{3}}$, and let B', D' be the intersection of this circles. So since $\sqrt{3}.AB < AC$ and $\sqrt{3}.CB < AC$, the point B is in the area between the two circles shown below (Similarly for D). So $60° = \angle B'AD' > \angle BAD$ and $60° = \angle B'CD' > \angle BCD$ (Notice that the triangles $B'AD', B'CD'$ are both equilateral triangles). So $\angle ABD + \angle CBD + \angle ADB + \angle CDB > 240°$, hence one of these 4 angles is bigger than 60, WLOG assume $\angle ABD > 60°$. Therefore $\angle ABD > 60° > \angle BAD$ and so $AD > BD$ which is a contradiction.



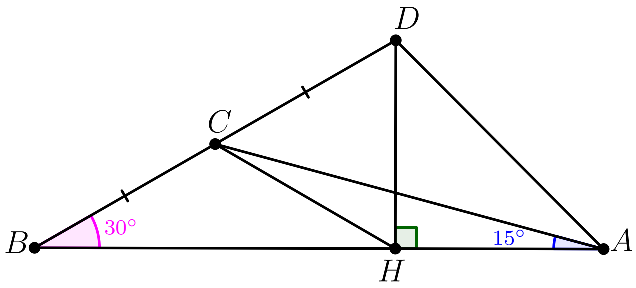
Problem 5. In triangle ABC with $\angle CAB = 15^\circ$ and $\angle CBA = 30^\circ$, points X and Y lie inside the angle $\angle BCA$ such that $\angle BCX = \angle ACY = 45^\circ$ and $BC = CY, AC = CX$. Let the line XY meet AB at point Z. Prove that $AZ = BC$.

Proposed by Mahdi Etesamifard - Iran

Solution. Let D be the point on BC such that $BC = CD$.

Lemma. $\angle DAC = 30^\circ$

Proof. Let H be the foot of the altitude for D in the triangle $\triangle ABD$. Since CH is the median in the right $\triangle BHD$, therefore $CB = CH = CD$ and since $\angle CDH = 60^\circ$, then the mentioned lengths are equal to DH too. Since $\angle ACH + \angle CAH = \angle CHB$, so $\angle ACH = \angle CAH = \angle 15^\circ$. So $\angle DAH = \angle ADH = 45^\circ$, hence $\angle DAC = 30^\circ$.



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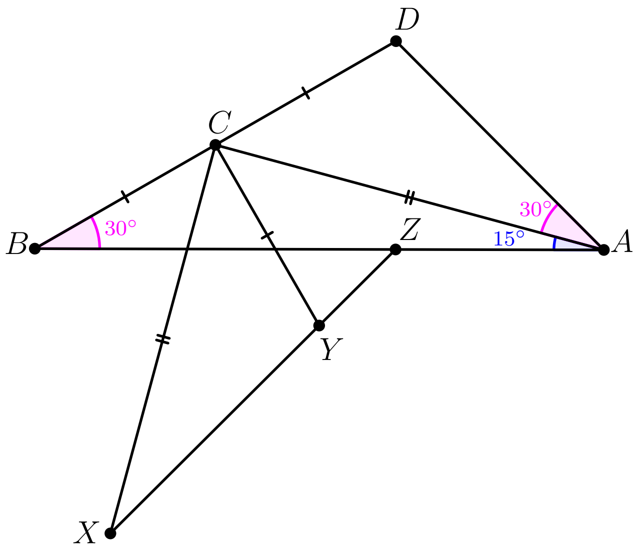
Since $\angle ACD = 45^\circ = \angle XCY$, $CD = CY$ and $CA = CX$, hence the triangles $\triangle ACD$ and $\triangle XCY$ are congruent. So $\angle CXY = 30^\circ$ and so since the triangle $\triangle BYX$ is an isosceles triangle with $BX = XY$ and $\angle BXY = 60^\circ$, then the triangle must be equilateral, so $BY = XY$. Again by symmetry, $XY = YA$, and since $\angle CYA = \angle CYX = \angle CDA = 105^\circ$, so $\angle XYA = 150^\circ$, and since $\angle BYX = 60^\circ$, then $\angle BYA = 150^\circ$, therefore the triangles $\triangle BYA$ and $\triangle XYA$ are congruent, hence $AB = AX$. Now notice that $\angle ZXA = \angle CXA - \angle CXY = 45^\circ - 30^\circ = 15^\circ = \angle CAB$, and $\angle XZA = \angle ZXC + 90^\circ + 15^\circ = 135^\circ = \angle ACB$ and $AB = XA$, so $\triangle ABC \cong \triangle XAZ$, hence $AZ = BC$



Intermediate Level

Problems

Problem 1. A right trapezoid $PYXQ$ ($PY \perp PQ \perp QX$) is given. Points A and B lie on the line PQ such that $\angle AYQ = \angle BXP = 90°$. If S is the intersection point of the diagonals of the trapezoid, prove that $\triangle AYS \sim \triangle BXS$.

(→ p.15)

Problem 2. A square $ABCD$ is given. Point E is the midpoint of side BC and F lies on the side AB such that $DE \perp EF$. Point G lies inside the square such that $GF = EF$ and $GF \perp EF$. Lines AC, DE intersect at point X. Prove that the points G, B, E, X lie on a circle.

(→ p.16)

Problem 3. Triangle ABC and its circumcircle ω are given. Point T is the midpoint of arc BC of circle ω (The arc that doesn't include A). Line BT intersects the external angle bisector of angle BAC at point P. H is the foot of the perpendicular line from A onto the line tangent to ω at T and M is the midpoint of segment AP. Prove that $\angle AHM = \angle ACP$.

(→ p.17)

Problem 4. In the convex hexagon $ABCYXD$ we have

$$\angle ACY = \angle BDX = 90°$$

$$\angle BAC = 2\angle CAY, \quad \angle ABD = 2\angle DBX$$

$$XY = DX + CY$$

Prove that

$$\sqrt{(CD - DX)(CD - CY)} \leq \frac{AC + BD - AB}{2}$$

(→ p.18)

Problem 5. Let ABC be a triangle with $AB < AC$, and let ω be an arbitrary circle passing through B and C. Denote by F and E the intersections of ω with AB and AC, respectively. Let M and N be the intersections of the perpendicular bisectors of segments BF and CE with side BC, respectively. Let P be the intersection of the perpendicular bisector of MN with EF. Prove that as ω varies, the point P lies on a fixed line.

(→ p.20)

Solutions

Problem 1. A right trapezoid $PYXQ$ ($PY \perp PQ \perp QX$) is given. Points A and B lie on the line PQ such that $\angle AYQ = \angle BXP = 90°$. If S is the intersection point of the diagonals of the trapezoid, prove that $\triangle AYS \sim \triangle BXS$.

Proposed by Lenoid Shatunov - Russia

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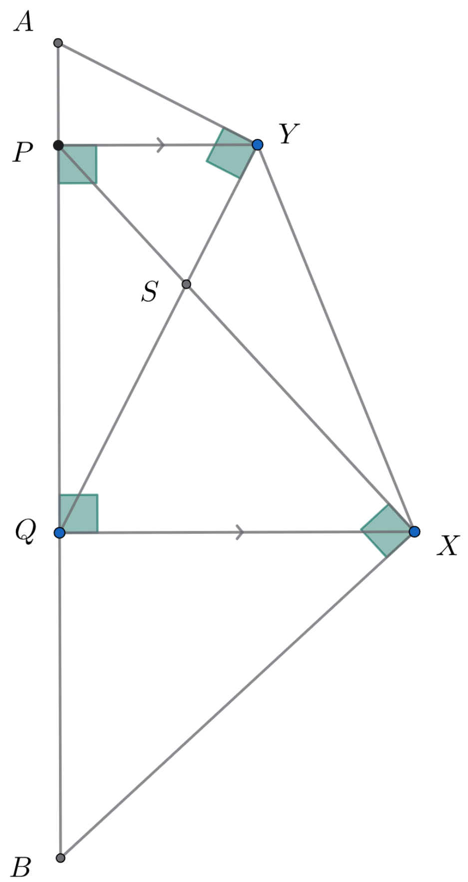
Solution. Since $\angle AYS = \angle BXS = 90°$, it suffices to show that $\frac{AY}{BX} = \frac{YS}{XS}$. Note that $\triangle APY \sim \triangle YPQ$, hence $\frac{AY}{YQ} = \frac{PY}{PQ}$. Similarly, we have $\frac{PX}{BX} = \frac{PQ}{QX}$. Now we have

$$\frac{AY}{BX} = \frac{AY}{QY} \cdot \frac{PX}{BX} \cdot \frac{QY}{PX} = \frac{PY}{PQ} \cdot \frac{PQ}{QX} \cdot \frac{QY}{PX} = \frac{PY}{QX} \cdot \frac{QY}{PX}$$

Note that $\triangle PSY \sim \triangle XSQ$, therefore $\frac{PY}{QX} = \frac{PS}{XS} = \frac{SY}{SQ}$. So

$$\frac{AY}{BX} = \frac{YS}{SQ} \cdot \frac{QY}{PX} = \frac{YS}{PX} \cdot \frac{QY}{SQ} = \frac{YS}{PX}(1 + \frac{SY}{SQ}) = \frac{YS}{PX}(1 + \frac{SP}{SX}) = \frac{YS}{PX} \cdot \frac{PX}{SX} = \frac{YS}{XS}$$

And so we are done.



Problem 2. A square $ABCD$ is given. Point E is the midpoint of side BC and F lies on the side AB such that $DE \perp EF$. Point G lies inside the square such that $GF = EF$ and $GF \perp EF$. Lines AC, DE intersect at point X. Prove that the points G, B, E, X lie on a circle.

Proposed by Ercole Suppa - Italy

Solution. To prove the statement, it suffices to show that $\angle BEG = \angle BXG$. We know that

$$\angle BEG = \angle BEF + \angle FEG = \angle BEF + 45°,$$

and since $\angle BEF + \angle DEC = 90°$, we have

$$\angle CBX = \angle CDE = \angle BEF.$$

Hence, the statement is equivalent to proving that

$$\angle BXG = \angle CBX + 45° = \angle BXA,$$

or equivalently, that the points A, G, C are collinear.
We know that $\angle BEF = \angle CDE$, therefore the triangles $\triangle BEF$ and $\triangle CDE$ are similar, and we have

$$\frac{BF}{CE} = \frac{BE}{CD} = \frac{1}{2} \quad \Rightarrow \quad BF = \frac{CE}{2} = \frac{BC}{4} = \frac{AB}{4}.$$

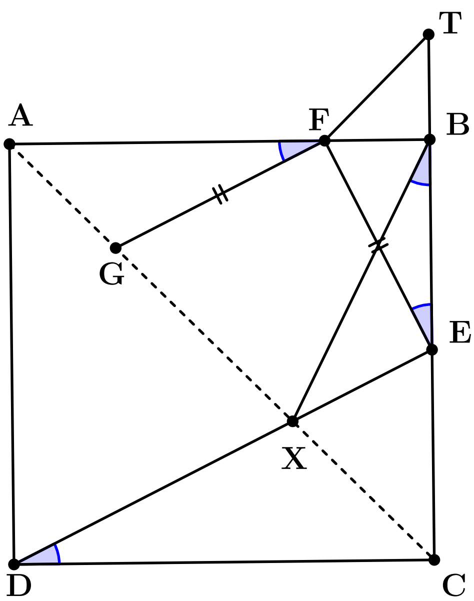
Thus,

$$AF = \frac{3AB}{4}.$$

Let T be a point on the extension of CB such that $BF = BT$. By the problem's assumption, we know that $EF = FG$. We also proved that $\angle TEF = \angle AFG$, and

$$TE = TB + BE = BF + BE = \frac{AB}{4} + \frac{AB}{2} = \frac{3AB}{4} = AF.$$

Hence, the triangles $\triangle TEF$ and $\triangle AFG$ are congruent, and consequently, $\angle FAG = 45°$.
The proof is complete.



Problem 3. Triangle ABC and its circumcircle ω are given. Point T is the midpoint of arc BC of circle ω (The arc that doesn't include A). Line BT intersects the external angle bisector of angle BAC at point P. H is the foot of the perpendicular line from A onto the line tangent to ω at T and M is the midpoint of segment AP. Prove that $\angle AHM = \angle ACP$.

Proposed by Iman Maghsoudi - Iran

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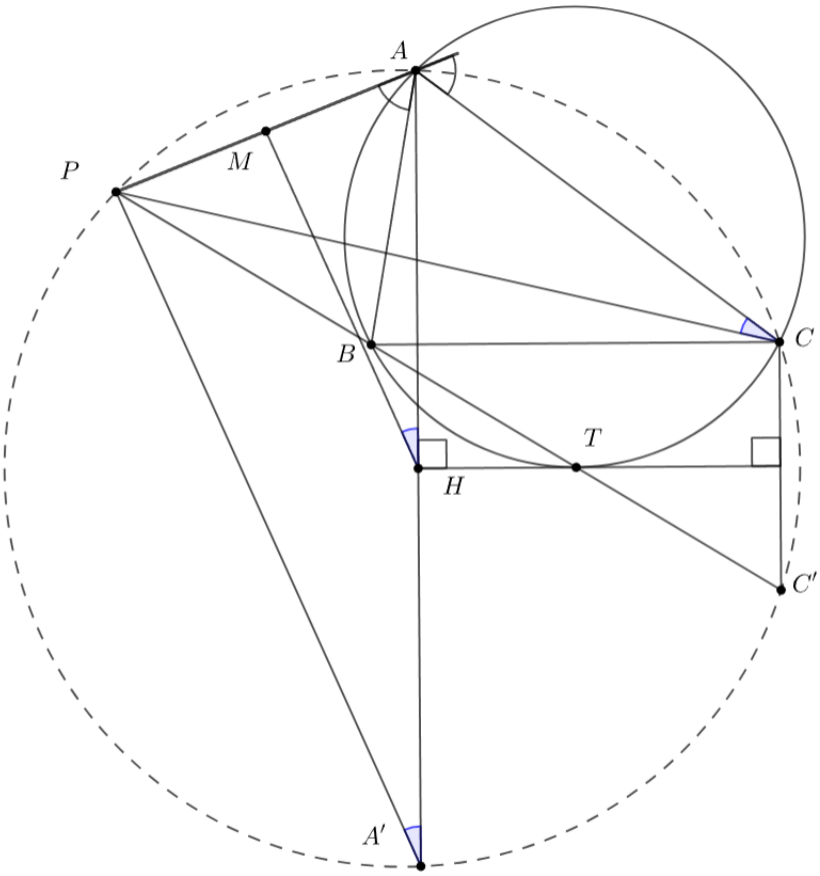
Let A' and C' be the reflections of points A, C across the tangent line to ω at T. Note that $ACC'A'$ is an isoceles trapezoid thus it is cyclic. Since $\triangle TC'C$ is isoceles we have :

$$\angle CTC' = 180 - 2\angle TCC' = 180 - 2(90 - \frac{1}{2}\angle A)$$

hence C' lies on the line BT. Now having $\angle PAC = \angle A + 90 - \frac{1}{2}\angle A = 90 + \frac{1}{2}\angle A = 180 - \angle TC'C$, the quadrilateral $PACC'$ is cyclic. Therefore all points P, A, C, C', A' lie on a circle and thus $\angle AHM = \angle AA'P = \angle ACP$ and we are done.



Problem 4. In the convex hexagon $ABCYXD$ we have

$$\angle ACY = \angle BDX = 90^\circ$$

$$\angle BAC = 2\angle CAY, \quad \angle ABD = 2\angle DBX$$

$$XY = DX + CY$$

Prove that

$$\sqrt{(CD - DX)(CD - CY)} \leq \frac{AC + BD - AB}{2}$$

Proposed by Iman Maghsoudi - Iran

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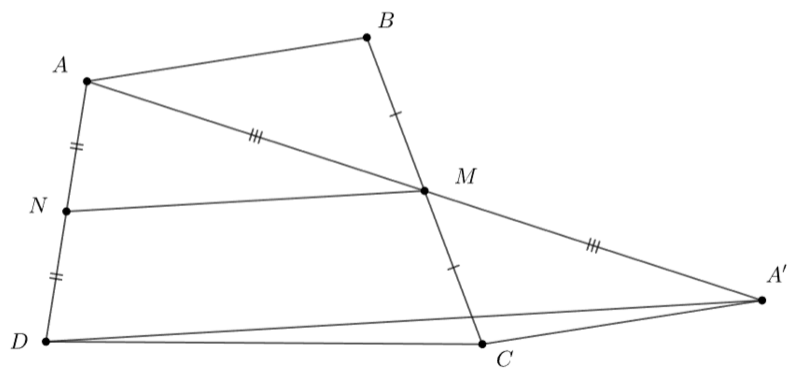
Solution. We begin by the following lemma.

Lemma . *In the convex quadrilateral $ABCD$, let M, N be the midpoints of sides BC, AD respectively. then $\frac{1}{2}(AB + CD) \geq MN$*

Proof. Let A' be the reflection of A with respect to point M. By thales' thoerem $DA' = 2MN$.Note that $ABA'C$ is a parallelogram thus $CA' = AB$. Now by triangle inequality we have

$$AB + CD = CA' + CD \geq DA' = 2MN$$



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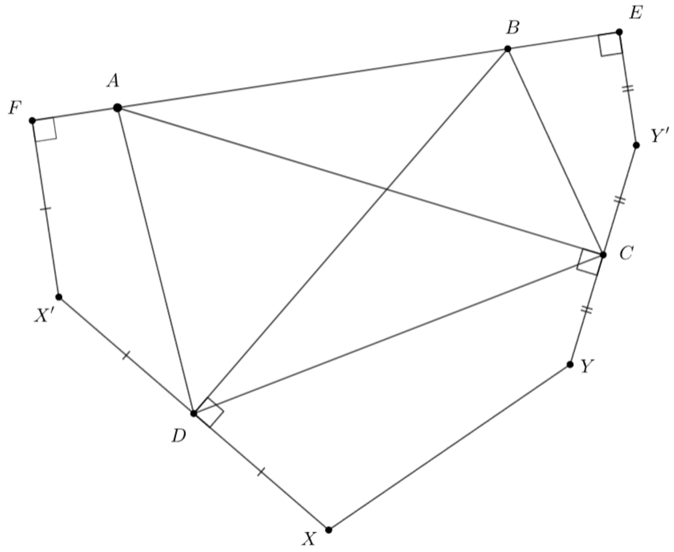
Let X', Y' be the reflections of X, Y with respect to the points C, D, respectively. Since $\angle CAB = 2\angle YAC$ and $\angle ACY = 90^\circ$, point Y' lies on the angle bisector of angle BAC. Letting E be the foot of perpendicular of Y onto AB, we have $YC = YC' = Y'E$ and $AC = AE$. Similarly letting F be the foot of perpendicular of X' onto AB we have $XD = DX' = X'F$ and $BD = BF$. Note that $FX'Y'E$ is a right trapezoid.



By the previous **Lemma**, we have

$$2CD \leq X'Y' + XY \quad \Rightarrow \quad 2CD - XY \leq X'Y'.$$

Then, we compute:

$$\begin{aligned}
EF^2 &= X'Y'^2 - (FX' - EY')^2 \\
&= X'Y'^2 - (DX - CY)^2 \\
&\geq (2CD - XY)^2 - (DX - CY)^2 \\
&= (2CD - XY - (DX - CY))\,(2CD - XY + (DX - CY)) \\
&= (2CD - (DX + CY) - (DX - CY))\,(2CD - (DX + CY) + (DX - CY)) \\
&= (2CD - 2DX)\,(2CD - 2CY) \\
&= 4\,(CD - DX)\,(CD - CY).
\end{aligned}$$

Since

$$\frac{AC + BD - AB}{2} = \frac{EF}{2},$$

we obtain

$$\frac{AC + BD - AB}{2} \geq \sqrt{(CD - DX)\,(CD - CY)}.$$

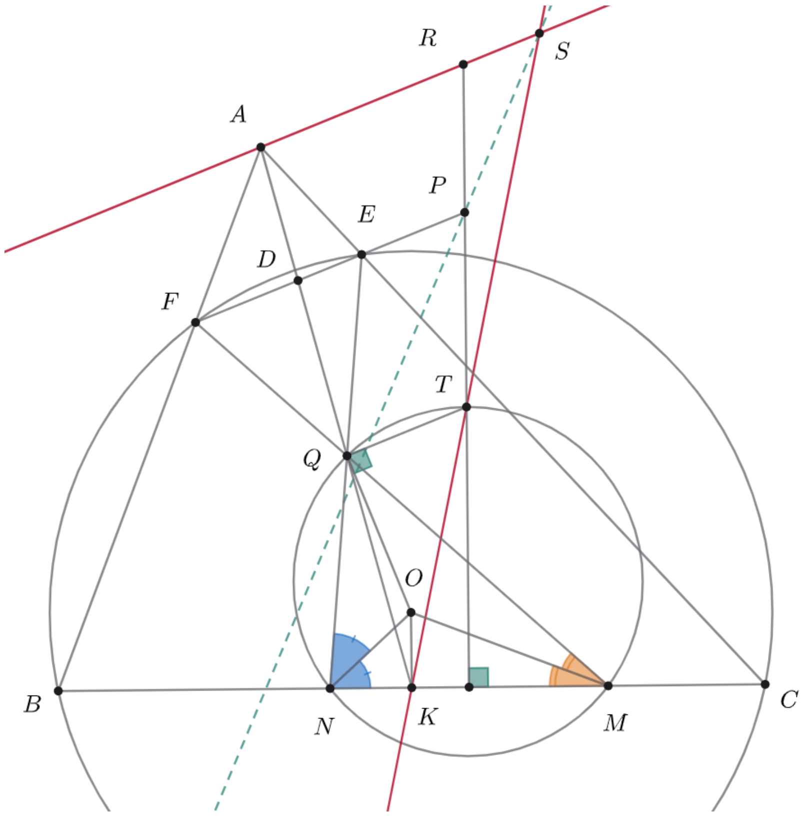
Hence, the proof is complete.

Problem 5. Let ABC be a triangle with $AB < AC$, and let ω be an arbitrary circle passing through B and C. Denote by F and E the intersections of ω with AB and AC, respectively. Let M and N be the intersections of the perpendicular bisectors of segments BF and CE with side BC, respectively. Let P be the intersection of the perpendicular bisector of MN with EF. Prove that as ω varies, the point P lies on a fixed line.

Proposed by Dang Dinh Trung - Vietnam

Solution. Let K, D, Q, O be the midpoint of BC, the intersection of AK and EF, the intersection of FM and EN, and the center of ω, respectively. we have $\angle FEQ = 180° - \angle AEF - \angle NEC = 180° - \angle B - \angle C = \angle A$, so QE is tangent to (AEF) and similarly, QF is also tangent to (AEF), therefore Q lies on $A - symmedian$. Note that $\triangle AEF \sim \triangle ABC$, hence Q lies on $A - median$, so Q lies on AK. Also, the ratio $\frac{AD}{DQ}$ is fixed as ω varies.
Since O lies on the perpendicular bisectors of BF, CE, O is the incenter of QMN, and K is the tangency point of the incircle with side MN.

Now let T be the midpoint of $\overset{\frown}{MQN}$, and R be a point on the perpendicular bisector of MN such that $AR || EF$. Note that $QT || EF$, since both QT and EF are perpendicular to QO. As ω varies, all of QMN are homothetic with respect to K, so T lies on a fixed line which passes through K. Now, since $AR || PD || TQ$, we have $\frac{RP}{PT} = \frac{AD}{DQ}$, which means that $\frac{RP}{PT}$ is constant. Let S be the intersection of AR and KT. Note that S is a fixed point, since the rays KT and AR are fixed. Now, by homothety with respect to S, P lies on a fixed line which passes through S.



Advanced Level

Problems

Problem 1. Given an isosceles ABC triangle with $AB = AC$, let points X and Y lie on side BC with X between B and Y. We denote the circumcircle of triangle AYB by ω_1. The circle ω_2 passes through points C and X and is tangent to AC. The points M and N are the intersections of the two circles ω_1 and ω_2. Prove that $\angle AMX = \angle BNX$.

(→ p.25)

Problem 2. Quadrilateral $ABCD$ is inscribed in a circle ω. Distinct points X, Y lie on the rays DB, CA such that $DA = DX, CB = CY$. Lines AX, BY intersect the side CD at points P, Q respectively. Prove that the radical axis of the two circumcircles of triangles APC, BQD and the perpendicular bisector of side AB intersect at a point on ω.

(→ p.26)

Problem 3. Point M is the midpoint of side BC of triangle $ABC(AB \neq AC)$. X is an arbitrary point on the segment AM. Point A' lies on the circumcircle of triangle $\triangle ABC$ such that $AA'||BC$. Circumcircle of triangle AXA' intersects the sides AB, AC at points F, E and P is the intersection of lines $BC, A'X$. Prove that points P, M, E, F lie on a circle.

(→ p.28)

Problem 4. Given an isosceles triangle ABC with $AB = AC$, let points M and N lie on side BC such that $\angle MAN = \frac{1}{2}\angle BAC$, and M lies between B and N. P is the center of the circumcircle of triangle AMN. The perpendicular bisectors of AM and AN intersect BC at points R and Q, respectively. Points S and T lie on PR and PQ, such that $ST \perp PA$. K and L are the reflections of A with respect to lines QS and RT, respectively. Prove that the circumcircles (CMK) and (BNL) intersect on the perpendicular bisector of side BC.

(→ p.29)

Problem 5. The incircle of triangle ABC touches the sides AC, AB at points E, F respectively. Circle ω_1 is tangent to the segments BE, CE and it also touches the circumcircle of triangle ABC. Circle ω_2 is tangent to the segments CF, BF and it also touches the circumcircle of triangle ABC. Prove that the exsimilicenter of two circles ω_1, ω_2 lies on the radical axis of the incircle and circumcircle of triangle ABC.(The exsimilicenter of two circles is the intersection point of their external common tangents.)

(→ p.31)

Solutions

Problem 1. Given an isosceles ABC triangle with $AB = AC$, let points X and Y lie on side BC with X between B and Y. We denote the circumcircle of triangle AYB by ω_1. The circle ω_2 passes through points C and X and is tangent to AC. The points M and N are the intersections of the two circles ω_1 and ω_2. Prove that $\angle AMX = \angle BNX$.

Proposed by Iman Maghsoudi - Iran

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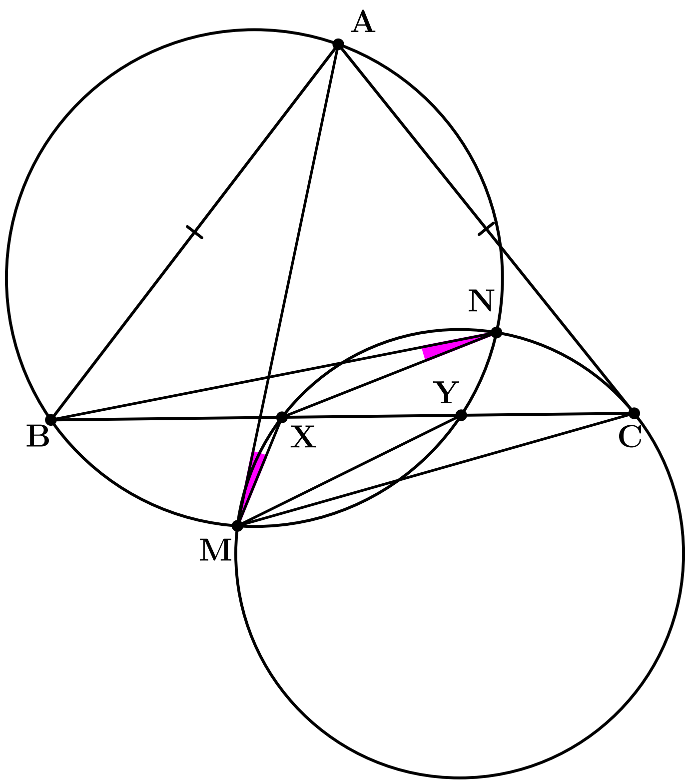
Solution. Since $ABMN$ is cyclic and ω_2 is tangent to AC, we have

$$\angle XMC = \angle XCA = \angle ABC = \angle ABY = \angle AMY.$$

Hence $\angle AMX = \angle YMC$.
Moreover,

$$\begin{aligned}\angle BNX &= \angle BNM - \angle XNM \\ &= \angle BYM - \angle XCM \\ &= \angle XYM - \angle YCM \\ &= \angle YMC = \angle AMX.\end{aligned}$$

Thus $\angle AMX = \angle BNX$, and we are done.



Problem 2. Quadrilateral $ABCD$ is inscribed in a circle ω. Distinct points X, Y lie on the rays DB, CA such that $DA = DX, CB = CY$. Lines AX, BY intersect the side CD at points P, Q respectively. Prove that the radical axis of the two circumcircles of triangles APC, BQD and the perpendicular bisector of side AB intersect at a point on ω.

Proposed by Amirparsa Hosseini - Iran

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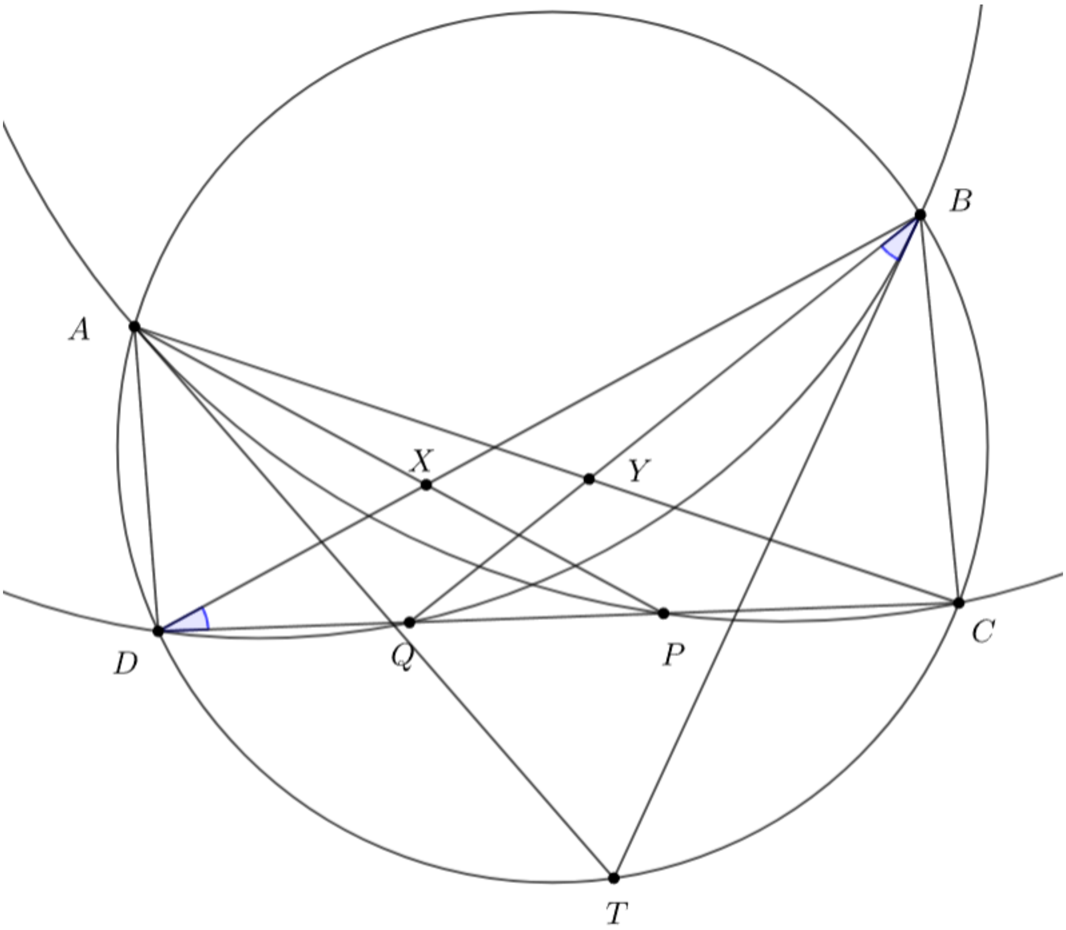
Solution. Let T be the midpoint of arc AB(the arc AB that contains C, D). Note that $\angle TBQ = \angle YBC - \angle TBC = (90 - \frac{1}{2}\angle BCA) - \angle TBC = \angle BDC$. Hence the line TB is tangent to circumcircle BQD. Similarly TA is tangent to circumcircle (APC) and since $TB = TA$, T lies on the radical axis of the two circles.



Second solution.
Let $AC \cap BD = E$, $AP \cap BQ = F$, and let the perpendicular bisector of AB intersect the arc ADB at T. From the problem conditions we have $\angle EBQ = \angle EAP$, hence

$$\begin{aligned} \angle APQ &= \angle PAC + \angle ACP \\ &= \angle EBY + \angle ABE \\ &= \angle ABQ. \end{aligned}$$

Thus, the quadrilateral $ABPQ$ is cyclic.
Since $AE \times EC = BE \times ED$ and $BF \times FQ = AF \times FP$, it follows that EF is the radical axis of the circumcircles of $\triangle APC$ and $\triangle BQD$. Therefore, it suffices to prove that T lies on this line.
We have

$$\begin{aligned} \angle BDT &= 90^\circ - \tfrac{\angle ADB}{2} \\ &= \angle PAD \\ &= \angle AXD, \end{aligned}$$

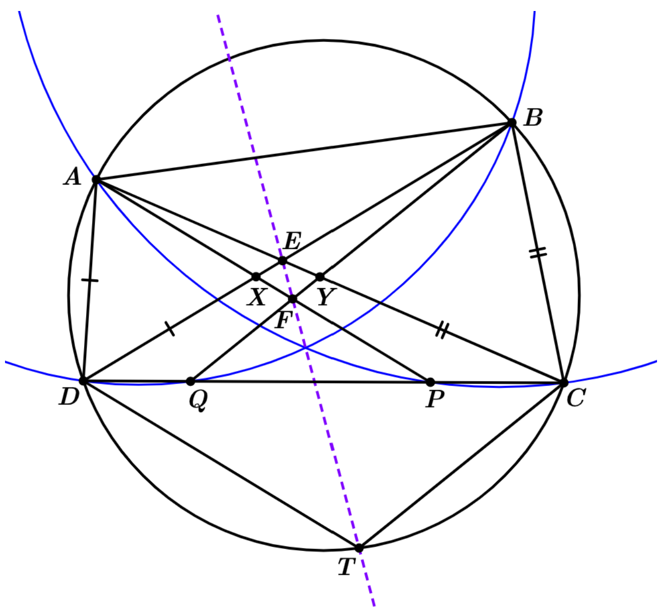
so $AP \parallel DT$. Similarly, $BQ \parallel CT$, hence the triangles PFQ and DTC are similar, and

$$\frac{FP}{FQ} = \frac{DT}{CT}.$$

Now, by the ratio lemma,

$$\begin{aligned} \frac{\sin \angle CET}{\sin \angle TED} &= \frac{CT}{TD} \times \frac{\sin \angle ECT}{\sin \angle EDT} \\ &= \frac{CT}{TD} = \frac{FQ}{FP} = \frac{FY}{FX} \\ &= \frac{FY}{FX} \times \frac{\sin \angle EYF}{\sin \angle EXF} \\ &= \frac{\sin \angle YEF}{\sin \angle FEX}. \end{aligned}$$

Hence the points E, F, and T are collinear, and we are done.



Problem 3. Point M is the midpoint of side BC of triangle $ABC(AB \neq AC)$. X is an arbitrary point on the segment AM. Point A' lies on the circumcircle of triangle $\triangle ABC$ such that $AA'||BC$. Circumcircle of triangle AXA' intersects the sides AB, AC at points F, E and P is the intersection of lines $BC, A'X$. Prove that points P, M, E, F lie on a circle.

Proposed by Mahdi Mashayekhi - Iran

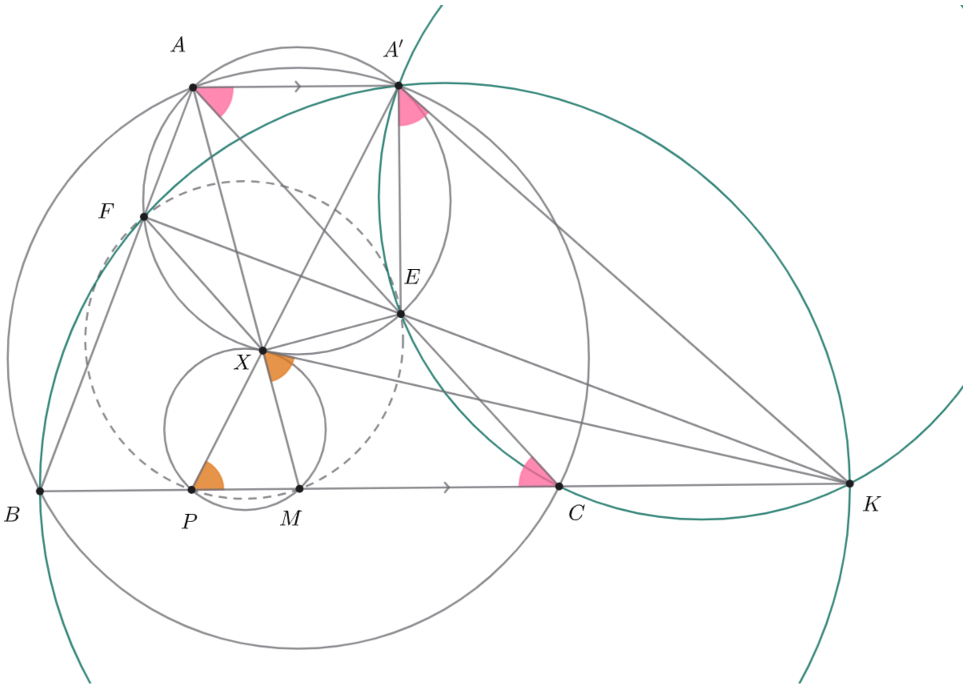
Solution. Let ω be the circumcircle of AXA' and K be the intersection of EF and BC. Note that A' is the Miquel point of $BFEC$, so $A'ECK, A'FBK$ are cyclic. Hence $\angle EA'K = \angle ECB = \angle EAA'$, therefore KA' is tangent to ω. Note that

$$-1 = (BC; M\infty) \stackrel{A}{=} (FE; XA')_\omega$$

So XK is also tangent to ω. So we have $\angle MXK = \angle AA'X = \angle XPM$, hence KX is tangent to (XPM) and we have

$$KP.KM = KX^2 = KE.KF$$

Therefore, $EPMF$ is cyclic.



Problem 4. Given an isosceles triangle ABC with $AB = AC$, let points M and N lie on side BC such that $\angle MAN = \frac{1}{2}\angle BAC$, and M lies between B and N. P is the center of the circumcircle of triangle AMN. The perpendicular bisectors of AM and AN intersect BC at points R and Q, respectively. Points S and T lie on PR and PQ, such that $ST \perp PA$. K and L are the reflections of A with respect to lines QS and RT, respectively. Prove that the circumcircles (CMK) and (BNL) intersect on the perpendicular bisector of side BC.

Proposed by Tran Quan Hung - Vietnam

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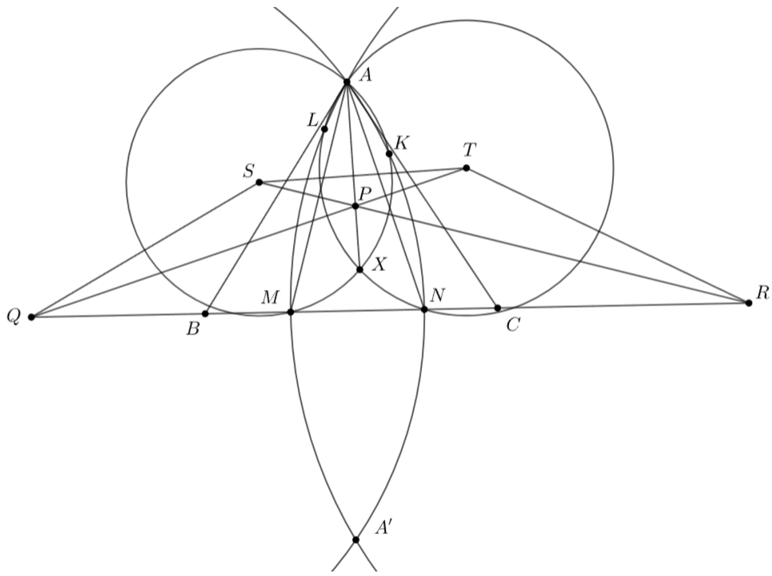
Solution. Let X be the reflection of A across the line ST. It's clear that $SM = SA = SK = SX$ and $TN = TA = TL = TX$. Define by A' the reflection of A across the line BC. Note that Q is the center of the circle passing through A, N, A'. Similarly R is the center of the circle passing through A, M, A'.



Let Ψ be the composition of an inversion centered at A and radius $\sqrt{AM.AN}$ followed by a reflection across the angle bisector of $\angle MAN$, and let $X^* = \Psi(X)$ for any point X in the plane. Note that $M^* = N$, $N^* = M$ and $P^* = A'$. Given $\angle MAN = \frac{1}{2}\angle ABC$ it's clear that $\angle ANB^* = \angle ABM = 90 - \angle MAN$. Similarly $\angle ANC^* = 90 - \angle MAN$ and since B^*, C^* lie on the circumcircle of triangle MAN, they are the reflections of the orthocenter H of triangle MAN across sides AN, AM. Note that X lies on the line AP hence X^* is a point on line AH. It's easy to see that K is the second intersection of circumcircle of triangles ANA', AXM thus K^* is the intersection of NX^*, PM. Similarly L^* is the intersection of MX^*, PN. We need to show that circumcircles of triangles B^*ML^*, C^*NK^* intersect on AP.

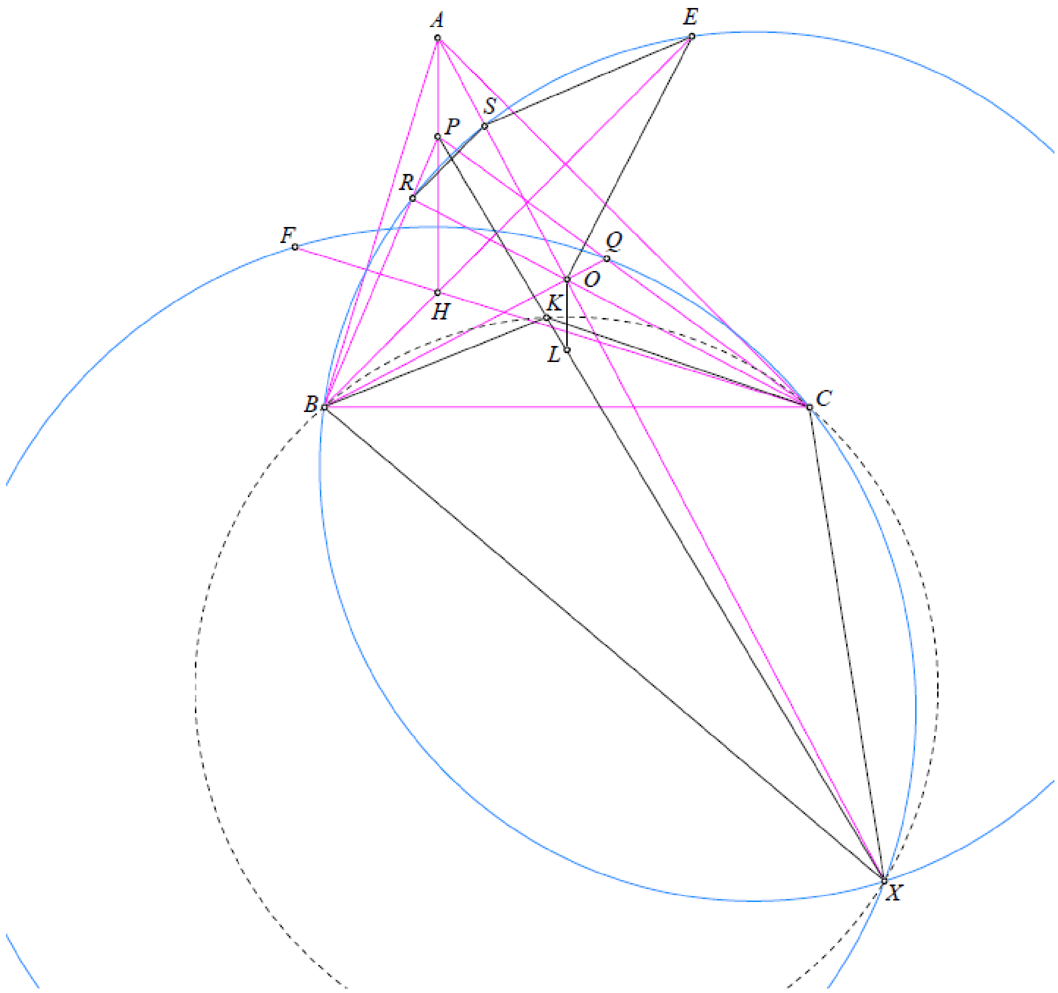
Now we translate the problem to standard notation as follows:

Problem. Given an acute triangle ABC with orthocenter H and circumcenter O. Let E and F be the reflections of H across CA and AB respectively. P is an arbitrary point on line AH. The lines OB, OC intersect PC, PB at Q, R, respectively. Prove that the circles $(BRE), (CQF)$ have a common point with AO.

Let K be the isogonal conjugate of H in the triangle PBC. Let X be the second intersection of PK

and the circumcircle (KBC). We have the following directed angle modulo 180° transformation :

$$\begin{aligned}(BP, BX) &= (BP, BC) + (BC, BX) = (BP, BC) + (KC, KX)\\ &= (PB, PK) + (CK, CB) = (PH, PC) + (CP, CH) = (HP, HC)\\ &= (BC, BA)\end{aligned}$$



Thus, the lines BP, BA are isogonal conjugates in angle $\angle XBC$. Similarly, lines CP, CA are isogonal conjugates in angle XCB. Hence, A, P are isogonal conjugates with respect to triangle XBC. Therefore, if L is the circumcenter of triangle PBC, then XO and XL are isogonal conjugates in angle $\angle BXC$. It is easy to see that PK passese through L, or XP passes through L, hence XA passes through O. Thus, X lies on OA.
Since E is the reflection of H across CA, the angles formed by the pairs of lines (OE, OB) and (OA, OC) share a common angle bisector. Therefore, if S is the reflection of R across the perpendicular bisector of BE (which is also the bisector of (OE, OB)), then S lies on OA. Thus, $BRSE$ is an isosceles trapezoid. We have the following directed angle modulo 180° transformation :

$$\begin{aligned}(XB, XS) &= (XP, XC) = (XK, XC) = (BK, BC)\\ &= (BP, BH) = (BR, BE) = (EB, ES)\end{aligned}$$

Thus, X lies on (BRE). By similar reasoning, X also lies on (CQF). This compeletes the proof.

Problem 5. The incircle of triangle ABC touches the sides AC, AB at points E, F respectively. Circle ω_1 is tangent to the segments BE, CE and it also touches the circumcircle of triangle ABC. Circle ω_2 is tangent to the segments CF, BF and it also touches the circumcircle of triangle ABC. Prove that the exsimilicenter of two circles ω_1, ω_2 lies on the radical axis of the incircle and circumcircle of triangle ABC.(The exsimilicenter of two circles is the intersection point of their external common tangents.)

Proposed by Amirmahdi Mohseni - Iran

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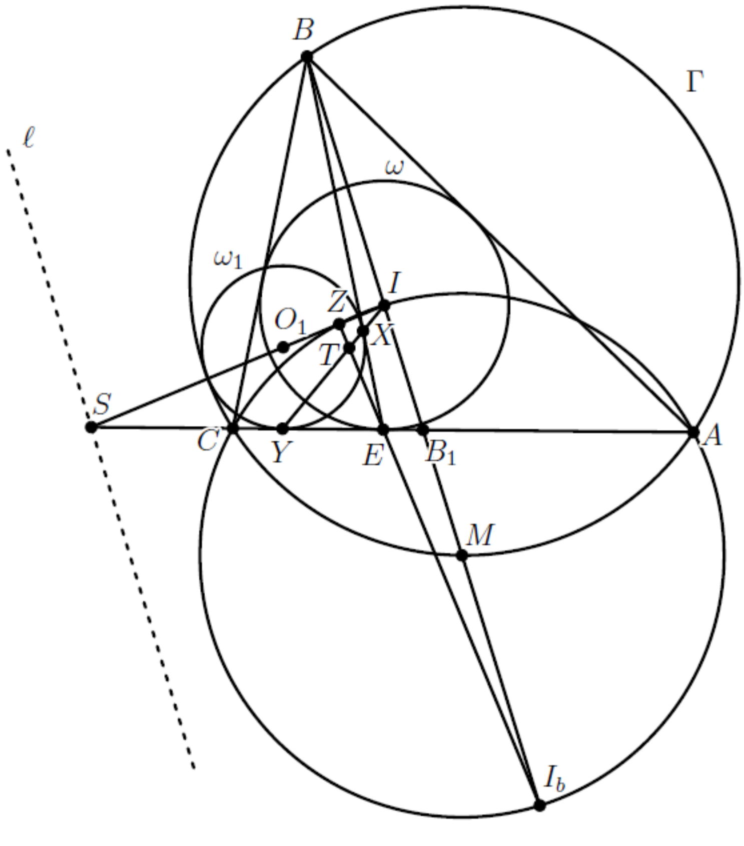
Solution.

Let ω, Γ be the incircle and circumcircle of triangle ABC respectively.Denote by l the radical axis of ω, Γ. We claim that the external homothety center of ω_1, ω lies on l. Similarly the external homothety center of ω_2, ω lies on l and the Monge's theorem concludes the proof.

Let I, I_b be the incenter and the B-excenter of triangle ABC. Letting B_1 be the foot of the internal angle bisector of B it is known that $(BB_1, II_b) = -1$. On the other hand if ω_1 touches BE, CE at points X, Y, then I, X, Y are collinear due to Sawayama's theorem. Let T be the intersection of EI_b, XY. Then

$$-1 = (BB_1, II_b) = E(BB_1, II_b) = (XY, IT)$$



Thus the polar line of I with respect to circle ω_1 passes through T. Note that XY is the polar line of E with respect to circle ω_1 and I lies on this line, therefore E lies on the polar of I with respect to ω_1 and ET is this line. Hence letting O_1 be the center of ω_1 we have $O_1I \perp ET$. Let Z, S be the intersection points of IO_1 with TE and BC respectively. It's clear that S is the external homothety center of ω_1, ω. Now since $\angle IZI_b = 90°$, Z lies on the circle with diameter II_b. Hence $CZIAI_b$ is cyclic and

$$SA.SC = SZ.SI = SE^2$$

since $\angle SZE = \angle IES = 90°$. This means that S lies on l and the claim concludes.