First Iranian Geometry Olympiad

September 2014

Solutions of Junior Level

1. In a right triangle ABC we have $\angle A = 90^\circ$, $\angle C = 30^\circ$. Denot by C the circle passing through A which is tangent to BC at the midpoint. Assume that C intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

Mahdi Etesami Fard

Proof. Let K midpoint of side BC. Therefore:

$$AK = KC \Rightarrow \angle KAC = \angle NKC = 30^\circ$$

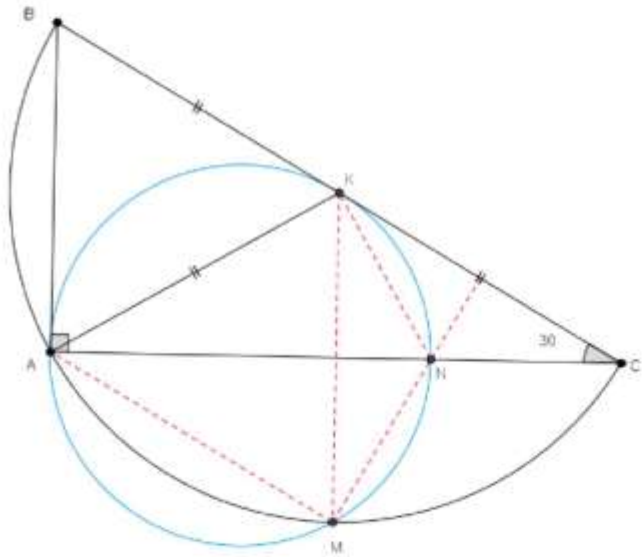
$$\angle ANK = \angle NKC + \angle ACB = 60^\circ$$

A, K, N, M lie on circle (C). Therefore:

$$\angle KAN = \angle KMN = 30^\circ, \angle AMK = 60^\circ$$

We know that K is the circumcenter of $\triangle ABC$. So we can say $KM = KC = AK$. Therefore $\triangle AKM$ is equilateral.(because of $\angle AMK = 60^\circ$). So $\angle AKM = 60^\circ$. We know that $\angle AKB = 60^\circ$, so we have $\angle MKC = 60^\circ$. On the other hand:

$$\angle KMN = 30^\circ \Rightarrow MN \perp BC$$


B
K
30
A
N
C
M

□

2. The inscribed circle of $\triangle ABC$ touches BC, AC and AB at D, E and F respectively. Denote the perpendicular foots from F, E to BC by K, L respectively. Let the second intersection of these perpendiculars with the incircle be M, N respectively. Show that $\frac{S_{\triangle BMD}}{S_{\triangle CND}} = \frac{DK}{DL}$.

Mahdi Etesami Fard

Proof. Let I be the incenter of $\triangle ABC$. We know that

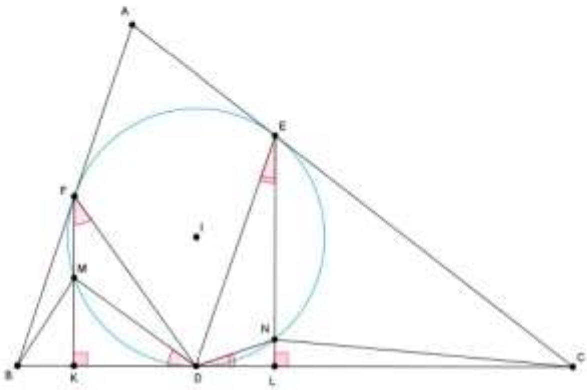
$$\left.\begin{array}{l}\angle BFK = 90^\circ - \angle B \\ \angle BFD = 90^\circ - \frac{1}{2}\angle B\end{array}\right\} \Rightarrow \angle DFM = \frac{1}{2}\angle B$$

But $\angle DFM = \angle MDK$. Therefore

$$\angle MDK = \frac{1}{2}\angle B$$

hence $\triangle MDK$ and $\triangle BID$ are similar (same angles) and $\frac{MK}{DK} = \frac{r}{BD}$. In the same way we have $\frac{NL}{DL} = \frac{r}{CD}$. Therefore

$$r = \frac{MK \cdot BD}{DK} = \frac{NL \cdot CD}{DL} \Rightarrow \frac{area\ of\ \triangle BMD}{area\ of\ \triangle CND} = \frac{MK \cdot BD}{NL \cdot CD} = \frac{DK}{DL}$$


A
E
F
I
M
N
B
K
D
L
C

□

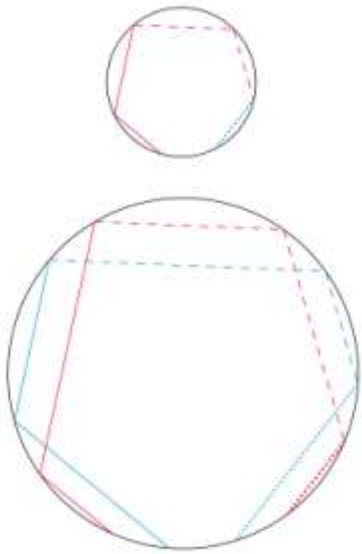
3. Each of Mahdi and Morteza has drawn an inscribed 93-gon. Denote the first one by $A_1A_2...A_{93}$ and the second by $B_1B_2...B_{93}$. It is known that $A_iA_{i+1} \parallel B_iB_{i+1}$ for $1 \leqslant i \leqslant 93$ ($A_{93} = A_1, B_{93} = B_1$). Show that $\frac{A_iA_{i+1}}{B_iB_{i+1}}$ is a constant number independent of i.

Morteza Saghafian

Proof. We draw a 93-gon similar with the second 93-gon in the circumcircle of the first 93-gon (so the sides of the second 93-gon would be multiplying by a constant number c). Now we have two 93-gons witch are inscribed in the same circle and apply the problem's conditions. We name this 93-gons $A_1A_2...A_{93}$ and $C_1C_2...C_{93}$.

We know that $A_1A_2 \parallel C_1C_2$. Therefore $\overset{\frown}{A_1C_1}=\overset{\frown}{A_2C_2}$ but they lie on the opposite side of each other. In fact, $\overset{\frown}{A_iC_i}=\overset{\frown}{A_{i+1}C_{i+1}}$ and they lie on the opposite side of each other for all $1 \leqslant i \leqslant 93$ ($\overset{\frown}{A_{94}C_{94}}=\overset{\frown}{A_1C_1}$). Therefore $\overset{\frown}{A_1C_1}$ and $\overset{\frown}{A_1C_1}$ lie on the opposite side of each other. So $\overset{\frown}{A_1C_1}= 0°$ or $180°$. This means that the 93-gons are coincident or reflections of each other across the center. So $A_iA_{i+1} = C_iC_{i+1}$ for

$1 \leqslant i \leqslant 93$. Therefore, $\frac{A_iA_{i+1}}{B_iB_{i+1}} = c$.



$\square$

4. In a triangle ABC we have $\angle C = \angle A + 90°$. The point D on the continuation of BC is given such that $AC = AD$. A point E in the side of BC in which A doesnt lie is chosen such that

$$\angle EBC = \angle A, \angle EDC = \frac{1}{2}\angle A$$

Prove that $\angle CED = \angle ABC$.

Morteza Saghafian

Proof. Suppose M is the midpoint of CD. hence AM is the perpendicular bisector of CD. AM intersects DE and BE at P, Q respectively. Therefore, $PC = PD$. We have

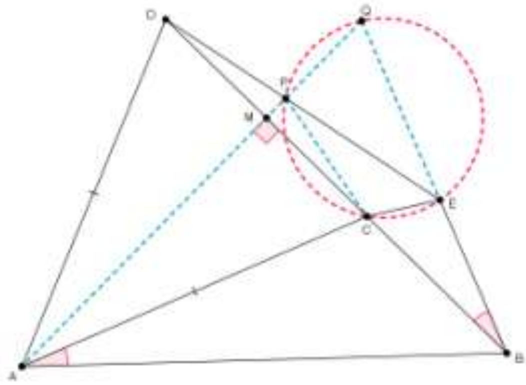
$$\angle EBA + \angle CAB = \angle A + \angle B + \angle A = 180° - \angle C + \angle A = 90°$$

hence $AC \perp BE$. Thus in $\triangle ABQ$, BC, AC are altitudes. This means C is the orthocenter of this triangle and

$$\angle CQE = \angle CQB = \angle A = \frac{1}{2}\angle A + \frac{1}{2}\angle A = \angle PDC + \angle PCD = \angle CPE$$

hence $CPQE$ is cyclic. Therefore

$$\angle CED = \angle CEP = \angle CQP = \angle CQA = \angle CBA = \angle B.$$


D
Q
P
M
E
C
A
B

□

First Iranian Geometry Olympiad

September 2014

Solutions of Senior Level

1. In a right triangle ABC we have $\angle A = 90^\circ$, $\angle C = 30^\circ$. Denot by C the circle passing through A which is tangent to BC at the midpoint.Assume that C intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

Mahdi Etesami Fard

Proof. Let K midpoint of side BC. Therefore:

$$AK = KC \Rightarrow \angle KAC = \angle NKC = 30^\circ$$

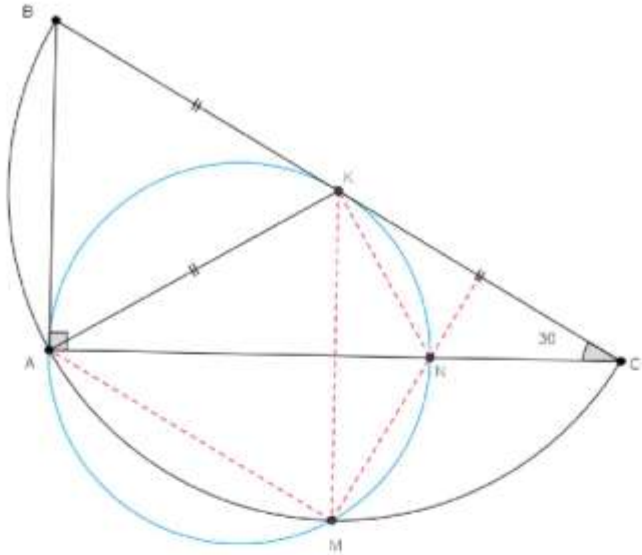
$$\angle ANK = \angle NKC + \angle ACB = 60^\circ$$

A, K, N, M lie on circle (C). Therefore:

$$\angle KAN = \angle KMN = 30^\circ, \angle AMK = 60^\circ$$

We know that K is the circumcenter of $\triangle ABC$. So we can say $KM = KC = AK$. Therefore $\triangle AKM$ is equilateral.(because of $\angle AMK = 60^\circ$). So $\angle AKM = 60^\circ$. We know that $\angle AKB = 60^\circ$, so we have $\angle MKC = 60^\circ$. On the other hand:

$$\angle KMN = 30^\circ \Rightarrow MN \perp BC$$


B
K
30
A
N
C
M

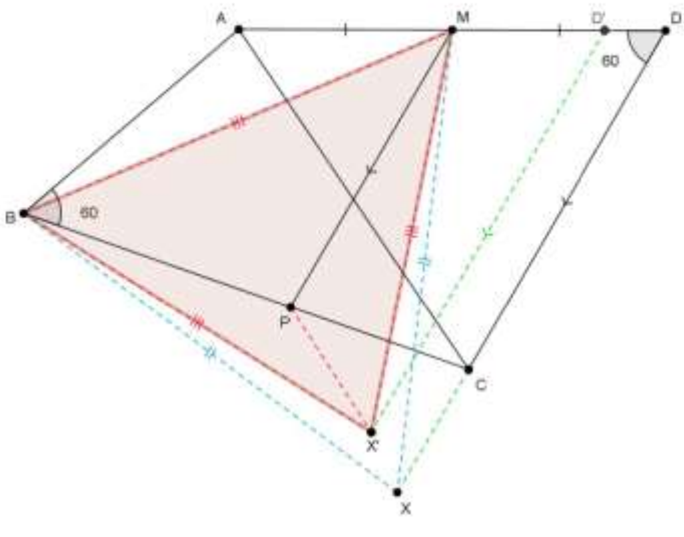
□

2. In a quadrilateral $ABCD$ we have $\angle B = \angle D = 60°$. Consider the line whice is drawn from M, the midpoint of AD, parallel to CD. Assume this line intersects BC at P. A point X lies on CD such that $BX = CX$. Prove that $AB = BP \Leftrightarrow \angle MXB = 60°$.

Davood Vakili

Proof. Suppose X' is a point such that $\triangle MBX'$ is equilateral.(X' and X lie on the same side of MB) It's enough to show that:

$$AB = BP \;\Leftrightarrow\; X' \equiv X$$



We want to prove that if $AB = BP$ then $\angle MXB = 60°$. $AB = BP$ therefore $\triangle ABP$ is equilateral. We know that $\angle ABP =$

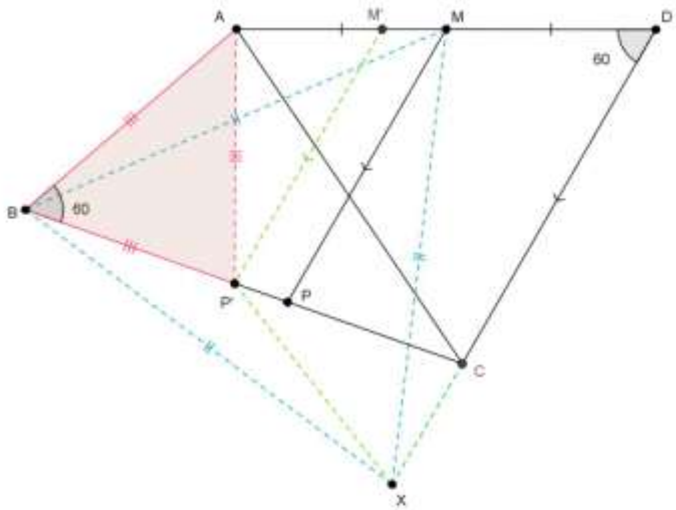
$\angle MBX' = 60°$, Therefore $\angle ABM = \angle PBX'$. On the other hand $AB = BP, BM = BX'$ therefore $\triangle BAM$ and $\triangle BPX'$ are equal.

$$\angle X'PM = 360° - \angle MPB - \angle BPX' = 360° - \angle DCB - \angle BAM' = 120°$$

$MP \parallel DC$, so we can say $\angle PMD = 120°$. If we draw the line passing through X' such that be parallel with CD and this line intersects AD in D', then quadrilateral $MPX'D'$ is isosceles trapezoid. Therefore $PX' = MD'$. In the other hand $PX' = AM = MD$ (becauese $\triangle BAM$ and $\triangle BPX'$ are equal.) According to the statements we can say $MD' = MD$. In other words, $D' \equiv D$ and X' lie on CD. Therefore both of X and X' lie on intersection of DC and perpendicular bisector of MB, so $X' \equiv X$.

Now we prove if $\angle MXB = 60°$ then $AB = BP$.
Let P' such that $\triangle MP'X$ be equilateral.(P' and X be on the same

side of AB) It's enough to show that $P' \equiv P$.



Draw the line passing through P' such that be parallel with CD. Suppose that this line intersects AD in M'.

$$\angle XP'M' = 360^\circ - \angle M'P'B - \angle BP'X = 360^\circ - \angle DCA - \angle BAM = 120^\circ$$

Also $\angle P'M'D = 120^\circ$. Therefore quadrilateral $XP'M'D$ is isosceles trapezoid and $DM' = P'X = AM = DM$. So we can say $M' \equiv M \Rightarrow P' \equiv P$. $\square$

3. An acute-angled triangle ABC is given. The circle with diameter BC intersects AB, AC at E, F respectively. Let M be the midpoint of BC and P the intersection point of AM and EF. X is a point on the arc EF and Y the second intersection point of XP with circle mentioned above. Show that $\angle XAY = \angle XYM$.

Ali Zooelm

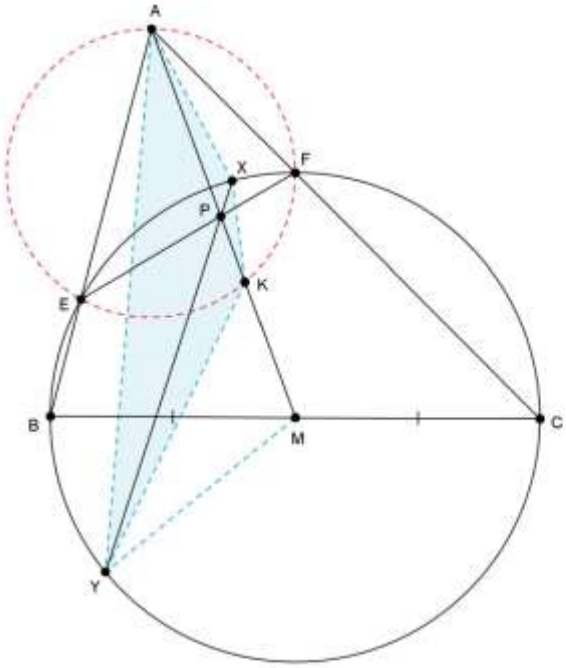
Proof. Suppose point K is intersection AM and circumcircle of $\triangle AEF$. MF tangent to circumcircle of $\triangle AEF$ at F.
(because of $\angle MFC = \angle MCF = \angle AEF$). Therefore $MF^2 = MK.MA$
. In the other hand, $MY = MF$ so $MY^2 = MK.MA$. It means

$$\angle MYK = \angle YAM \tag{1}$$

Also $AP.PK = PE.PF = PX.PY$ therefore $AXKY$ is(...??) .Therefore

$$\angle XAY = \angle XYK \tag{2}$$

According to equation 1 and 2 we can say $\angle XAY = \angle XYM$.



A
X
F
P
K
E
B
M
C
Y

□

4. The tangent line to circumcircle of the acute-angled triangle ABC $(AC > AB)$ at A intersects the continuation of BC at P. We denote by O the circumcenter of ABC. X is a point OP such that $\angle AXP = 90°$. Two points E, F respectively on AB, AC at the same side of OP are chosen such that

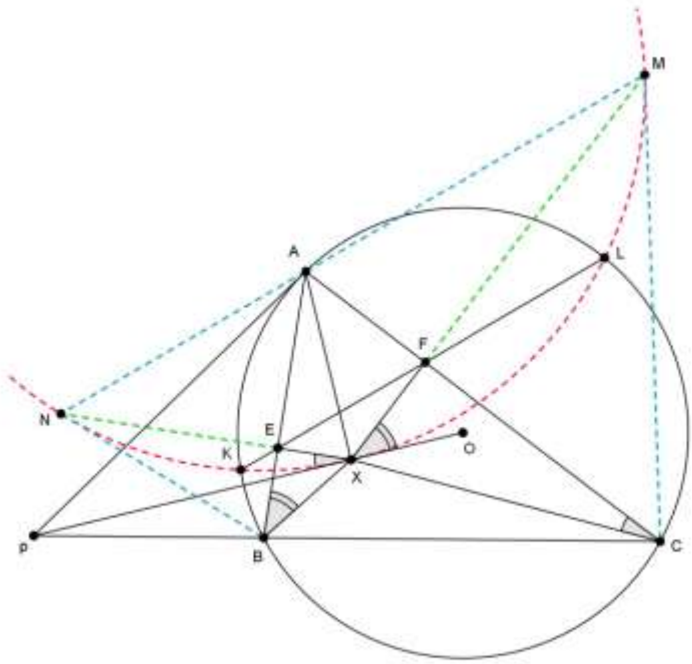
$$\angle EXP = \angle ACX, \ \angle FXO = \angle ABX$$

If K, L denote the intersection points of EF with the circumcircle of $\triangle ABC$, show that OP is tangent to the circumcircle of $\triangle KLX$.

Mahdi Etesami Fard

Proof. Let M and N on continuation of XF and XE such that M, L, X, N, K lie on same circle. We have to prove $\angle AMX = \angle ACX$. In other hand,

$\angle ACX = \angle NXP$ so we have to prove $\angle ACX = \angle NMX$.



We know:

$$XF.FM = FL.FK = AF.FC$$

Therefore $AMCX$ is (...?) and $\angle AMX = \angle ACX$. similarly we can say $ANBX$ is (...?). Now it's enough to show that $\angle AMX = \angle NMX$. In other words, we have to show that A, N, M lie on same line. we know that $ANBX$ is (...??) therefore:

$$\angle NAM = \angle NAE + \angle A + \angle FAM = \angle EXB + \angle A + \angle CXF$$

$$= \angle A + 180^\circ - \angle BXC + \angle ABX + \angle ACX$$

$$= \angle A + 180^\circ - \angle BXC + \angle BXC - \angle A = 180^\circ \qquad \square$$

5. Two points P, Q lie on the side BC of triangle ABC and have the same distance to the midpoint. The pependiculars from$P$, Q tp BC intesects AC, AB at E, F respectively. LEt M be the intersection point of $PF$and EQ. If H_1 and H_2 denote the orthocenter of $\triangle BFP$ and $\triangle CEQ$ recpectively, show that $AM \perp H_1H_2$.

Mahdi Etesami Fard

Proof. First we show that if we move P and Q, the line AM doesn't move. To show that we calculate $\frac{\sin\angle A_1}{\sin\angle A_2}$. By the law of sines in $\triangle AFM$ and $\triangle AEM$ we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle F_1}{\sin\angle E_1} \cdot \frac{FM}{EM} \tag{3}$$

also, for $\triangle FBP$ and $\triangle CEQ$ we have

$$\left.\begin{array}{l} \sin\angle F_1 = \frac{BP}{PF} \cdot \sin\angle B \\ \sin\angle E_1 = \frac{CQ}{EQ} \cdot \sin\angle C \end{array}\right\} \Rightarrow \frac{\sin\angle F_1}{\sin\angle E_1} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{EQ}{FP} \tag{4}$$

from (3) and (4) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{EQ}{FP} \cdot \frac{FM}{EM} \tag{5}$$

$\triangle FMQ$ and $\triangle EMP$ are similar, thus

$$\frac{FM}{FP} = \frac{FQ}{FQ+EP}, \ \frac{EQ}{EM} = \frac{FQ+EP}{EP}$$

with putting this into (5) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{FQ}{EP} \tag{6}$$

on the other hand

$$\left.\begin{array}{l} \tan\angle B = \frac{FQ}{BQ} \\ \tan\angle C = \frac{EP}{CP} \\ BQ = CP \end{array}\right\} \Rightarrow \frac{FQ}{EP} = \frac{\tan\angle B}{\tan\angle C}$$

if we put this in (6) we have

$$\frac{\sin\angle A_1}{\sin\angle A_2} = \frac{\sin\angle B}{\sin\angle C} \cdot \frac{\tan\angle B}{\tan\angle C}$$

wich is constant.

now we show that H_1H_2s are parallel. consider α the angle between H_1H_2 and BC. hence we have

$$\tan\alpha = \frac{H_2P - H_1Q}{QP} \tag{7}$$

H_1 and H_2 are the orthometers of $\triangle BFP$ and $\triangle CQE$ respectively. Thus we have

$$QF \cdot H_1Q = BQ \cdot QP \Rightarrow H_1Q = \frac{BQ \cdot QP}{FQ}$$

$$EP \cdot H_2P = CP \cdot PQ \Rightarrow H_2P = \frac{CP \cdot PQ}{EP}$$

but $CP = BQ$. Thus

$$H_2P - H_1Q = \frac{PQ \cdot BQ \cdot (FQ - EP)}{EP \cdot FQ}$$

by putting this in (7) :

$$\tan\alpha = \frac{BQ \cdot (FQ - EP)}{EP \cdot FQ} = \frac{BQ}{EP} - \frac{BQ}{FQ} = \frac{CP}{EP} - \frac{BQ}{FQ}$$

$$\Rightarrow \tan\alpha = \cot\angle B - \cot\angle C \tag{8}$$

hence $\tan\alpha$ is constant, thus H_1H_2s are parallel.

Soppuse θ is the angle between AM and BC. we have to show

$$\tan\alpha \cdot \tan\theta = 1$$

let AM intersects with BC at X. We have

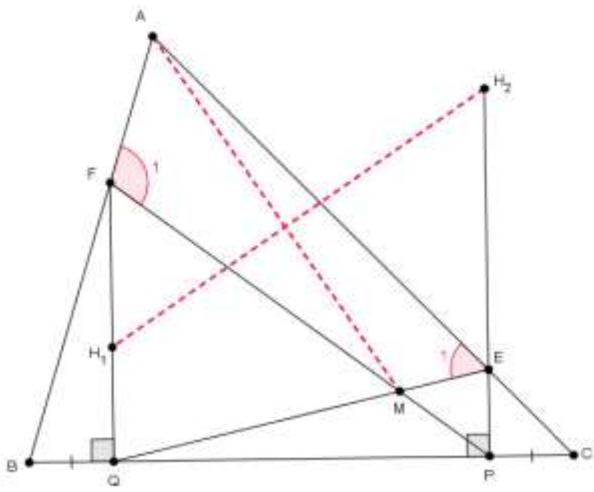
$$\frac{BX}{CX} = \frac{\sin\angle A_1}{\sin\angle A_2} \cdot \frac{\sin\angle C}{\sin\angle B} \Rightarrow \frac{BX}{CX} = \frac{\tan\angle B}{\tan\angle C}$$

let D be the foot of the altitude drawn from A. We have

$$\frac{BX}{CX} = \frac{\tan\angle B}{\tan\angle C} = \frac{\frac{AD}{BD}}{\frac{AD}{CD}} = \frac{CD}{BD} \Rightarrow BD = CX$$

$$\tan\theta = \frac{AD}{DX} = \frac{AD}{CD - CX} = \frac{AD}{CD - BD} = \frac{1}{\frac{CD}{AD} - \frac{BD}{AD}} = \frac{1}{\cot\angle B - \cot\angle C}$$

this equality and (8) implies that $AM \perp H_1H_2$.



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5. Two points X, Y lie on the arc BC of the circumcircle of $\triangle ABC$ (this arc does not contain A) such that $\angle BAX = \angle CAY$. Let M denotes the midpoint of the chord AX. Show that $BM + CM > AY$.

Mahan Tajrobekar

Proof. O is the circumcenter of $\triangle ABC$, so $OM \perp AX$. We draw a perpendicular line from B to OM. This line intersects with the circumcircle at Z. Since $OM \perp BZ$, OM is the perpendicular bisector of BZ. This means $MZ = MB$. By using triangle inequality we have

$$BM + MC = ZM + MC > CZ$$

But $BZ \parallel AX$, thus

$$\overset{\frown}{AZ} = \overset{\frown}{BX} = \overset{\frown}{CY} \Rightarrow Z\overset{\frown}{AC} = Y\overset{\frown}{CA} \Rightarrow CZ = AY$$

hence $BM + CM > AY$.