On the extraction of roots of nonics

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Abstract. *We present a novel decomposition method, which decomposes the given nonic equation into quartic and quintic polynomials as factors, eventually leading to the extraction of its four roots in radicals. The conditions to be satisfied by the coefficients of such partially solvable nonic are derived.*

Key words: *Nonic equation, decic equation, solvable equations, polynomial decomposition*

1. Introduction

As known from the works of Abel and Galois, general polynomial equations of degree higher than four cannot be solved in radicals [1]. However they can be solved algebraically using symbolic coefficients [1 - 2]. By imposing certain conditions, these equations lose their generality and become solvable in radicals. Such equations are known as solvable equations. There is another class of equations, which under some conditions becomes decomposable (reducible over complex field in general) but not completely solvable in radicals. In this paper we deal with such polynomial equation (of degree nine), which under certain conditions, gets decomposed into two polynomial factors: one quartic and one quintic. From the quartic polynomial factor four roots of the nonic are extracted. However, the quintic polynomial cannot be solved in radicals unless one more condition is imposed.

The criteria to be satisfied by the coefficients of such partially solvable nonic equation are derived. In the end, a numerical example is given, which uses the proposed method to extract four roots of nonic equation.

2. Formulation of equations

Consider the following nonic equation in x:

$$x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad (1)$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8 are the real coefficients. In the proposed method, we add a root at the origin to the above equation, which is equivalent

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to multiplying (1) by x. This action converts the nonic equation (1) to a decic equation as shown below.

$$x^{10} + a_8x^9 + a_7x^8 + a_6x^7 + a_5x^6 + a_4x^5 + a_3x^4 + a_2x^3 + a_1x^2 + a_0x = 0 \qquad (2)$$

Our aim is to decompose the above decic equation into two quintic factors. For this purpose consider another decic equation as shown below.

$$(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)^2 - (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)^2 = 0 \quad (3)$$

where b_0, b_1, b_2, b_3, b_4 and c_0, c_1, c_2, c_3, c_4 are the unknown coefficients in the quintic and the quartic polynomial terms respectively, in (3). Notice that the above decic equation [(3)] can be easily decomposed into two quintic factors as shown below.

$$\begin{aligned}&\{(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)\}\\&\{(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) + (c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)\}\\&= 0 \qquad (4)\end{aligned}$$

Equation (4) is further simplified as:

$$\begin{aligned}&\{x^5 + (b_4 - c_4)x^4 + (b_3 - c_3)x^3 + (b_2 - c_2)x^2 + (b_1 - c_1)x + b_0 - c_0\}\\&\{x^5 + (b_4 + c_4)x^4 + (b_3 + c_3)x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + b_0 + c_0\}\\&= 0 \qquad (5)\end{aligned}$$

Thus if the decic equation (2) can be represented in the form of decic equation (3), then it can be easily decomposed into two quintic factors as given in (5). In order that (3) represents decic equation (2), the coefficients of (2) and (3) should be equal. However, note that the coefficients of (3) are not explicitly available. Therefore, to facilitate comparison of coefficients of (2) and (3), the decic equation (3) is expanded and rearranged in descending powers of x, as shown below.

$$\begin{aligned}&x^{10} + 2b_4x^9 + (b_4^2 + 2b_3 - c_4^2)x^8 + 2(b_2 + b_3b_4 - c_3c_4)x^7\\&+(b_3^2 + 2b_1 + 2b_2b_4 - c_3^2 - 2c_2c_4)x^6 + 2(b_0 + b_1b_4 + b_2b_3 - c_1c_4 - c_2c_3)x^5\\&+(b_2^2 + 2b_0b_4 + 2b_1b_3 - c_2^2 - 2c_0c_4 - 2c_1c_3)x^4\\&+2(b_0b_3 + b_1b_2 - c_0c_3 - c_1c_2)x^3 + (b_1^2 + 2b_0b_2 - c_1^2 - 2c_0c_2)x^2\\&+2(b_0b_1 - c_0c_1)x + b_0^2 - c_0^2 = 0 \qquad (6)\end{aligned}$$

Equating the coefficients of (2) and (6), we obtain the following ten equations.

$$2b_4 = a_8 \qquad (7)$$

$$b_4^2 + 2b_3 - c_4^2 = a_7 \qquad (8)$$

$$2(b_2 + b_3b_4 - c_3c_4) = a_6 \qquad (9)$$

$$b_3^2 + 2b_1 + 2b_2b_4 - c_3^2 - 2c_2c_4 = a_5 \qquad (10)$$

$$2(b_0 + b_1b_4 + b_2b_3 - c_1c_4 - c_2c_3) = a_4 \qquad (11)$$

$$b_2^2 + 2b_0b_4 + 2b_1b_3 - c_2^2 - 2c_0c_4 - 2c_1c_3 = a_3 \tag{12}$$

$$2(b_0b_3 + b_1b_2 - c_0c_3 - c_1c_2) = a_2 \tag{13}$$

$$b_1^2 + 2b_0b_2 - c_1^2 - 2c_0c_2 = a_1 \tag{14}$$

$$2(b_0b_1 - c_0c_1) = a_0 \tag{15}$$

$$b_0^2 - c_0^2 = 0 \tag{16}$$

Equation (16) results into two expressions for c_0 as: $c_0 = \pm b_0$, and we choose:

$$c_0 = b_0 \tag{17}$$

From (7) b_4 is evaluated as: $b_4 = a_8/2$, and substituting this in equations, (8) - (12), we obtain expressions as shown below.

$$2b_3 - c_4^2 = a_7 - (a_8^2/4) \tag{18}$$

$$2b_2 + a_8b_3 - 2c_3c_4 = a_6 \tag{19}$$

$$b_3^2 + 2b_1 + a_8b_2 - c_3^2 - 2c_2c_4 = a_5 \tag{20}$$

$$2b_0 + a_8b_1 + 2b_2b_3 - 2c_1c_4 - 2c_2c_3 = a_4 \tag{21}$$

$$b_2^2 + a_8b_0 + 2b_1b_3 - c_2^2 - 2c_0c_4 - 2c_1c_3 = a_3 \tag{22}$$

We now use (17) to eliminate c_0 from the equations, (22), (13), (14), and (15), to obtain the following new expressions.

$$b_2^2 - c_2^2 + 2b_1b_3 - 2c_1c_3 + b_0(a_8 - 2c_4) = a_3 \tag{23}$$

$$2[b_1b_2 - c_1c_2 + b_0(b_3 - c_3)] = a_2 \tag{24}$$

$$b_1^2 - c_1^2 + 2b_0(b_2 - c_2) = a_1 \tag{25}$$

$$2b_0(b_1 - c_1) = a_0 \tag{26}$$

The latest equations to be considered for determining the unknowns are, (18), (19), (20), (21), (23), (24), (25), and (26). The unknowns to be determined are: $b_0, b_1, b_2, b_3, c_1, c_2, c_3$, and c_4. Notice that there are eight equations to determine eight unknowns. However, as we know from the works of Abel and Galois, we need some more equations, called supplementary equations, to determine the unknowns, and thereby to decompose the given nonic equation.

3. Use of supplementary equations

Our aim is to decompose the nonic using minimum possible number of supplementary equations. Therefore, we introduce one supplementary equation as shown below, and see whether it is sufficient to decompose the nonic equation (1).

$$c_4 = 0 \tag{27}$$

The above value of c_4 is substituted in equations, (18), (19), (20), (21), and (23), resulting in the following new equations.

$$b_3 = [a_7 - (a_8^2/4)]/2 \tag{28}$$

$$b_2 = (a_6 - a_8 b_3)/2 \tag{29}$$

$$2b_1 = a_5 - b_3^2 - a_8 b_2 + c_3^2 \tag{30}$$

$$2b_0 = a_4 - 2b_2 b_3 - a_8 b_1 + 2c_2 c_3 \tag{31}$$

$$a_8 b_0 = a_3 - b_2^2 - 2b_1 b_3 + 2c_1 c_3 + c_2^2 \tag{32}$$

¿From (28) and (29) we notice that b_3 and b_2 are determined. Therefore, now there are five unknowns (b_0, b_1, c_1, c_2, and c_3) to be determined from six equations [(30), (31), (32), (24), (25), and (26)]. We attempt to determine the unknowns using these equations by means of elimination method. However, after some algebraic manipulations it becomes clear that it is not possible to solve these equations. Hence one more supplementary equation is introduced as shown below, to facilitate the determination of unknowns.

$$c_3 = 0 \tag{33}$$

Using (33) we substitute the value of c_3 in equations, (30), (31), (32), and (24), to obtain the following new expressions.

$$b_1 = (a_5 - b_3^2 - a_8 b_2)/2 \tag{34}$$

$$b_0 = (a_4 - 2b_2 b_3 - a_8 b_1)/2 \tag{35}$$

$$c_2^2 = a_8 b_0 + b_2^2 + 2b_1 b_3 - a_3 \tag{36}$$

$$2c_1 c_2 = 2(b_1 b_2 + b_0 b_3) - a_2 \tag{37}$$

¿From (34) and (35) we observe that b_1 and b_0 are determined. Therefore, we are left with two unknowns (c_1 and c_2), and four equations [(36), (37), (25) and (26)]. Expression (26) is rearranged as shown below, to facilitate determination of c_1.

$$c_1 = b_1 - [a_0/(2b_0)] \tag{38}$$

Similarly, (37) is rearranged to obtain an expression for c_2 as shown below.

$$c_2 = [2(b_1 b_2 + b_0 b_3) - a_2]/(2c_1) \tag{39}$$

Since all the unknowns have been determined, readers may wonder what to do with the remaining two equations, (36) and (25). We discuss the utility of these equations later.

4. Decomposition of nonic

We have determined all the unknowns and, therefore, we conclude that the decic equation (2) can be decomposed in the form of (5). Substituting for c_0, c_3, and c_4, equation (5) gets modified as follows.

$$x\{x^4 + b_4x^3 + b_3x^2 + (b_2 - c_2)x + b_1 - c_1\}$$
$$\{x^5 + b_4x^4 + b_3x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + 2b_0\} = 0 \quad (40)$$

Remember that we added a root at the origin to the given nonic equation (1) to convert it into a decic equation [(2)]. Also note that the decic equation (2) is now represented in the form of factored equation (40). The added root is visible as a linear factor, x, in (40). After removing this linear factor, what is left is the factored nonic equation, representing the given nonic equation (1), as shown below.

$$\{x^4 + b_4x^3 + b_3x^2 + (b_2 - c_2)x + b_1 - c_1\}$$
$$\{x^5 + b_4x^4 + b_3x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + 2b_0\} = 0 \quad (41)$$

Thus we have successfully decomposed the given nonic equation into two polynomial factors, one quartic and one quintic. When quartic polynomial factor is equated to zero and solved either by the well-known Ferrari's method or by the method published recently [3], four roots of the nonic are extracted. However, the roots of quintic polynomial factor cannot be obtained in radicals unless one condition is imposed on the coefficients of the quintic polynomial, and making it solvable. Since obtaining the roots of solvable quintic has been discussed elsewhere [4, 5], we do not dwell upon it here.

5. Conditions for coefficients

Notice that equations, (36) and (25), were left unused so far, and with the determination of all unknowns in terms of coefficients of nonic (1), these equations [(36) and (25)] will now serve as the two conditions for the coefficients of nonic (1) to satisfy, so that it is decomposable. In other words two of the nine coefficients of the nonic (1) are dependent coefficients, which are determined from the remaining coefficients. For example, rearranging (36), we obtain an expression for a_3 in terms of other coefficients, as shown below.

$$a_3 = a_8b_0 + b_2^2 + 2b_1b_3 - c_2^2 \quad (42)$$

Note that b_0, b_1, b_2, b_3, and c_2 [appearing in (42)] are all functions of coefficients of nonic (1). Similarly, from expression (25) a_1 is determined from the other coefficients.

6. Numerical example

Let us decompose the following nonic equation using the proposed method.

$$x^9 + 2x^8 + 3x^7 + 4x^6 + 6x^5 + 7x^4 - 3x^3 + 2x^2 - 2x + 2 = 0$$

The first step is to check whether the coefficients of the above nonic satisfy the expressions (36) and (25). It is found that the coefficients satisfy these expressions. Therefore, we conclude that the nonic is decomposable by the proposed method. Next we determine $b_0, b_1, b_2, b_3, b_4, c_1$, and c_2 from the respective expressions as: 1, 1.5, 1, 1, 1, 1, 0.5, and 3. The above nonic is then decomposed in the form of (41) as:

$$(x^4 + x^3 + x^2 - 2x + 1)(x^5 + x^4 + x^3 + 4x^2 + 2x + 2) = 0$$

The four roots of the quartic equation, $x^4 + x^3 + x^2 - 2x + 1 = 0$, are determined using the methods available in literature [2, 3], as:

$$0.5106635 + 0.33248i, 0.5106635 - 0.33248i$$

$$-1.0106635 + 1.292934i, -1.0106635 - 1.292934i$$

where $i = \sqrt{-1}$.

7. Conclusions

The method of decomposition of certain nonic equations is presented, wherein the given nonic equation is factored into quartic and quintic polynomials using minimum number of supplementary equations. Four roots of nonic are extracted in radicals from the quartic polynomial factor.

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