JBMO Shortlist 2011

– Algebra

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1 Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$\prod(a^5 + a^4 + a^3 + a^2 + a + 1) \geq 8(a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)$

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2 Let x, y, z be positive real numbers. Prove that: $\frac{x+2y}{z+2x+3y} + \frac{y+2z}{x+2y+3z} + \frac{z+2x}{y+2z+3x} \leq \frac{3}{2}$

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3 A3 If a, b be positive real numbers, show that:

$$\sqrt{\frac{a^2 + ab + b^2}{3}} + \sqrt{ab} \leq a + b$$

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4 A4 Let x, y be positive reals satisfying the condition $x^3 + y^3 \leq x^2 + y^2$.Find the maximum value of xy.

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5 A5 Determine all positive integers a, b such that $a^2b^2 + 208 = 4([a, b] + (a, b))^2$ where $[a, b]$-lcm of a, b and (a, b)-gcd of a, b.

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6 Let $x_i > 1, \forall i \in \{1, 2, 3, \ldots, 2011\}$. Show that:

$$\frac{x_1^2}{x_2 - 1} + \frac{x_2^2}{x_3 - 1} + \frac{x_3^2}{x_4 - 1} + \ldots + \frac{x_{2010}^2}{x_{2011} - 1} + \frac{x_{2011}^2}{x_1 - 1} \geq 8044$$

When the equality holds?

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7 A7 Let a, b, c be positive reals such that $abc = 1$.Prove the inequality $\sum \frac{2a^2 + \frac{1}{a}}{b + \frac{1}{a} + 1} \geq 3$

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8 Decipher the equality $(\overline{LARN} - \overline{ACA}) : (\overline{CYP} + \overline{RUS}) = C^{Y^P} \cdot R^{U^S}$ where different symbols correspond to different digits and equal symbols correspond to equal digits. It is also supposed that all these digits are different from 0.

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9 Let $x_1, x_2, ..., x_n$ be real numbers satisfying $\sum_{k=1}^{n-1} min(x_k; x_{k+1}) = min(x_1; x_n)$.
Prove that $\sum_{k=2}^{n-1} x_k \geq 0$.

–	Combinatorics
1	Inside of a square whose side length is 1 there are a few circles such that the sum of their circumferences is equal to 10. Show that there exists a line that meets at least four of these circles.
2	Can we divide an equilateral triangle $\triangle ABC$ into 2011 small triangles using 122 straight lines? (there should be 2011 triangles that are not themselves divided into smaller parts and there should be no polygons which are not triangles)
3	We can change a natural number n in three ways: a) If the number n has at least two digits, we erase the last digit and we subtract that digit from the remaining number (for example, from 123 we get $12-3=9$); b) If the last digit is different from 0, we can change the order of the digits in the opposite one (for example, from 123 we get 321); c) We can multiply the number n by a number from the set $\{1,2,3,...,2010\}$. Can we get the number 21062011 from the number 1012011?
4	In a group of n people, each one had a different ball. They performed a sequence of swaps, in each swap, two people swapped the ball they had at that moment. Each pair of people performed at least one swap. In the end each person had the ball he/she had at the start. Find the least possible number of swaps, if: a) $n=5$, b) $n=6$.
5	A set S of natural numbers is called *good*, if for each element $x \in S, x$ does not divide the sum of the remaining numbers in S. Find the maximal possible number of elements of a *good* set which is a subset of the set $A=\{1,2,3,...,63\}$.
6	Let $n>3$ be a positive integer. Equilateral triangle ABC is divided into n^2 smaller congruent equilateral triangles (with sides parallel to its sides). Let m be the number of rhombuses that contain two small equilateral triangles and d the number of rhombuses that contain eight small equilateral triangles. Find the difference $m-d$ in terms of n.
7	Consider a rectangle whose lengths of sides are natural numbers. If someone places as many squares as possible, each with area 3, inside of the given rectangle, such that the sides of the squares are parallel to the rectangle sides, then the maximal number of these squares fill exactly half of the area of the rectangle. Determine the dimensions of all rectangles with this property.
8	Determine the polygons with n sides $(n \geq 4)$, not necessarily convex, which satisfy the property that the reflection of every vertex of polygon with respect to every diagonal of the polygon does not fall outside the polygon.

Note: Each segment joining two non-neighboring vertices of the polygon is a diagonal. The reflection is considered with respect to the support line of the diagonal.

9 Decide if it is possible to consider 2011 points in a plane such that the distance between every two of these points is different from 1 and each unit circle centered at one of these points leaves exactly 1005 points outside the circle.

– Geometry

1 Let ABC be an isosceles triangle with $AB = AC$. On the extension of the side CA we consider the point D such that $AD < AC$. The perpendicular bisector of the segment BD meets the internal and the external bisectors of the angle $\angle BAC$ at the points $E$and Z, respectively. Prove that the points A, E, D, Z are concyclic.

2 Let AD, BF and CE be the altitudes of $\triangle\ ABC$. A line passing through D and parallel to ABintersects the line EFat the point G. If H is the orthocenter of $\triangle\ ABC$, find the angle $\angle CGH$.

3 Let ABC be a triangle in which ($BL$is the angle bisector of $\angle ABC$ $(L \in AC)$, AH is an altitude of$\triangle\ ABC$ $(H \in BC)$ and $M$is the midpoint of the side AB. It is known that the midpoints of the segments BL and MH coincides. Determine the internal angles of triangle $\triangle\ ABC$.

4 Point D lies on the side BC of $\triangle\ ABC$. The circumcenters of $\triangle\ ADC$ and $\triangle\ BAD$ are O_1 and O_2, respectively and $O_1O_2 \parallel AB$. The orthocenter of $\triangle\ ADC$is H and $AH = O_1O_2$. Find the angles of $\triangle\ ABC$ if $2m\left(\angle C\right) = 3m\left(\angle B\right)$.

5 Inside the square $ABCD$, the equilateral triangle $\triangle\ ABE$ is constructed. Let M be an interior point of the triangle $\triangle\ ABE$ such that $MB = \sqrt{2}, MC = \sqrt{6}, MD = \sqrt{5}$ and $ME = \sqrt{3}$. Find the area of the square $ABCD$.

6 Let $ABCD$ be a convex quadrilateral and points E and F on sides AB, CD such that

$$\frac{AB}{AE} = \frac{CD}{DF} = n$$

If S is the area of $AEFD$ show that $S \leq \frac{AB\cdot CD+n(n-1)AD^2+n^2DA\cdot BC}{2n^2}$

– Number Theory

1 Solve in positive integers the equation $1005^x + 2011^y = 1006^z$.

2 Find all primes p such that there exist positive integers x, y that satisfy $x(y^2-p)+y(x^2-p) = 5p$

3 Find all positive integers n such that the equation $y^2 + xy + 3x = n(x^2 + xy + 3y)$ has at least a solution (x, y) in positive integers.

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4 N4 Find all primes p, q such that $2p^3 - q^2 = 2(p + q)^2$.

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5 Find the least positive integer such that the sum of its digits is 2011 and the product of its digits is a power of 6.
