# JBMO Shortlist 2014

– Algebra

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**1** Solve in positive real numbers: $n + \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor = 2014$

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**2** Let $a, b, c$ be positive real numbers such that $abc = \dfrac{1}{8}$. Prove the inequality:

$$a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 \geq \frac{15}{16}$$

When the equality holds?

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**3** For positive real numbers $a, b, c$ with $abc = 1$ prove that $\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \geq 3(a + b + c + 1)$

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**4** With the conditions $a, b, c \in \mathbb{R}^+$ and $a + b + c = 1$, prove that

$$\frac{7+2b}{1+a} + \frac{7+2c}{1+b} + \frac{7+2a}{1+c} \geq \frac{69}{4}$$

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**5** Let $x, y$ and $z$ be non-negative real numbers satisfying the equation $x + y + z = xyz$. Prove that $2(x^2 + y^2 + z^2) \geq 3(x + y + z)$.

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**6** Let $a, b, c$ be positive real numbers. Prove that

$$\left((3a^2+1)^2 + 2\left(1 + \frac{3}{b}\right)^2\right)\left((3b^2+1)^2 + 2\left(1 + \frac{3}{c}\right)^2\right)\left((3c^2+1)^2 + 2\left(1 + \frac{3}{a}\right)^2\right) \geq 48^3$$

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**7** $a, b, c \in \mathbb{R}^+$ and $a^2 + b^2 + c^2 = 48$. Prove that

$$a^2\sqrt{2b^3+16} + b^2\sqrt{2c^3+16} + c^2\sqrt{2a^3+16} \leq 24^2$$

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**8** Let $x, y, z$ be positive real numbers such that $xyz = 1$. Prove the inequality:

$$\frac{1}{x\left(ay+b\right)} + \frac{1}{y\left(az+b\right)} + \frac{1}{z\left(ax+b\right)} \geq 3$$

if:
(A) $a = 0, b = 1$
(B) $a = 1, b = 0$
(C) $a + b = 1,\ a, b > 0$
When the equality holds?

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**9** Let $n$ a positive integer and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ real positive numbers such that $x_1 + \ldots + x_n = y_1 + \ldots + y_n = 1$. Prove that:

$$|x_1 - y_1| + \ldots + |x_n - y_n| \leq 2 - \underset{1\leq i\leq n}{min} \frac{x_i}{y_i} - \underset{1\leq i\leq n}{min} \frac{y_i}{x_i}$$

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**–** Combinatorics

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**1** There are some real numbers on the board (at least two). In every step we choose two of them, for example $a$ and $b$, and then we replace them with $\frac{ab}{a+b}$. We continue until there is one number. Prove that the last number does not depend on which order we choose the numbers to erase.

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**2** In a country with $n$ towns, all the direct flights are of double destinations (back and forth). There are $r > 2014$ rootes between different pairs of towns, that include no more than one intermediate stop (direction of each root matters). Find the minimum possible value of $n$ and the minimum possible $r$ for that value of $n$.

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**3** For a positive integer $n$, two payers $A$ and $B$ play the following game: Given a pile of $s$ stones, the players take turn alternatively with $A$ going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a positive multiple of $n$ stones. The winner is the one who takes the last stone. Assuming both $A$ and $B$ play perfectly, for how many values of $s$ the player $A$ cannot win?

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**4** $A = 1 \cdot 4 \cdot 7 \cdots 2014$.Find the last non-zero digit of $A$ if it is known that $A \equiv 1 \mod 3$.

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**–** Geometry

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**1** Let $ABC$ be a triangle with $m\left(\angle B\right) = m\left(\angle C\right) = 40^\circ$ Line bisector of $\angle B$ intersects $AC$ at point $D$. Prove that $BD + DA = BC$.

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**2** Acute-angled triangle $ABC$ with $AB < AC < BC$ and let be $c(O, R)$ its circumcircle. Diameters $BD$ and $CE$ are drawn. Circle $c_1(A, AE)$ interescts $AC$ at $K$. Circle $c_2(A, AD)$ intersects $BA$ at $L$ .($A$ lies between $B$ $L$). Prove that lines $EK$ and $DL$ intersect at circle $c$ .

by Evangelos Psychas (Greece)

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**3** Consider an acute triangle $ABC$ of area $S$. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by $H_1$ and $H_2$ the orthocentres of the triangles $MNC$, respectively $MND$. Find the area of the quadrilateral $AH_1BH_2$ in terms of $S$.

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**4** Let $ABC$ be an acute triangle such that $AB \neq AC$.Let $M$ be the midpoint $BC, H$ the orthocenter of $\triangle ABC, O_1$ the midpoint of $AH$ and $O_2$ the circumcenter of $\triangle BCH$. Prove that $O_1AMO_2$ is a parallelogram.

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**5** Let $ABC$ be a triangle with $AB \neq BC$; and let $BD$ be the internal bisector of $\angle ABC, ,(D \in AC)$. Denote by $M$ the midpoint of the arc $AC$ which contains point $B$. The circumscribed circle of the triangle $\triangle\ BDM$ intersects the segment $AB$ at point $K \neq B$. Let $J$ be the reflection of $A$ with respect to $K$. If $DJ \cap AM = \{O\}$, prove that the points $J, B, M, O$ belong to the same circle.

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**6** Let $ABCD$ be a quadrilateral whose diagonals are not perpendicular and whose sides $AB$ and $CD$ are not parallel.Let $O$ be the intersection of its diagonals.Denote with $H_1$ and $H_2$ the orthocenters of triangles $AOB$ and $COD$, respectively.If $M$ and $N$ are the midpoints of the segment lines $AB$ and $CD$, respectively.Prove that the lines $H_1H_2$ and $MN$ are parallel if and only if $AC = BD$.

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**–** Number Theory

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**1** All letters in the word $VUQAR$ are different and chosen from the set $\{1, 2, 3, 4, 5\}$. Find all solutions to the equation

$$\frac{(V+U+Q+A+R)^2}{V-U-Q+A+R} = V^{U^{Q^{A^R}}}.$$

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**2** Find all triples of primes $(p, q, r)$ satisfying $3p^4 - 5q^4 - 4r^2 = 26$.

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**3** Find all integer solutions to the equation $x^2 = y^2(x + y^4 + 2y^2)$ .

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**4** Prove that there are not intgers $a$ and $b$ with conditions,
i) $16a - 9b$ is a prime number.
ii) $ab$ is a perfect square.
iii) $a + b$ is also perfect square.

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**5** Find all non-negative solutions to the equation $2013^x + 2014^y = 2015^z$

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**6** Vukasin, Dimitrije, Dusan, Stefan and Filip asked their teacher to guess three consecutive positive integers, after these true statements:
Vukasin: " The sum of the digits of one number is prime number. The sum of the digits of another of the other two is, an even perfect number.($n$ is perfect if $\sigma(n) = 2n$). The sum of the

digits of the third number equals to the number of it's positive divisors".
Dimitrije:"Everyone of those three numbers has at most two digits equal to $1$ in their decimal representation".
Dusan:"If we add $11$ to exactly one of them, then we have a perfect square of an integer"
Stefan:"Everyone of them has exactly one prime divisor less than $10$".
Filip:"The three numbers are square free".
Professor found the right answer. Which numbers did he mention?

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