MEDITERRANEAN MATHEMATICAL COMPETITION

Problem 1

Let $S = \{1, \ldots, 999\}$. Determine the smallest integer m, for which there exist m two-sided cards $C_1, \ldots, C_m$ with the following properties:

- Every card C_i has an integer from S on one side and another integer from S on the other side.
- For all $x, y \in S$ with $x \neq y$, it is possible to select a card C_i that shows x on one of its sides and another card C_j (with $i \neq j$) that shows y on one of its sides.

Solution

The answer is $m = 666$.

First, we construct a set of $m = 666$ cards with the desired property: For every triple $\{3k-2, 3k-1, 3k\}$ with $1 \leq k \leq 333$, we introduce one card with numbers $3k-2$ and $3k-1$, and one card with numbers $3k-2$ and $3k$. This altogether yields $2 \cdot 333 = 666$ cards. Now consider $x, y \in S$ with $x < y$. If $x, y \in \{3k-2, 3k-1, 3k\}$ for some k, then the card with numbers $3k-2$ and $3k-1$ shows x and the card with numbers $3k-2$ and $3k$ whows y. If x and y belong to different triples, it is trivial to select two cards C_i and C_j that show x and y.

Next, we show that $m \geq 666$ must hold. Consider a card system $C_1, \ldots, C_m$ with the desired property, and divide S into two parts:

- S_1 contains all elements of S that show up on exactly one card.
- $S_{\geq 2}$ contains all elements of S that show up on at least two cards.

Clearly $|S_1| + |S_{\geq 2}| = 999$. As every card has two sides, the m cards altogether show $2m$ numbers. This implies

$$|S_1| + 2|S_{\geq 2}| \;\leq\; 2m. \tag{1}$$

Next, observe that no card can show two numbers x and y from S_1: In that case it would be impossible to select a card C_i that shows x and another card C_j (with $i \neq j$) that shows y. This implies

$$|S_1| \;\leq\; m. \tag{2}$$

By adding up (1) and (2), we derive

$$2 \cdot 999 \;=\; 2(|S_1| + |S_{\geq 2}|) \;\leq\; m + 2m,$$

which implies the desired lower bound $m \geq 2 \cdot 999/3$.

Problem 2

(a) Decide whether there exist two decimal digits a and b, such that every integer with decimal representation $ab222\dots231$ is divisible by 73.

(b) Decide whether there exist two decimal digits c and d, such that every integer with decimal representation $cd222\dots231$ is divisible by 79.

Solution

(a) Suppose that such digits a and b do exist. Then $x = ab2231$ and $y = ab231$ are divisible by 73, and so is $10y - x = 79$. A contradiction.

(b) There are many ways of settling this. First, one has to guess that $c = 7$ and $d = 0$ might work. (Look for a small integer z, such that the decimal representation of $79z$ ends with the digits 31. As the unit digit of z must be 9, we try $z = 10n + 9$ and $79z = 790n + 711$. As $79z - 1 = 790n + 710$ ends with 30, we conclude that $79n + 71$ should end with 3, which means that $9n + 1$ should end with 3. Hence we try $n = 8$ and $z = 89$, which yields $79z = 7031$. Done with guessing.)

First proof that $c = 7$ and $d = 0$ works: Every integer $70222\dots231$ can be written as

$$70 \cdot 10^m + \frac{2}{9}(10^m - 1) + 9 \quad = \quad \frac{1}{9}(632 \cdot 10^m + 79) \quad = \quad 79 \cdot (8 \cdot 10^m + 1)\frac{1}{9}.$$

Second proof that $c = 7$ and $d = 0$ works: Induction. Clearly $7031 = 89 \cdot 79$. In the inductive step, express the $(m+1)$-digit number $x = 70222\dots231$ in terms of the m-digit number $y = 7022\dots231$. Since $x = 10y + 79$, the inductive assumption on y implies that also x is divisible by 79.

Problem 3

Let a, b, c, d be four positive real numbers. Prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{(b+c+d)^3}{b^3+c^3+d^3} + \frac{(c+d+a)^4}{c^4+d^4+a^4} + \frac{(d+a+b)^5}{d^5+a^5+b^5} \leq 120.$$

Solution. For $n \geq 2$, consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = x^n$. Since $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2} > 0$, then f is convex and we have, on account of Jensen's inequality,

$$f\left(\frac{x+y+z}{3}\right) \leq \frac{1}{3}\left(f(x) + f(y) + f(z)\right).$$

That is,

$$\frac{(x+y+z)^n}{3^n} \leq \frac{1}{3}\left(x^n + y^n + z^n\right) \Leftrightarrow \frac{(x+y+z)^n}{x^n+y^n+z^n} \leq 3^{n-1}.$$

We have,

- For $n = 2$, $\dfrac{(a+b+c)^2}{a^2+b^2+c^2} \leq 3.$

- For $n = 3$, $\dfrac{(b+c+d)^3}{b^3+c^3+d^3} \leq 9.$

- For $n = 4$, $\dfrac{(c+d+a)^4}{c^4+d^4+a^4} \leq 27.$

- For $n = 5$, $\dfrac{(d+a+b)^5}{d^5+a^5+b^5} \leq 81.$

Adding up the preceding, yields

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{(b+c+d)^3}{b^3+c^3+d^3} + \frac{(c+d+a)^4}{c^4+d^4+a^4} + \frac{(d+a+b)^5}{d^5+a^5+b^5} \leq 120.$$

Equality holds when $a = b = c = d$, and we are done.

Problem 4

The triangle ABC is inscribed in a circle γ of center O, with $AB < AC$. A point D on the angle bisector of $\angle BAC$ and a point E on segment BC satisfy OE is parallel to AD and $DE \perp BC$. Point K lies on the extension line of EB such that $EA = EK$. A circle pass through points A, K, D meets the extension line of BC at point P, and meets the circle of center O at point $Q \neq A$. Prove that the line PQ is tangent to the circle γ.

Solution

Draw the segments OA, OD. Let Q′ be a point on the circle γ such that $\angle Q'OD = 90^\circ$, and Q′ lies on different sides of D with respect to line AO. We prove that Q′=Q.

Draw the segments AK,KD,AQ′,Q′D. Suppose that line AD meet the circle γ again at $M \neq A$

$M \neq A$, then M is the midpoint of arc $\widehat{BC}$, and $MC \perp BC$. Notice that $DE \perp BC$, we have OM parallel to DE. Since OE is parallel to MD, the quadrilateral OMDE is a parallelogram, so DE=OM=OA, therefore the quadrilateral AOED is an isosceles trapezoid, then we can see that DE=AO=OQ′, EK=EA=OD. Notice that $\angle DEK = \angle Q'OD = 90^\circ$, we have $\triangle DEK \cong \triangle Q'OD$. Then $\angle OQ'D + \angle DKE = \angle EDK + \angle DKE = 90^\circ$.

Moreover $\angle AQ'O = 90^\circ - \frac{1}{2}\angle AOQ' = 90^\circ - \frac{1}{2}(270^\circ - \angle AOD) = \frac{1}{2}(\angle AOD - 90^\circ) =$

$= \frac{1}{2}(\angle AED - 90^\circ) = \frac{1}{2}\angle AEK = 90^\circ - \angle AKE$. So,

$\angle AQ'D + \angle AKD = (\angle AQ'O + \angle AKE) + (\angle OQ'D + \angle DKE) = 90^\circ + 90^\circ = 180^\circ$.

Then A,K,D,Q′ are concyclic. Since $Q' \neq A$, this shows that Q′=Q.

From $\triangle DEK \cong \triangle QOD$, we get DK=DQ, then $\angle KAD = \angle QAD$, and therefore $\angle KAB = \angle QAC$ since $\angle BAD = \angle CAD$.

Finally, by $\angle PQC = \angle BCQ - \angle KPQ = (180^\circ - \angle BAQ) - (180 - \angle KAQ) = \angle KAB = \angle QAC$, and the line PQ is tangent to circle γ.