10-th Mediterranean Mathematical Competition 2007

1. Let $x \le y \le z$ be real numbers such that $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the value of constant $\frac{1}{2}$?

2. The diagonals AC and BD of a convex cyclic quadrilateral $ABCD$ intersect at point E. Given that $AB = 39$, $AE = 45$, $AD = 60$ and $BC = 56$, determine the length of CD.

3. In the triangle ABC, the angle $\alpha = \angle BAC$ and the side $a = BC$ are given. Assume that $a = \sqrt{rR}$, where r is the inradius and R the circumradius. Compute all possible lengths of sides AB and AC.

4. Let $x > 1$ be a non-integer number. Prove that

$$\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}} \right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]} \right) > \frac{9}{2}.$$