

Neke konačne sume

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Sažetak. *Razmatraju se konačne sume, koje su ilustrirane na nizu zanimljivih zadataka prilagođenim učenicima srednjih škola.*

Ključne riječi: *konačne sume*

Some finite sums

Abstract. *Finite sums are considered. These applications are illustrated on a number of interesting tasks adapted for high school students.*

Key words: *finite sums*

Zadanoj n -torki brojeva a_1, a_2, \dots, a_n pridružimo sumu

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k.$$

Primjerice:

$$1 + 2 + 3 + \dots + n = \sum_{k=1}^n k,$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \sum_{k=1}^n k^2,$$

$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = \sum_{k=1}^n (2k-1)^3,$$

$$c + c + \dots + c = nc = \sum_{k=1}^n c,$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} = \sum_{k=1}^5 \frac{1}{k(k+1)}.$$

Lako se dokazuje da vrijede sljedeće formule:

1. $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k,$

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$$2. \sum_{k=1}^n (c \cdot a_k) = c \cdot \sum_{k=1}^n a_k,$$

$$3. \sum_{k=1}^n (a_k - a_{k-1}) = a_n - a_0.$$

Sa $S_k(n)$ označimo sumu k -tih potencija prvih n prirodnih brojeva, tj.

$$S_k(n) = 1^k + 2^k + \dots + n^k, \quad k \in \mathbb{N}$$

i odredimo formule pomoću kojih ćemo izračunavati te sume.

Primjer 1. *Odredimo sumu prvih n prirodnih brojeva.*

Polazimo od jednakosti $(a+1)^2 = a^2 + 2a + 1$ u koju umjesto a redom uvrštavamo brojeve $0, 1, 2, \dots, n-1, n$:

$$(0+1)^2 = 0^2 + 2 \cdot 0 + 1,$$

$$(1+1)^2 = 1^2 + 2 \cdot 1 + 1,$$

$$(2+1)^2 = 2^2 + 2 \cdot 2 + 1,$$

$$\vdots$$

$$(n-1+1)^2 = (n-1)^2 + 2(n-1) + 1,$$

$$(n+1)^2 = n^2 + 2n + 1.$$

Zbrajanjem ovih jednakosti dobivamo

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + n^2 + (n+1)^2 &= 1^2 + 2^2 + \dots + (n-1)^2 + n^2 \\ &\quad + 2(1 + 2 + \dots + (n-1) + n) \\ &\quad + n + 1, \end{aligned}$$

odakle slijedi

$$(n+1)^2 = 2S_1(n) + n + 1,$$

tj.

$$S_1(n) = \frac{n(n+1)}{2}.$$

Primjer 2. *Odredimo sumu kvadrata prvih n prirodnih brojeva.*

U jednakost $(a+1)^3 = a^3 + 3a^2 + 3a + 1$ umjesto a redom uvrštavamo brojeve $0, 1, 2, \dots, n-1, n$. Dobivamo:

$$1^3 = 0^3 + 3 \cdot 0^2 + 3 \cdot 0 + 1,$$

$$2^3 = 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1,$$

$$3^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1,$$

$$\vdots$$

$$n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1,$$

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1.$$

Zbrajanjem dobivamo

$$(n+1)^3 = 3S_2(n) + 3S_1(n) + (n+1).$$

Odatle imamo, redom,

$$(n+1)^3 = 3S_2(n) + 3 \cdot \frac{n(n+1)}{2} + (n+1),$$

$$3S_2(n) = (n+1)\left((n+1)^2 - \frac{3}{2}n - 1\right),$$

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

Primjer 3. *Odredimo sumu kubova prvih n prirodnih brojeva.*

U jednakost $(a+1)^4 = a^4 + 4a^3 + 6a^2 + 4a + 1$ umjesto a redom uvrštavamo brojeve $0, 1, 2, \dots, n-1, n$. Zbrajanjem dobivamo

$$(n+1)^4 = 4S_3(n) + 6S_2(n) + 4S_1(n) + (n+1),$$

odakle je

$$S_3(n) = \left(\frac{n(n+1)}{2}\right)^2.$$

Analogno bismo dobili formulu za sumu četvrtih, petih, ... potencija prvih n prirodnih brojeva.

Primjer 4. *Odredimo rekurzivnu formulu za sumu k -tih potencija prvih n prirodnih brojeva, tj.*

$$S_k(n) = 1^k + 2^k + \dots + n^k, \quad k \in \mathbb{N}.$$

Prema binomnom poučku je

$$\begin{aligned} (a+1)^{k+1} &= \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k + \binom{k+1}{2}a^{k-1} + \dots \\ &\quad + \binom{k+1}{k-1}a^2 + \binom{k+1}{k}a + \binom{k+1}{k+1} \cdot 1 \\ &= a^{k+1} + \binom{k+1}{1}a^k + \binom{k+1}{2}a^{k-1} + \dots \\ &\quad + \binom{k+1}{2}a^2 + \binom{k+1}{1}a + 1. \end{aligned}$$

Uvrštavajući umjesto a redom $0, 1, 2, \dots, n-2, n-1, n$ dobivamo

$$1^{k+1} = 0 + 0 + 0 + \dots + 0 + 0 + 1,$$

$$2^{k+1} = 1^{k+1} + \binom{k+1}{1} \cdot 1^k + \binom{k+1}{2} \cdot 1^{k-1} + \dots + \binom{k+1}{2} \cdot 1^2 + \binom{k+1}{1} \cdot 1 + 1,$$

$$3^{k+1} = 2^{k+1} + \binom{k+1}{1} \cdot 2^k + \binom{k+1}{2} \cdot 2^{k-1} + \dots + \binom{k+1}{2} \cdot 2^2 + \binom{k+1}{1} \cdot 2 + 1,$$

$$\vdots$$

$$(n-1)^{k+1} = (n-2)^{k+1} + \binom{k+1}{1} \cdot (n-2)^k + \binom{k+1}{2} \cdot (n-2)^{k-1} + \dots \\ + \binom{k+1}{2} \cdot (n-2)^2 + \binom{k+1}{1} \cdot (n-2) + 1,$$

$$n^{k+1} = (n-1)^{k+1} + \binom{k+1}{1} \cdot (n-1)^k + \binom{k+1}{2} \cdot (n-1)^{k-1} + \dots \\ + \binom{k+1}{2} \cdot (n-1)^2 + \binom{k+1}{1} \cdot (n-1) + 1,$$

$$(n+1)^{k+1} = n^{k+1} + \binom{k+1}{1} \cdot n^k + \binom{k+1}{2} \cdot n^{k-1} + \dots \\ + \binom{k+1}{2} \cdot n^2 + \binom{k+1}{1} \cdot n + 1.$$

Zbrajanjem dobivamo

$$(n+1)^{k+1} = \binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \dots \\ + \binom{k+1}{2} S_2(n) + \binom{k+1}{1} S_1(n) + S_0(n),$$

gdje je $S_0(n) = n + 1$. Odatle lako nalazimo $S_k(n)$ kada znamo $S_1(n)$, $S_2(n)$, \dots , $S_{k-1}(n)$.

Primjer 5. *Odredimo sumu četvrtih potencija prvih n prirodnih brojeva.*

Prema rekurzivnoj formuli je

$$(n+1)^5 = \binom{5}{1} S_4(n) + \binom{5}{2} S_3(n) + \binom{5}{2} S_2(n) + \binom{5}{1} S_1(n) + n + 1.$$

Dalje imamo redom

$$(n+1)^5 = 5S_4(n) + 10 \cdot \frac{n^2(n+1)^2}{4} + 10 \cdot \frac{n(n+1)(2n+1)}{6} + 5 \cdot \frac{n(n+1)}{2} + n + 1,$$

$$5S_4(n) = (n+1) \cdot \left((n+1)^4 - \frac{5n^2(n+1)}{2} - \frac{5n(2n+1)}{3} - \frac{5n}{2} - 1 \right),$$

$$5S_4(n) = (n+1) \cdot \left(n^4 + 4n^3 + 6n^2 + 4n + 1 - \frac{5n^3 + 5n^2}{2} - \frac{10n^2 + 5n}{3} - \frac{5n}{2} - 1 \right),$$

$$5S_4(n) = (n+1) \cdot \frac{6n^4 + 9n^3 + n^2 - n}{6},$$

$$S_4(n) = \frac{(n+1)n(6n^3+9n^2+n-1)}{30}.$$

Rastavimo sada polinom $6n^3+9n^2+n-1$ na proste faktore. Moguća racionalna multočka tog polinoma je iz skupa kvocijenata djelitelja slobodnog člana -1 i vodećeg koeficijenta 6 , tj. $n \in \{\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}\}$. Probanjem se vidi da je $-\frac{1}{2}$ multočka, pa dijeljenjem $6n^3+9n^2+n-1$ sa $(n+\frac{1}{2})$ dobivamo $6n^2+6n-2$, pa je $6n^3+9n^2+n-1 = (2n+1)(3n^2+3n-1)$. Stoga je

$$S_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Riješimo sada nekoliko zadataka.

U zadacima 1–14 izračunajte zadanu konačnu sumu.

Zadatak 1. $1 + 3 + 5 + \dots + (2n - 1)$

Rješenje.

$$\begin{aligned} 1 + 3 + 5 + \dots + (2n - 1) &= \sum_{k=1}^n (2k - 1) \\ &= 2 \sum_{k=1}^n k - \sum_{k=1}^n 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2 \end{aligned}$$

Zadatak 2. $2^2 + 4^2 + 6^2 + \dots + (2n)^2$.

Rješenje.

$$\begin{aligned} 2^2 + 4^2 + 6^2 + \dots + (2n)^2 &= \sum_{k=1}^n (2k)^2 = 4 \sum_{k=1}^n k^2 \\ &= 4 \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n(n+1)(2n+1)}{3} \end{aligned}$$

Zadatak 3. $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2$.

Rješenje.

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 &= \sum_{k=1}^n (2k - 1)^2 = \sum_{k=1}^n (4k^2 - 4k + 1) \\ &= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + \sum_{k=1}^n 1 = 4 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + n \\ &= \frac{4n^3 - n}{3} = \frac{n(2n-1)(2n+1)}{3} \end{aligned}$$

Zadatak 4. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)$.

Rješenje.

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) &= \sum_{k=1}^n k(k+1) = \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3} \end{aligned}$$

Zadatak 5. $1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \dots + n(3n-1)$.

Rješenje.

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8 + \dots + n(3n-1) &= \sum_{k=1}^n k(3k-1) = \sum_{k=1}^n (3k^2 - k) \\ &= 3 \sum_{k=1}^n k^2 - \sum_{k=1}^n k = 3 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = n^2(n+1) \end{aligned}$$

Zadatak 6. $\sum_{k=1}^n (2k)^3$.

Rješenje.

$$\sum_{k=1}^n (2k)^3 = 8 \sum_{k=1}^n k^3 = 8 \cdot \left(\frac{n(n+1)}{2} \right)^2 = 2n^2(n+1)^2$$

Zadatak 7. $\sum_{k=1}^n (2k-1)^3$.

Rješenje.

$$\begin{aligned} \sum_{k=1}^n (2k-1)^3 &= \sum_{k=1}^n (8k^3 - 12k^2 + 6k - 1) \\ &= 8 \sum_{k=1}^n k^3 - 12 \sum_{k=1}^n k^2 + 6 \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= 8 \cdot \left(\frac{n(n+1)}{2} \right)^2 - 12 \cdot \frac{n(n+1)(2n+1)}{6} + 6 \cdot \frac{n(n+1)}{2} - n \\ &= n^2(2n^2 - 1) \end{aligned}$$

Zadatak 8. $\sum_{k=1}^n k(k+1)(k+2)$.

Rješenje.

$$\begin{aligned} \sum_{k=1}^n k(k+1)(k+2) &= \sum_{k=1}^n (k^3 + 3k^2 + 2k) \\ &= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\ &= \left(\frac{n(n+1)}{2}\right)^2 + 3 \cdot \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(n+2)(n+3)}{4} \end{aligned}$$

Zadatak 9. $1 + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n)$.

Rješenje. Kako je suma prvih n prirodnih brojeva jednaka $\frac{n(n+1)}{2}$, to je

$$\begin{aligned} &1 + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n) \\ &= \sum_{k=1}^n \frac{k(k+1)}{2} = \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{2} \sum_{k=1}^n k \\ &= \frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \cdot \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(n+2)}{6}. \end{aligned}$$

Zadatak 10. $1+(1+3)+(1+3+5)+(1+3+5+7)+\dots+(1+3+5+\dots+(2n-1))$.

Rješenje. Kako je suma prvih n neparnih prirodnih brojeva jednaka n^2 , to je

$$\begin{aligned} &1 + (1+3) + (1+3+5) + (1+3+5+7) + \dots \\ &+ (1+3+5+\dots+(2n-1)) \\ &= \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Zadatak 11. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + (1^2 + 2^2 + 3^2 + \dots + n^2)$.

Rješenje. Kako je suma kvadrata prvih n prirodnih brojeva jednaka $\frac{n(n+1)(2n+1)}{6}$, to je

$$\begin{aligned}
 & 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + (1^2 + 2^2 + 3^2 + \dots + n^2) \\
 = & \sum_{k=1}^n \frac{k(k+1)(2k+1)}{6} = \frac{1}{6} \sum_{k=1}^n (2k^3 + 3k^2 + k) \\
 = & \frac{1}{6} \left(2 \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + \sum_{k=1}^n k \right) \\
 = & \frac{1}{6} \cdot \left(2 \cdot \frac{n^2(n+1)^2}{4} + 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) \\
 = & \frac{n(n+1)^2(n+2)}{12}.
 \end{aligned}$$

Zadatak 12. $\sum_{k=1}^n \frac{2k^2+2k+1}{k(k+1)}$.

Rješenje.

$$\begin{aligned}
 & \sum_{k=1}^n \frac{2k^2 + 2k + 1}{k(k+1)} = \sum_{k=1}^n \left(2 + \frac{1}{k(k+1)} \right) \\
 = & \sum_{k=1}^n 2 + \sum_{k=1}^n \frac{1}{k(k+1)} = 2n + \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
 = & 2n + \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\
 & + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
 = & 2n + \left(1 - \frac{1}{n+1} \right) = 2n + \frac{n}{n+1} = \frac{n(2n+3)}{n+1}
 \end{aligned}$$

Zadatak 13. $\sum_{k=1}^n \frac{k^2}{(2k-1)(2k+1)}$.

Rješenje.

$$\begin{aligned}
 & \sum_{k=1}^n \frac{k^2}{(2k-1)(2k+1)} = \sum_{k=1}^n \frac{k^2}{4k^2-1} = \frac{1}{4} \sum_{k=1}^n \frac{4k^2}{4k^2-1} \\
 &= \frac{1}{4} \sum_{k=1}^n \frac{4k^2-1+1}{4k^2-1} = \frac{1}{4} \left(\sum_{k=1}^n 1 + \frac{1}{4k^2-1} \right) \\
 &= \frac{1}{4} \left(n + \sum_{k=1}^n \frac{1}{(2k-1)(2k+1)} \right) = \frac{1}{4} \left(n + \sum_{k=1}^n \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \right) \\
 &= \frac{1}{4} n + \frac{1}{8} \sum_{k=1}^n \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\
 &= \frac{1}{4} n + \frac{1}{8} \left(\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots \right. \\
 &\quad \left. + \left(\frac{1}{2n-3} - \frac{1}{2n-1} \right) + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right) \\
 &= \frac{1}{4} n + \frac{1}{8} \left(1 - \frac{1}{2n+1} \right) = \frac{1}{4} n + \frac{1}{8} \cdot \frac{2n}{2n+1} \\
 &= \frac{1}{4} n \cdot \left(1 + \frac{1}{2n+1} \right) = \frac{1}{4} n \cdot \frac{2n+2}{2n+1} = \frac{n(n+1)}{2(2n+1)}
 \end{aligned}$$

Zadatak 14. $\sum_{k=1}^n (n^2 - (2k-1)n)$.

Rješenje.

$$\begin{aligned}
 \sum_{k=1}^n (n^2 - (2k-1)n) &= \sum_{k=1}^n n^2 - \sum_{k=1}^n (2k-1)n \\
 &= n^2 \sum_{k=1}^n 1 - n \sum_{k=1}^n (2k-1) = n^2 \cdot n - n \cdot n^2 = 0
 \end{aligned}$$

Zadaci za vježbu

Izračunajte sljedeće sume:

1. $\sum_{k=1}^n k(2k-1)$

Rješenje. $\frac{n(n+1)(4n-1)}{6}$

2. $\sum_{k=1}^n 2(3k-1)$

Rješenje. $n(3n+1)$

$$3. \sum_{k=1}^n n(3k-1)$$

$$Rješenje. n^2(n+1)$$

$$4. \sum_{k=1}^n k(k+1)^2$$

$$Rješenje. \frac{n(n+1)(n+2)(3n+5)}{12}$$

$$5. \sum_{k=1}^n (k+1)k^2$$

$$Rješenje. \frac{n(n+1)(n+2)(3n+1)}{12}$$

$$6. \sum_{k=1}^n \frac{k(3k+1)}{2}$$

$$Rješenje. \frac{n(n+1)^2}{2}$$

$$7. \sum_{k=1}^n (2k-1)(k+2)$$

$$Rješenje. \frac{n(4n^2+15n-1)}{6}$$

$$8. \sum_{k=1}^n (k^2+2k-1)$$

$$Rješenje. \frac{n(2n^2+9n+1)}{6}$$

$$9. \sum_{k=1}^n (k+1)(2k+1)$$

$$Rješenje. \frac{n(4n^2+15n+17)}{6}$$

$$10. \frac{1^2}{1} + \frac{1^2+2^2}{2} + \dots + \frac{1^2+2^2+\dots+n^2}{n}$$

$$Rješenje. \frac{n(4n^2+15n+17)}{36}$$

$$11. \sum_{k=1}^n (2k-1)(2k+1)(2k+3)$$

$$Rješenje. n(n+2)(2n^2+4n-1)$$

$$12. \sum_{k=1}^n \frac{k^2+k+1}{k(k+1)}$$

$$Rješenje. \frac{n(n+2)}{n+1}$$

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