

THE 1996 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours

NO calculators are to be used.

Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Question 2

Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n .$$

Question 3

Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incentre of the triangle $P_2P_3P_4$; I_2 be the incentre of the triangle $P_1P_3P_4$; I_3 be the incentre of the triangle $P_1P_2P_4$; I_4 be the incentre of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Question 4

The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of a group must be of the same sex; i.e. they are either all male or all female.
2. The difference in the size of any two groups is 0 or 1.
3. All groups have at least 1 member.
4. Each person must belong to one and only one group.

Find all values of n , $n \leq 1996$, for which this is possible. Justify your answer.

Question 5

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} ,$$

and determine when equality occurs.

Solutions

Problem 1. Let $M'N'$ and $P'Q'$ be two segments perpendicular to BD at a distance d . We need to prove that the perimeters $AMNCQP$ and $AM'N'CQ'P'$ are equal. Denote by S' the projection of N into $M'N'$ and by S the projection of P' into PQ . The triangles $NS'N'$ and $P'SP$ are equal. (right triangles such that $NS' = P'S$ and equal angles $\angle P'PS = \angle NN'S'$.) So $PP' = NN'$. Since it is clear that $MM' = NN'$, we have $MM' = NN' = PP' = QQ'$. On the other hand, since $SP = S'N'$, using T' and T the projections of M and Q' into $M'N'$ and PQ , respectively, we have

$$T'M' = S'N' = PS = TQ.$$

$$\begin{aligned} \text{So } P'Q' + M'N' &= ST + M'T' + T'S' + S'N' \\ &= ST + PS + TQ + MN \\ &= PQ + MN. \end{aligned}$$

(Without loss of generality M' lies between M and A)

So the perimeter of $AMNCQP$ is:

$$\begin{aligned} AM + MN + NC + CQ + QP + PA &= \\ AM' + M'M + MN + NN' + N'C + CQ + QP + PA &= \\ AM' + PP' + MN + QQ' + N'C + CQ + QP + PA &= \\ AM' + MN + QP + PP' + PA + QQ' + CQ + N'C &= \\ AM' + MN + QT + TS + SP + PP' + PA + QQ' + CQ + N'C &= \\ AM' + M'T' + T'S' + S'N' + N'C + CQ + QQ' + Q'P' + P'P + PA &= \\ AM' + M'N' + N'C + CQ' + Q'P' + P'A, & \end{aligned}$$

which is the perimeter of $AM'N'CQ'P'$.

Points to be given for showing that:

$$MM' = NN' = PP' = QQ'$$

1 point

$$T'M' = S'N' = PS = TQ.$$

1 point

$$P'Q' + M'N' = PQ + MN.$$

2 points

The perimeter of $AMNCQP$ is the same as the perimeter of $AM'N'CQ'P'$

3 points

Problem 2. We first prove by induction on n that:

$$\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^n (m^2 + m - i^2 + i).$$

$$1. \frac{(m+1)!}{(m-1)!} = m(m+1) = m^2 + m.$$

$$2. \frac{(m+n+1)!}{(m-n-1)!} = \left(\prod_{i=1}^n (m^2 + m - i^2 + i) \right) (m+n+1)(m-n) \quad (\text{by induction})$$

$$= \left(\prod_{i=1}^n (m^2 + m - i^2 + i) \right) (m^2 + m - n^2 - n)$$

$$= \prod_{i=1}^{n+1} (m^2 + m - i^2 + i)$$

But $m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$.

Therefore

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2 + m)^n.$$

Points to be given for showing that:

the innequities hold for special values

up to 1 point

$$\frac{(m+n)!}{(m-n)!} = \prod_{i=1}^n (m^2 + m - i^2 + i) \text{ for all } n$$

4 points

$m^2 + m \geq m^2 + m - i^2 + i \geq i^2 + i - i^2 + i = 2i$, for $i \geq m$,
and therefore the innequities hold

2 points

Problem 3. Let C be the given circle. Draw four circles $C_{12}, C_{23}, C_{34}, C_{41}$ with centers $O_{12}, O_{23}, O_{34}, O_{41}$, respectively, on the circle C such that C_{12} passes through P_1 and P_2 , C_{23} passes through P_2 and P_3 , C_{34} passes through P_3 and P_4 , C_{41} passes through P_4 and P_1 . Let the other point of intersection of C_{12} and C_{23} be Q_4 , the other point of intersection of C_{23} and C_{34} be Q_1 , the other point of intersection of C_{34} and C_{41} be Q_2 , and the other point of intersection of C_{41} and C_{12} be Q_3 . Then

$$\angle Q_4 P_1 P_2 = \frac{1}{2} \angle Q_4 O_{12} P_2 \text{ and } \angle O_{23} P_1 P_2 = \frac{1}{2} \angle P_3 O_{12} P_2.$$

Clearly, O_{23}, Q_4 and P_1 are collinear.

It follows that $\angle Q_4 Q_3 P_2 = \angle Q_4 P_1 P_2 = \angle O_{23} P_1 P_2 = \angle O_{23} O_{41} P_2$. Since also O_{41}, Q_3 and P_2 are collinear, it follows that $Q_3 Q_4$ and $O_{41} O_{23}$ are parallel.

Since O_{ij} bisects the arcs $P_i P_j$, for $(i, j) = (1, 2), (2, 3), (3, 4), (4, 1)$ we conclude that $O_{41} O_{23}$ and $O_{12} O_{34}$ are perpendicular, and hence $Q_3 Q_4$ and $O_{12} O_{34}$ are perpendicular.

Since both (O_{12}, Q_3, P_4) and (O_{12}, Q_4, P_3) are collinear triples of points, we have $\angle P_4 O_{12} P_3 = \angle Q_3 O_{12} Q_4$, and this angle is bisected by $O_{12} O_{34}$.

Thus Q_3 and Q_4 are reflections through the axis $O_{12} O_{34}$, and so are, by a similar argument Q_1 and Q_2 .

We have thus shown: Q_1, Q_2, Q_3, Q_4 form a rectangle. But as Q_4 lies on both the angle bisector $O_{12} P_3$ and the angle bisector $O_{23} P_1$ of the triangle $P_1 P_2 P_3$, the point Q_4 must coincide with the incenter I_4 of the triangle $P_1 P_2 P_3$, and by a similar argument, $Q_1 = I_1, Q_2 = I_2$ and $Q_3 = I_3$.

Points to be given for showing that

Q_3Q_4 and $O_{41}O_{23}$ are parallel 2 points

Q_3Q_4 and $O_{12}O_{34}$ are perpendicular 1 points

Q_1, Q_2, Q_3, Q_4 form a rectangle 2 points

$Q_4 = I_4, Q_1 = I_1, Q_2 = I_2$ and $Q_3 = I_3$ 1 point

Problem 4. . We may assume that the n couples will form x male groups and y female groups. Without loss of generality, let $x \geq y$, and

(*)
$$x + y = 17.$$

Then, by the pigeonhole theorem, there exists a male group of size $\leq \left\lceil \frac{n}{x} \right\rceil$ and a female group of size $\geq \left\lfloor \frac{n}{y} \right\rfloor$. By condition (2), we have

(**)
$$\left\lfloor \frac{n}{y} \right\rfloor - \left\lceil \frac{n}{x} \right\rceil \leq 1$$

From the conditions $x + y = 17$ and $x \geq y$ follows that $x \geq 9$, and $y \leq 8$, which in turn implies

$$\left\lfloor \frac{n}{y} \right\rfloor - \left\lceil \frac{n}{x} \right\rceil > \left\lfloor \frac{n}{8} \right\rfloor - \left\lceil \frac{n}{9} \right\rceil \quad /$$

Therefore, we only need to exclude those n such that $\left\lfloor \frac{n}{8} \right\rfloor - \left\lceil \frac{n}{9} \right\rceil > 1$. Let $n = 9u + s$, $0 \leq s < 9$.

Then

$$\left\lfloor \frac{n}{8} \right\rfloor - \left\lceil \frac{n}{9} \right\rceil > 1 \Leftrightarrow \left\lfloor \frac{u+s}{8} \right\rfloor > 1.$$

By analyzing this condition it is clear that the only values of n that are allowed are
 $n = 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30,$
 $31, 32, 36, 37, 38, 39, 40, 45, 46, 46, 47, 48, 54, 55, 56, 63, 63, 72.$

Conversely, conditions (*) and (**) give rise to a set of discussing groups according to the following description:

Let $\left\lfloor \frac{n}{x} \right\rfloor = p$ and $\left\lceil \frac{n}{y} \right\rceil = q$, then $p \leq q \leq p + 1$

We have

$$n = px + \alpha, 0 \leq \alpha < x \quad \text{and} \quad n = qy - \beta \quad 0 \leq \beta < y.$$

So we can arrange the males in α discussing groups of size $p + 1$ and $x - \alpha$ groups of p elements. The females are distributed in β discussing groups of size $q - 1$, and $(y - \beta)$ discussing groups of size q .

Points to be given for showing that

$$x \geq 9, \text{ and } y \leq 8$$

1 points

one only needs to exclude those n such that $[\frac{n}{8}] - [\frac{n}{9}] > 1$

2 points

analyzing the condition ian giving the values of n that are allowed

2 points

given p and q as described in the solution

one can arrange the males in α discussion groups of size $p + 1$ and $x - \alpha$ groups of p elements and the females in β discussion groups of size $q - 1$, and $(y - \beta)$ groups of size q .

2 points

Problem 5. Without loss of generality we can assume that $a \geq b \geq c$. Note that if $x \geq y > 0$, then $\sqrt{y} \leq 1/2(\sqrt{x} + \sqrt{y})$, i.e.,

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \geq 1$$

Similarly, if $y \geq x > 0$, then

$$\frac{1}{2\sqrt{y}}(\sqrt{x} + \sqrt{y}) \leq 1$$

Multiplying both inequalities by $\sqrt{x} - \sqrt{y}$ we obtain

$$\sqrt{x} - \sqrt{y} \leq \frac{1}{2\sqrt{y}}(x - y),$$

for every $x, y > 0$. Moreover, it is easily seen that equality occurs if and only if $x = y$. By applying this last inequality we obtain

$$\sqrt{a + b - c} - \sqrt{a} \leq \frac{1}{2\sqrt{a}}(b - c)$$

$$(**) \quad \begin{aligned} \sqrt{c+a-b} - \sqrt{b} &\leq \frac{1}{2\sqrt{b}}(c+a-2b) \\ \sqrt{b+c-a} - \sqrt{c} &\leq \frac{1}{2\sqrt{c}}(b-a) \end{aligned}$$

and by adding up the left hand and the right hand sides of these inequalities we have

$$\begin{aligned} &\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} - (\sqrt{a} + \sqrt{b} + \sqrt{c}) \\ &\quad \frac{1}{2} \left(\frac{1}{\sqrt{a}}(b-c) + \frac{1}{\sqrt{b}}(c+a-2b) + \frac{1}{\sqrt{c}}(b-a) \right) \\ &\quad \frac{1}{2} \left((b-c) \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) + (a-b) \left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}} \right) \right) \leq 0, \end{aligned}$$

i.e.,

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c},$$

and equality occurs if and only if the three relations in (**) are equalities, i.e., if and only if $a = b = c$.

Points to be given for showing that

$$\sqrt{x} - \sqrt{y} \leq \frac{1}{2\sqrt{y}}(x-y) \quad 2 \text{ points}$$

the relations in (**) hold true 1 point each

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} - (\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq$$

1 point

equality occurs if and only if the three relations in (**) are equalities, i.e., if and only if $a = b = c$

1 point