TWO INTERESTING INEQUALITIES FOR ACUTE TRIANGLES

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Abstract. The paper considers the proofs of two interesting geometric inequalities for acute triangles.
Keywords: acute triangle; inequality; orthocentre of a triangle

In this paper we will prove two interesting inequalities for acute triangles, namely:

$$\frac{AH}{a}+\frac{BH}{b}+\frac{CH}{c}\geq\sqrt{3} \tag{1}$$

and

$$\frac{AH}{a}\cdot\frac{BH}{b}\cdot\frac{CH}{c}\leq\frac{1}{3\sqrt{3}}, \tag{2}$$

where the point H is the orthocentre of the ΔABC with sides a, b and c. For the purpose we will make use of the following two inequalities:

$$\sin\alpha\sin\beta\sin\gamma\leq\frac{3\sqrt{3}}{8} \tag{3}$$

and

$$a^2+b^2+c^2\geq 4F\sqrt{3}, \tag{4}$$

where $\alpha,\ \beta$ and γ are the interior angles of the ΔABC and F is the area of the triangle.

Remark 1. The proofs of the inequalities (3) and (4) can be found in (Bottema et al., 1969), (Bulajich Manfrino et al., 2009), (Grozdev, 2007) and (Cvetkovski, 2012). We will propose new proofs of the which are not included in the above references.

We start with the inequality (3).

Firstly, we will prove the inequality:

$$\sin x\sin y\leq\sin^2\frac{x+y}{2};\ \left(x,y\in[0,\pi]\right). \tag{5}$$

We have

$$\sin x\sin y\leq\sin^2\frac{x+y}{2}$$

$$\Leftrightarrow 2\sin\frac{x}{2}\cos\frac{x}{2}\cdot 2\sin\frac{y}{2}\cos\frac{y}{2}\le\left(\sin\frac{x}{2}\cos\frac{y}{2}+\cos\frac{x}{2}\sin\frac{y}{2}\right)^2$$

$$\Leftrightarrow 4\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{y}{2}\cos\frac{y}{2}\le\sin^2\frac{x}{2}\cos^2\frac{y}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{y}{2}\cos\frac{y}{2}+\cos^2\frac{x}{2}\sin^2\frac{y}{2}$$

$$\Leftrightarrow \sin^2\frac{x}{2}\cos^2\frac{y}{2}-2\sin\frac{x}{2}\cos\frac{y}{2}\cos\frac{x}{2}\sin\frac{y}{2}+\cos^2\frac{x}{2}\sin^2\frac{y}{2}\ge 0$$

$$\Leftrightarrow\left(\sin\frac{x}{2}\cos\frac{y}{2}-\cos\frac{x}{2}\sin\frac{y}{2}\right)^2\ge 0$$

$$\Leftrightarrow \sin^2\left(\frac{x}{2}-\frac{y}{2}\right)\ge 0 .$$

The last inequality is obvious, which proves the inequality (5). Note, that equality holds if and only if $x=y$.

The next two inequalities follow from the inequality (5):

$$\sin\alpha\sin\beta\le\sin^2\frac{\alpha+\beta}{2}$$

and

$$\sin\gamma\sin 60^0\le\sin^2\frac{\gamma+60^0}{2},$$

and from here after multiplying them:

$$\sin\alpha\sin\beta\sin\gamma\sin 60^0\le\left(\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+60^0}{2}\right)^2 . \qquad (6)$$

It follows from the inequality (5), that:

$$\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+60^0}{2}\le\sin^2\frac{\alpha+\beta+\gamma+60^0}{4},$$

and from here:

$$\left(\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+60^0}{2}\right)^2\le\sin^4\frac{\alpha+\beta+\gamma+60^0}{4}, \text{ i.e.}$$

$$\left(\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+60^0}{2}\right)^2\le\sin^4 60^0$$

or

$$\left(\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+60^0}{2}\right)^2\le\frac{9}{16} . \qquad (7)$$

Inequalities (6) and (7) follow from here:

$$\sin\alpha\sin\beta\sin\gamma\sin 60^0\le\frac{9}{16},$$

and further:

$$sin\alpha\, sin\beta\, sin\gamma \le \frac{\frac{9}{16}}{\frac{\sqrt{3}}{2}}, \text{ i.e.}$$

$$sin\alpha\, sin\beta\, sin\gamma \le \frac{3\sqrt{3}}{8}, \text{ q.e.d.}$$

Note, that equality holds if and only if $\alpha=\beta=\gamma=60^0$ (equilateral trinagle).

Now, we will prove the inequality (4).

We deduce from the Heron's formula $F^2=s(s-a)(s-b)(s-c)$, where $s=\frac{a+b+c}{2}$, a new formula for the area of the ΔABC:

$$16F^2=2\left(a^2b^2+b^2c^2+c^2a^2\right)-a^4-b^4-c^4. \tag{8}$$

After squaring, the inequality (4) turns to be equivalent to the inequality:

$$\left(a^2+b^2+c^2\right)^2\ge 3\cdot 16F^2 \tag{9}$$

$$\overset{(8)}{\Leftrightarrow}\left(a^2+b^2+c^2\right)^2\ge 6\left(a^2b^2+b^2c^2+c^2a^2\right)-3\left(a^4+b^4+c^4\right)$$

$$\Leftrightarrow 4\left(a^4+b^4+c^4\right)-4\left(a^2b^2+b^2c^2+c^2a^2\right)\ge 0 /:2$$

$$\Leftrightarrow 2\left(a^4+b^4+c^4\right)-2\left(a^2b^2+b^2c^2+c^2a^2\right)\ge 0$$

$$\Leftrightarrow\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2\ge 0.$$

The last is obvious. Thus, the inequality (9) is true and consequently the inequality (4) is also true. The equality in (4) holds if and only if $a=b=c$ (equilateral triangle).

Now, we will give a new proof of the inequality (4).

From the well known inequality:

$$a^2+b^2+c^2\ge ab+bc+ca$$

$$\left(\Leftrightarrow\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\ge 0\right),$$

using the well known formulae of the area of the triangle:

$$F=\frac{ab}{2}\sin\gamma=\frac{bc}{2}\sin\alpha=\frac{ac}{2}\sin\beta,$$

we get:

$$a^2+b^2+c^2\geq 2F\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\right). \tag{10}$$

We use the inequality between the arithmetic and the geometric means of three positive numbers:

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\geq 3\sqrt[3]{\frac{1}{\sin\alpha\sin\beta\sin\gamma}},$$

and from here we get the inequality (3):

$$\frac{1}{\sin\alpha\sin\beta\sin\gamma}\geq\frac{8}{3\sqrt{3}}:$$

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\geq 3\cdot\frac{2}{\sqrt{3}}, \text{ i.e.}$$

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\geq 2\sqrt{3}. \tag{11}$$

Now, the inequality (4) follows from (10) and (11).

Finally, we will prove the inequalities (1) and (2).

Proof (of the inequality (1)):

Since $AH\perp BC$ and $AC\perp BE$ (Fig.1), it follows that $\measuredangle HAE=\measuredangle EBC$ and consequently $\measuredangle AEH=\measuredangle BEC=90^0$. Thus, $\Delta AHE\sim\Delta BEC$. From here we have:

$$\frac{AH}{BC}=\frac{AE}{BE}, \text{ i.e.}$$

$$\frac{AH}{a}=\frac{AE}{h_b}. \tag{12}$$

At the same time the right-angle ΔABE gives, that:

$$\cos\alpha=\frac{AE}{AB}=\frac{AE}{c},$$

and from here by the cosine law:

$$AE=c\cdot\cos\alpha=c\cdot\frac{b^2+c^2-a^2}{2bc}, \text{ i.e.}$$

$$AE=\frac{b^2+c^2-a^2}{2b}. \tag{13}$$

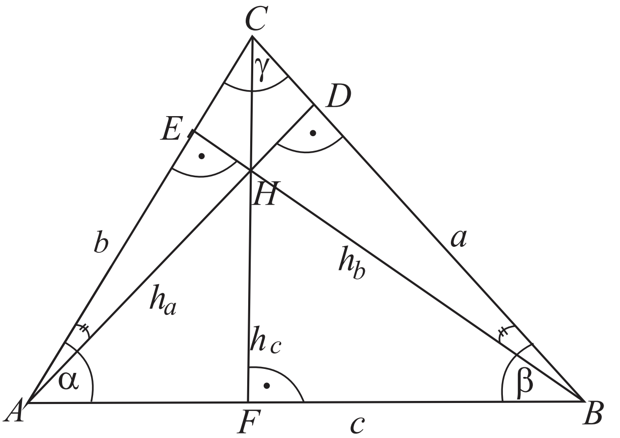

Fig. 1

Because $h_b = \frac{2F}{b}$, we get from (12) and (13), that:

$$\frac{AH}{a} = \frac{b^2 + c^2 - a^2}{4F} \tag{14}$$

Analogously

$$\frac{BH}{b} = \frac{c^2 + a^2 - b^2}{4F} \tag{15}$$

and

$$\frac{CH}{c} = \frac{a^2 + b^2 - c^2}{4F}. \tag{16}$$

Applying (14), (15) and (16), we deduce that:

$$\frac{AH}{a} + \frac{BH}{b} + \frac{CH}{c} = \frac{a^2 + b^2 + c^2}{4F}. \tag{17}$$

Finally, from (17) and (4) we have:

$$\frac{AH}{a} + \frac{BH}{b} + \frac{CH}{c} \geq \sqrt{3}, \text{ q.e.d.}$$

The equality holds if and only if $a = b = c$ (equilateral triangle).

Proof (of the inequlity (2)): Feom (12) we have:

$$\frac{AH}{a} = \frac{AE}{h_b} = ctg\alpha, \text{ i.e.}$$

$$\frac{AH}{a}=\frac{1}{tg\alpha}. \quad (18)$$

Analogously:

$$\frac{BH}{b}=\frac{1}{tg\beta} \quad (19)$$

and

$$\frac{CH}{c}=\frac{1}{tg\gamma}, \quad (20)$$

and from here after multiplying (18), (19) and (20), we obtain:

$$\frac{AH}{a}\cdot\frac{BH}{b}\cdot\frac{CH}{c}=\frac{1}{tg\alpha tg\beta tg\gamma}. \quad (21)$$

Note the well known equality

$$tg\alpha+tg\beta+tg\gamma=tg\alpha tg\beta tg\gamma .$$

It follows by the inequality between the arithmetic and the geometric means for three positive numbers ($tg\alpha, tg\beta, tg\gamma>0$, because the trinagle is acute), that:

$$tg\alpha tg\beta tg\gamma=tg\alpha+tg\beta+tg\gamma\geq 3\sqrt[3]{tg\alpha tg\beta tg\gamma} \; /^3$$

$$\Leftrightarrow tg^3\alpha tg^3\beta tg^3\gamma\geq 27 tg\alpha tg\beta tg\gamma$$

$$\Leftrightarrow \left(tg\alpha tg\beta tg\gamma\right)^2\geq 27$$

$$\Leftrightarrow tg\alpha tg\beta tg\gamma\geq 3\sqrt{3}$$

$$\Leftrightarrow \frac{1}{tg\alpha tg\beta tg\gamma}\leq\frac{1}{3\sqrt{3}}. \quad (22)$$

Finally, from (21) and (22) we obtain the inequality (2), q.e.d. The equality holds if and only if $\alpha=\beta=\gamma=\frac{\pi}{3}$ (equilateral triangle).

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