Balkan MO Shortlist 2011

– Algebra

A1 Given real numbers x, y, z such that $x + y + z = 0$, show that

$$\frac{x(x+2)}{2x^2+1} + \frac{y(y+2)}{2y^2+1} + \frac{z(z+2)}{2z^2+1} \geq 0$$

When does equality hold?

A2 Given an integer $n \geq 3$, determine the maximum value of product of n non-negative real numbers $x_1, x_2, \ldots, x_n$ when subjected to the condition

$$\sum_{k=1}^{n} \frac{x_k}{1+x_k} = 1$$

A3 Let n be an integer number greater than 2, let $x_1, x_2, \ldots, x_n$ be n positive real numbers such that

$$\sum_{i=1}^{n} \frac{1}{x_i+1} = 1$$

and let k be a real number greater than 1. Show that:

$$\sum_{i=1}^{n} \frac{1}{x_i^k+1} \geq \frac{n}{(n-1)^k+1}$$

and determine the cases of equality.

A4 Let $x, y, z \in \mathbb{R}^+$ satisfying $xyz = 3(x+y+z)$. Prove, that

$$\sum \frac{1}{x^2(y+1)} \geq \frac{3}{4(x+y+z)}$$

– Geometry

G1 Let $ABCD$ be a convex quadrangle such that $AB = AC = BD$ (vertices are labelled in circular order). The lines AC and BD meet at point O, the circles ABC and ADO meet again at point P, and the lines AP and BC meet at the point Q. Show that the angles COQ and DOQ are equal.

G2 Let ABC be a triangle and let O be its circumcentre. The internal and external bisectrices of the angle BAC meet the line BC at points D and E, respectively. Let further M and L respectively denote the midpoints of the segments BC and DE. The circles ABC and ALO meet again at point N. Show that the angles BAN and CAM are equal.

G3 Given a triangle ABC, let D be the midpoint of the side AC and let M be the point that divides the segment BD in the ratio $1/2$; that is, $MB/MD = 1/2$. The rays AM and CM meet the sides BC and AB at points E and F, respectively. Assume the two rays perpendicular: $AM \perp CM$. Show that the quadrangle $AFED$ is cyclic if and only if the median from A in triangle ABC meets the line EF at a point situated on the circle ABC.

G4 Given a triangle ABC, the line parallel to the side BC and tangent to the incircle of the triangle meets the sides AB and AC at the points A_1 and A_2 , the points B_1, B_2 and C_1, C_2 are de ned similarly. Show that

$$AA_1 \cdot AA_2 + BB_1 \cdot BB_2 + CC_1 \cdot CC_2 \geq \frac{1}{9}(AB^2 + BC^2 + CA^2)$$

C1 Let S be a finite set of positive integers which has the following property:if x is a member of S,then so are all positive divisors of x. A non-empty subset T of S is *good* if whenever $x, y \in T$ and $x < y$, the ratio y/x is a power of a prime number. A non-empty subset T of S is *bad* if whenever $x, y \in T$ and $x < y$, the ratio y/x is not a power of a prime number. A set of an element is considered both *good* and *bad*. Let k be the largest possible size of a *good* subset of S. Prove that k is also the smallest number of pairwise-disjoint *bad* subsets whose union is S.

C2 Let $ABCDEF$ be a convex hexagon of area 1, whose opposite sides are parallel. The lines AB, CD and EF meet in pairs to determine the vertices of a triangle. Similarly, the lines BC, DE and FA meet in pairs to determine the vertices of another triangle. Show that the area of at least one of these two triangles is at least $3/2$.

C3 Is it possible to partition the set of positive integer numbers into two classes, none of which contains an infinite arithmetic sequence (with a positive ratio)? What is we impose the extra condition that in each class $\mathcal{C}$ of the partition, the set of difference

$$\{\min\{n \in \mathcal{C} \mid n > m\} - m \mid m \in \mathcal{C}\}$$

be bounded?

N1 Given an odd number $n > 1$, let

$$S = \{k \mid 1 \leq k < n, \gcd(k, n) = 1\}$$

and let

$$T = \{k \mid k \in S, \gcd(k+1, n) = 1\}$$

For each $k \in S$, let r_k be the remainder left by $\frac{k^{|S|}-1}{n}$ upon division by n. Prove

$$\prod_{k \in T} (r_k - r_{n-k}) \equiv |S|^{|T|} \pmod{n}$$

N2 Let $n \in \mathbb{N}$ such that $p = 17^{2n} + 4$ is a prime. Show

$$p \mid 7^{\frac{p-1}{2}} + 1$$