SEEMOUS 2018

FEBRUARY 27-MARCH 4, IAŞI, ROMÂNIA

South Eastern European Mathematical Olympiad for University Students
Iaşi, Romania
March 1, 2018

COMPETITION PROBLEMS

Problem 1. Let $f : [0,1] \to (0,1)$ be a Riemann integrable function. Show that

$$\frac{2\displaystyle\int_0^1 x f^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 (f^2(x)+1)\,\mathrm{d}x} < \frac{\displaystyle\int_0^1 f^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 f(x)\,\mathrm{d}x}.$$

Problem 2. Let $m, n, p, q \geq 1$ and let the matrices $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $B \in \mathcal{M}_{n,p}(\mathbb{R})$, $C \in \mathcal{M}_{p,q}(\mathbb{R})$, $D \in \mathcal{M}_{q,m}(\mathbb{R})$ be such that

$$A^t = BCD, \quad B^t = CDA, \quad C^t = DAB, \quad D^t = ABC.$$

Prove that $(ABCD)^2 = ABCD$.

Problem 3. Let $A,\ B \in \mathcal{M}_{2018}(\mathbb{R})$ such that $AB = BA$ and $A^{2018} = B^{2018} = I$, where I is the identity matrix. Prove that if $\mathrm{Tr}(AB) = 2018$, then $\mathrm{Tr}\,A = \mathrm{Tr}\,B$.

Problem 4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function. Prove that

$$\int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x = f(0) + f'(0) + f''(0) + \cdots .$$

(b) Let f be a function which has a Taylor series expansion at 0 with radius of convergence $R = \infty$. Prove that if $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(0)$ converges absolutely then $\displaystyle\int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x$ converges and

$$\sum_{n=0}^{\infty} f^{(n)}(0) = \int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x.$$