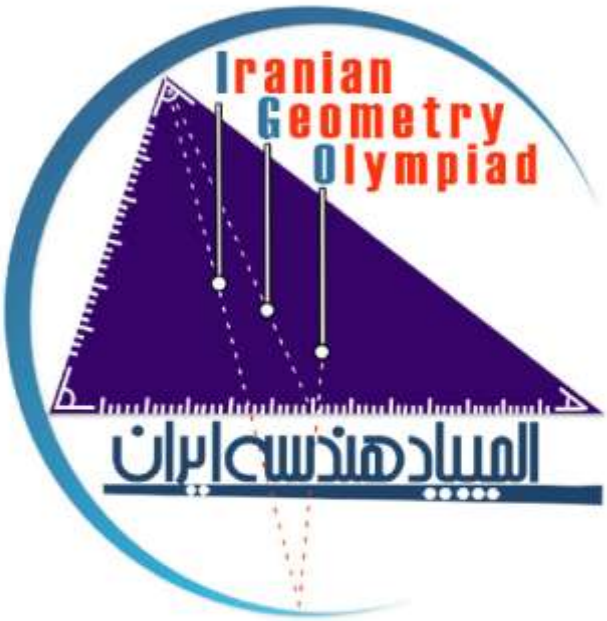


2nd Iranian Geometry Olympiad

September 2015

Problems of 2nd Iranian Geometry Olympiad (Advanced)

1. Two circles ω_1 and ω_2 (with center O_1 and O_2 respectively) intersect at A and B. The point X lies on ω_2. Let point Y be a point on ω_1 such that $\angle XBY = 90°$. The line O_1X intersects ω_2 at X' for second time. If $X'Y$ intersects ω_2 at K, prove that X lie on the midpoint of arc AK.

Proposed by Davood Vakili

2. Suppose that ABC be the equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC (the arc witch A doesn't lie). Tangent to ω in P intersects extension of AB and AC at K and L respectively. Show that: $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

3. In triangle ABC, H is the orthocenter of triangle. Let l_1 and l_2 be two lines such that pass through H and perpendicular to each other. The line l_1 intersects BC and extension of AB at D and Z respectively and the line l_2 intersects BC and extension of AC at E and X respectively. We draw the Line which passes through D and parallel to AC and we draw the Line which passes through E and parallel to AB. Suppose that intersectoin of these lines denote by Y. Prove that X, Y, Z are collinear.

Proposed by Ali Golmakani

4. In triangle ABC, we draw the circle with center A and radius AB. This circle intersects AC at two points. Also we draw the circle with center A and radius AC and this circle intersects AB at two points. Denote these four points by A_1, A_2, A_3, A_4. Find the points B_1, B_2, B_3, B_4 and C_1, C_2, C_3, C_4 similarly. Suppose that these 12 points lie on two circles. Prove that the triangle ABC is isosceles.

Proposed by Morteza Saghafian

5. Rectangles ABA_1B_2, BCB_1C_2, CAC_1A_2 lie otside triangle ABC. Let C' be such point that $C'A_1 \perp A_1C_2$ and $C'B_2 \perp B_2C_1$, points A', B' are defined similarly. Prove that lines AA', BB', CC' concur.

Proposed by Alexey Zaslavsky (Russia)

Solutions of 2nd Iranian Geometry Olympiad (Advanced)

1. Two circles ω_1 and ω_2 (with center O_1 and O_2 respectively) intersect at A and B. The point X lies on ω_2. Let point Y be a point on ω_1 such that $\angle XBY = 90°$. The line O_1X intersects ω_2 at X' for second time. If $X'Y$ intersects ω_2 at K, prove that X lie on the midpoint of arc AK.

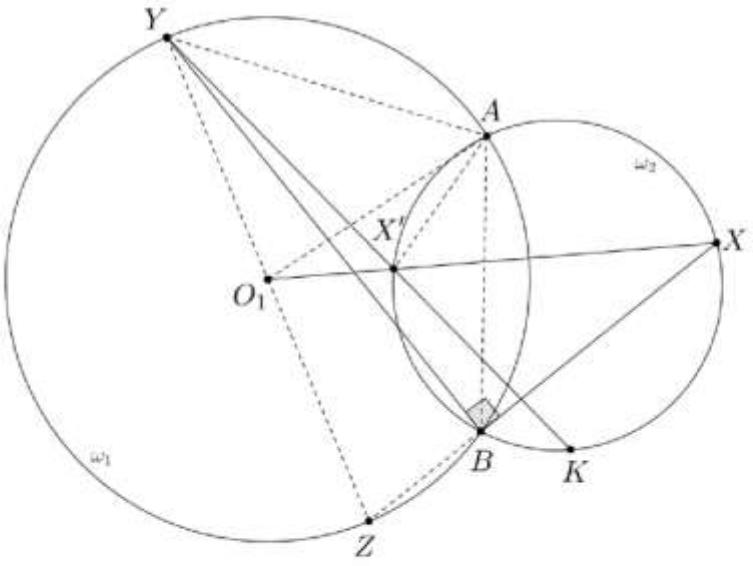
Proposed by Davood Vakili

Solution.

The center of circle ω_1 denote by O_1. Suppose that the point Z be the intersection of BX and circle ω_1. We know that $\angle YBZ = 90°$, therefore the points Y, O_1, Z are collinear.

$$\angle O_1YA = \angle ABX = \angle AX'X \quad \Rightarrow \quad YAX'O_1 : cyclic$$

In the other hand, we know that $AO_1 = YO_1$ so $\angle AX'X = \angle YX'O_1 = \angle XX'K$. Therefore the point X lie on the midpoint of arc AK.



2. Suppose that ABC be the equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC (the arc witch A doesn't lie). Tangent to ω in P intersects extension of AB and AC at K and L respectively. Show that: $\angle KOL > 90°$.

Proposed by Iman Maghsoudi

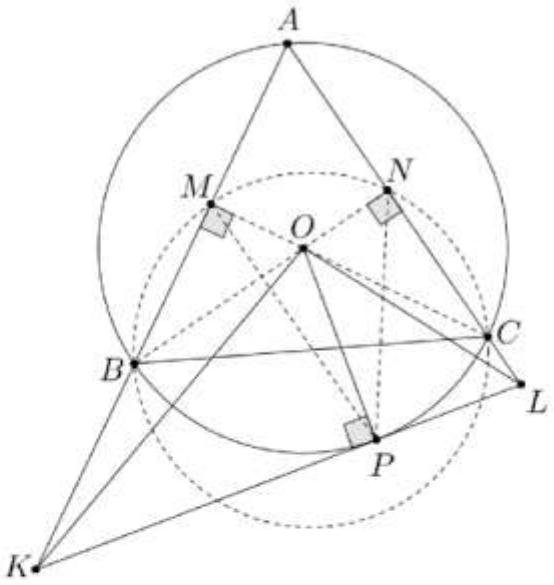
Solution-1.

Suppose that M and N be the midpoints of AB and AC respectively. We know that quadrilateral $BMNC$ is cyclic. Also $\angle BPC = 120° > 90°$, so we can say the point P is in the circumcircle of quadrilateral $BMNC$. Therefore: $\angle MPN > \angle MBN = 30°$

In the other hand, quadrilaterals $TMOP$ and $NOPC$ are cyclic. Therefore:

$$\angle MKO = \angle MPO \ , \ \angle NLO = \angle NPO \ \Rightarrow \ \angle AKO + \angle ALO = \angle MPN > 30°$$

$$\Rightarrow \angle KOL = \angle A + \angle AKO + \angle ALO > 90°$$



Solution-2.

Suppose that $\angle KOL \leq 90°$, therfore $KL^2 \leq OK^2 + OL^2$. Assume that R is the radius of a circumcircle $\triangle ABC$. Let $BK = x$ and $LC = y$ and $AB = AC = BC = a$. According to law of cosine in triangle AKL, we have:

$$KL^2 = AK^2 + AL^2 - AK.AL.cos(\angle A) \quad \Rightarrow \quad KL^2 = (a+x)^2 + (a+y)^2 - (a+x)(a+y)$$

In the other hand:

$$KB.KA = OK^2 - R^2 \quad \Rightarrow \quad OK^2 = R^2 + x(a+x)$$

$$LC.LA = OL^2 - R^2 \quad \Rightarrow \quad OL^2 = R^2 + y(a+y)$$

We know that $KL^2 \leq OK^2 + OL^2$ and $a = R\sqrt{3}$, therfore:

$$(a+x)^2 + (a+y)^2 - (a+x)(a+y) \leq 2R^2 + x(a+x) + y(a+y)$$

$$\Rightarrow \quad R^2 \leq xy \quad (1)$$

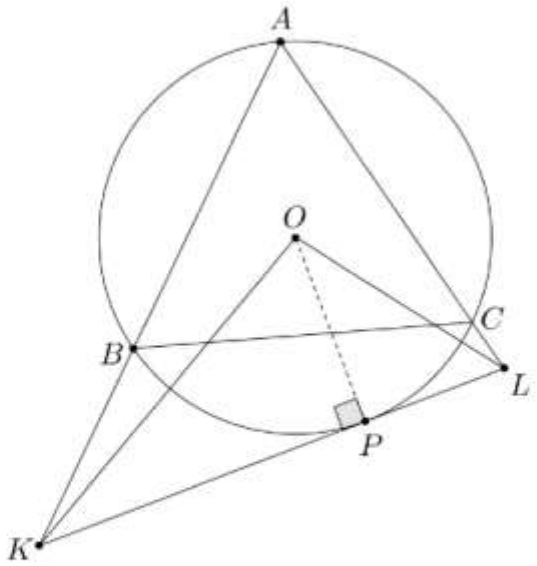
KL is tangent to circumcircle of $\triangle ABC$ at P. So we have:

$$KP^2 = KB.KA = x(a+x) > x^2 \quad \Rightarrow \quad KP > x \quad (2)$$

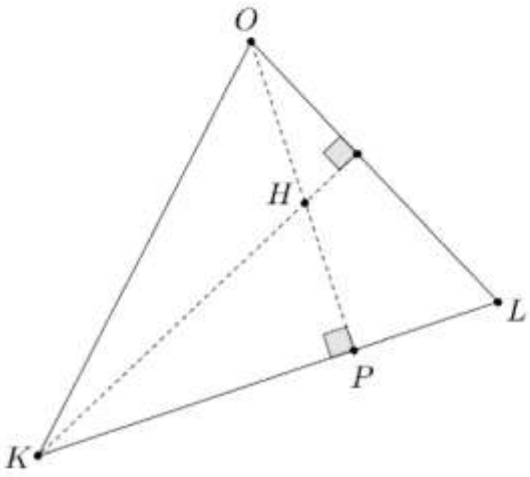
$$LP^2 = LC.LA = y(a+y) > y^2 \quad \Rightarrow \quad LP > y \quad (3)$$

According to inequality 2, 3 we can say: $xy < KP.LP$ (4)

Now According to inequality 1, 4 we have: $R^2 < KP.LP$ (5)



We know that $\angle KOL \leq 90°$, therefore KOL is acute-triangle. Suppose that H is orthocenter of $\triangle KOL$. So the point H lie on OP and we can say $HP \leq OP$.



In other hand, $\angle HKP = \angle POL$ and $\angle KHP = \angle OLP$, therefore two triangles THP and OPL are similar. So we have:

$$\frac{KP}{HP} = \frac{OP}{LP} \quad \Rightarrow \quad KP.LP = HP.OP \leq OP^2 = R^2$$

But according to inequality 5, we have $R^2 < KP.LP$ and it's a contradiction. Therfore $\angle KOL > 90°$.

3. In triangle ABC, H is the orthocenter of triangle. Let l_1 and l_2 be two lines such that pass through H and perpendicular to each other. The line l_1 intersects BC and extension of AB at D and Z respectively and the line l_2 intersects BC and extension of AC at E and X respectively. We draw the Line which passes through D and parallel to AC and we draw the Line which passes through E and parallel to AB. Suppose that intersectoin of these lines denote by Y. Prove that X, Y, Z are collinear.

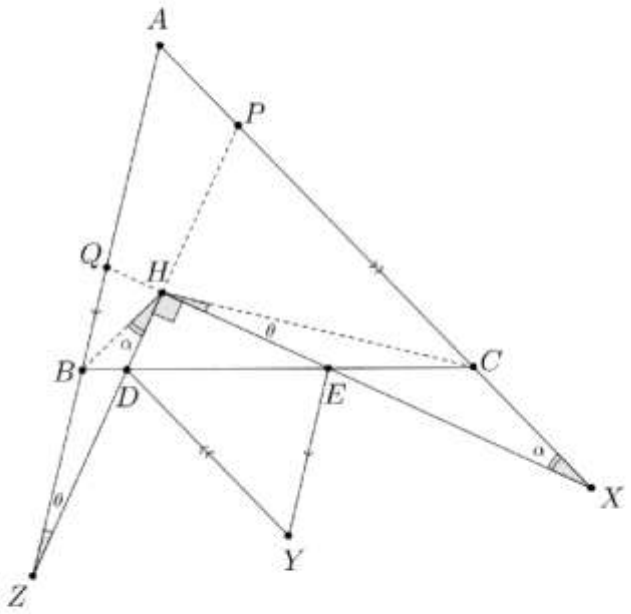
Proposed by Ali Golmakani

Solution.

Suppose that HZ intersects AC at P and HX intersects AB at Q. According to Menelaus theorem in two triangles AQX and APZ we can say:

$$\frac{CX}{AC}.\frac{AB}{BQ}.\frac{QE}{EX}=1 \quad (1) \quad \textit{and} \quad \frac{BZ}{AB}.\frac{AC}{PC}.\frac{PD}{DZ}=1 \quad (2)$$

In the other hand, H is the orthocenter of $\triangle ABC$. So $BH \perp AC$ and we know that $\angle DHE = 90°$, therefore $\angle HXA = \angle BHZ = \alpha$. Similarly we can say $\angle HZA = \angle CHX = \theta$.



According to law of sines in $\triangle HPC$, $\triangle HCX$ and $\triangle HPX$:

$$\frac{sin(90-\theta)}{PC} = \frac{sin(\angle HCP)}{HP} \quad , \quad \frac{sin(\theta)}{CX} = \frac{sin(\angle HCX)}{HX} \quad , \quad \frac{HP}{HX} = \frac{sin(\alpha)}{sin(90-\alpha)}$$

$$\Rightarrow \quad \frac{PC}{CX} = \frac{tan(\alpha)}{tan(\theta)}$$

Similarly, according to law of sines in $\triangle HBQ$, $\triangle HBZ$ and $\triangle HQZ$, we can show:

$$\Rightarrow \quad \frac{BZ}{BQ} = \frac{tan(\alpha)}{tan(\theta)} \quad \Rightarrow \quad \frac{BZ}{BQ} = \frac{PC}{CX} \quad \Rightarrow \quad \frac{PC}{BZ} = \frac{CX}{BQ} \qquad (3)$$

According to equality 1, 2 and 3, we can say:

$$\frac{XE}{EQ} = \frac{PD}{ZD} \qquad (4)$$

Suppose that the Line witch passes through E and parallel to AB, intersects ZX at Y_1 and the Line witch passes through D and parallel to AC, intersects ZX at Y_2. According to thales theorem we can say:

$$\frac{Y_1X}{ZY_1} = \frac{XE}{EQ} \quad , \quad \frac{Y_2X}{ZY_2} = \frac{PD}{ZD}$$

According to equality 4, we show that $Y_1 \equiv Y_2$, therefore the point Y lie on ZX.

4. In triangle ABC, we draw the circle with center A and radius AB. This circle intersects AC at two points. Also we draw the circle with center A and radius AC and this circle intersects AB at two points. Denote these four points by A_1, A_2, A_3, A_4. Find the points B_1, B_2, B_3, B_4 and C_1, C_2, C_3, C_4 similarly. Suppose that these 12 points lie on two circles. Prove that the triangle ABC is isosceles.

Proposed by Morteza Saghafian

Solution-1.

Suppose that triangle ABC isn't isosceles and $a > b > c$. In this case, there are four points (from these 12 points) on each side of $\triangle ABC$. Suppose that these 12 points lie on two circles ω_1 and ω_2. Therefore each one of the circles ω_1 and ω_2 intersects each side of $\triangle ABC$ exactly at two points. Suppose that $P(A, \omega_1), P(A, \omega_2)$ are power of the point A with respect to circles ω_1, ω_2 respectively. Now we know that:

$$P(A, \omega_1).P(A, \omega_2) = b.b.(a-c).(a+c) = c.c.(a-b)(a+b)$$

$$\Rightarrow \quad b^2(a^2-c^2) = c^2(a^2-b^2) \quad \Rightarrow \quad a^2(b^2-c^2) = 0 \quad \Rightarrow \quad b = c$$

But we know that $b > c$ and it's a contradiction. Therefore the triangle ABC is isosceles.

Solution-2.

Suppose that triangle ABC isn't isosceles. In this case, there are four points (from these 12 points) on each side of $\triangle ABC$. Suppose that these 12 points lie on two circles ω_1 and ω_2. Therefore each one of the circles ω_1 and ω_2 intersects each side of $\triangle ABC$ exactly at two points (and each one of the circles ω_1 and ω_2 doesn't pass through A, B, C). We know that the intersections of ω_1 and the sides of $\triangle ABC$ is even number. Also the intersections of ω_2 and the sides of $\triangle ABC$ is even number. But Among the these 12 points, just 3 points lie on the sides of $\triangle ABC$ and this is odd number. So it's a contradiction. Therefore the triangle ABC is isosceles.

5. Rectangles ABA_1B_2, BCB_1C_2, CAC_1A_2 lie otside triangle ABC. Let C' be such point that $C'A_1 \perp A_1C_2$ and $C'B_2 \perp B_2C_1$, points A', B' are defined similarly. Prove that lines AA', BB', CC' concur.

Proposed by Alexey Zaslavsky (Russia)

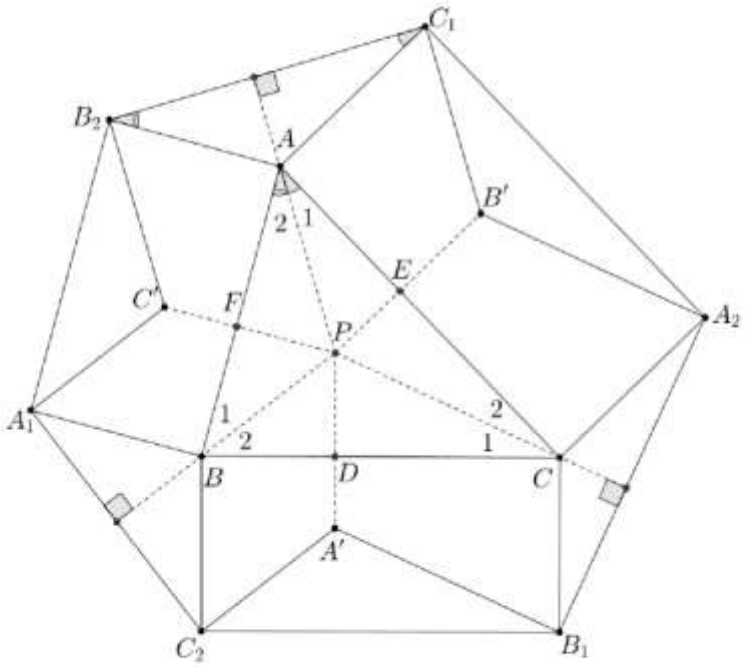
Solution.

Suppose that l_A is the line which passes through A and perpendicular to B_2C_1. Let l_B and l_C similarly. Suppose that $CB_1 = BC_2 = x$ and $BA_1 = AB_2 = y$ and $AC_1 = CA_2 = z$. According to angles equality, we can say:

$$\frac{sin(\angle A_1)}{sin(\angle A_2)} = \frac{y}{z} \quad , \quad \frac{sin(\angle B_1)}{sin(\angle B_2)} = \frac{x}{y} \quad , \quad \frac{sin(\angle C_1)}{sin(\angle C_2)} = \frac{z}{x}$$

According to sine form of Ceva's theorem in $\triangle ABC, l_A, l_B, l_C$ are concur. Suppose that l_A, l_B, l_C pass through the point P. We know that $\triangle PBC$ and $\triangle A'C_2B_1$ are equal. (because of $BP \parallel A'C_2$, $CP \parallel A'B_1$, $BC \parallel B_1C_2$ and $BC = B_1C_2$). So we have:

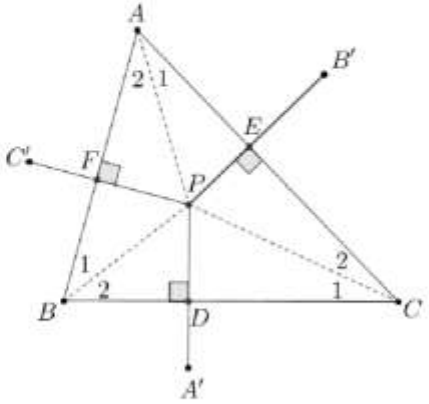
$$PA' = x \ , \ PC' = y \ , \ PB' = z \qquad PA' \perp BC \ , \ PB' \perp AC \ , \ PC' \perp AB$$



Suppose that PA', PB', PC' intersects BC, AC, AB at D, E, F respectively and $PD = m$, $PE = n$, $PF = t$. According to before picture, we have:

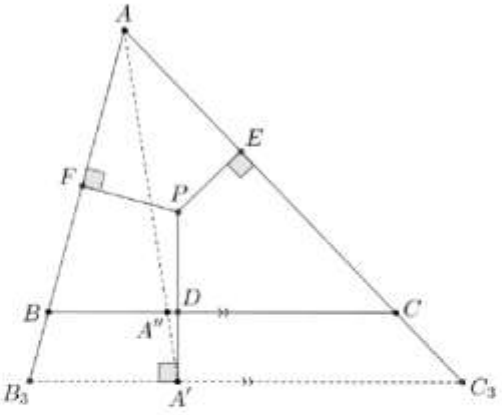
$$\frac{sin(\angle A_1)}{sin(\angle A_2)} = \frac{n}{t} = \frac{y}{z} \quad , \quad \frac{sin(\angle B_1)}{sin(\angle B_2)} = \frac{t}{m} = \frac{x}{y} \quad , \quad \frac{sin(\angle C_1)}{sin(\angle C_2)} = \frac{m}{n} = \frac{z}{x}$$

If $n = ky$, then: $t = kz$, $m = \frac{kyz}{x}$.



Now draw the line from A' such that be parallel to BC. The intersection of this line and extension AB and AC denote by B_3 and C_3 respectively. Let the point A'' be the intersection of AA' and BC. According to Thales theorem, we have:

$$\frac{BA''}{CA''} = \frac{B_3A'}{C_3A'}$$



Let $\angle B_3PA' = \alpha$ and $\angle C_3PA' = \theta$. We know that the quadrilaterals PFB_3A and PEC_3A' are cyclic. Therefore $\angle B_3FA' = \alpha$ and $\angle C_3EA' = \theta$.

According to law of sines in $\triangle PB_3A'$ and $\triangle PC_3A'$ and $\triangle PC_3B_3$:

$$\frac{B_3A'}{C_3A'} = \frac{tan(\alpha)}{tan(\theta)}$$

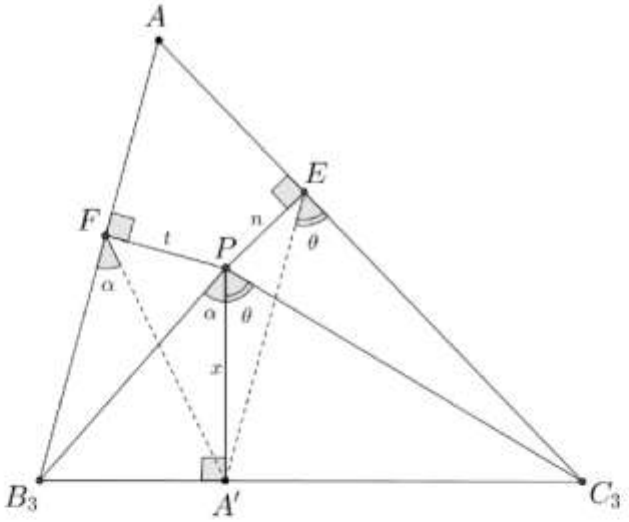
Also according to law of sines in $\triangle PFA'$:

$$\frac{t}{x} = \frac{sin(\angle B + \alpha - 90)}{cos(\alpha)} = \frac{cos(\angle B + \alpha)}{cos(\alpha)} = cos(\angle B) - tan(\alpha).sin(\angle B)$$

$$\Rightarrow \quad tan(\alpha) = \frac{cos(\angle B) - \frac{t}{x}}{sin(\angle B)}$$

Similarly we can say:

$$tan(\theta) = \frac{cos(\angle C) - \frac{n}{x}}{sin(\angle C)} \quad \Rightarrow \quad \frac{B_3A'}{C_3A'} = \frac{BA''}{CA''} = \frac{x.cos(\angle B) - t}{x.cos(\angle C) - n}.\frac{sin(\angle C)}{sin(\angle B)}$$



Similarly, two other fractions can be calculated.
According to Ceva's theorem in $\triangle ABC$, we have to that:

$$\frac{x.cos(\angle B) - t}{x.cos(\angle C) - n}.\frac{sin(\angle C)}{sin(\angle B)}.\frac{z.cos(\angle C) - m}{z.cos(\angle A) - t}.\frac{sin(\angle A)}{sin(\angle C)}.\frac{y.cos(\angle A) - n}{y.cos(\angle B) - m}.\frac{sin(\angle B)}{sin(\angle A)} = 1$$

$$\iff \frac{x.cos(\angle B) - t}{x.cos(\angle C) - n}.\frac{z.cos(\angle C) - m}{z.cos(\angle A) - t}.\frac{y.cos(\angle A) - n}{y.cos(\angle B) - m} = 1$$

In other hand, we know that:

$$n = ky \quad , \quad t = kz \quad , \quad m = \frac{kyz}{x}$$

$$\Longleftrightarrow \frac{x.cos(\angle B) - kz}{x.cos(\angle C) - ky}.\frac{x.cos(\angle C) - ky}{x.cos(\angle A) - kx}.\frac{x.cos(\angle A) - kx}{x.cos(\angle B) - kz} = 1$$

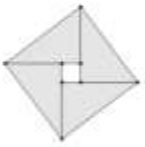
Therfore, we show that AA', BB', CC' are concur.

2nd Iranian Geometry Olympiad

September 2015

Problems of 2nd Iranian Geometry Olympiad (Elementary)

1. We have four wooden triangles with sides 3, 4, 5 centimeters. How many convex polygons we can make by all of these triangles?(Just draw the polygons without any proof)
A convex polygon is a polygon which all of the angles of it are less than 180° and there isn't any hole in it. For example:



This polygon isn't convex

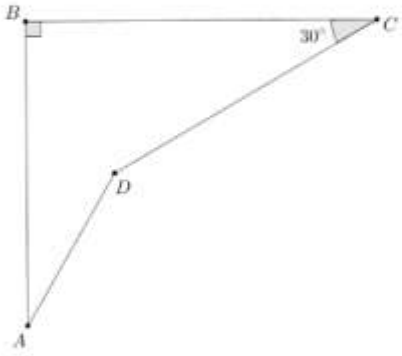
This polygon is convex

Proposed by Mahdi Etesami Fard

2. Let ABC be a triangle with $\angle A = 60^\circ$. The points M, N, K lie on BC, AC, AB respectively such that $BK = KM = MN = NC$. If $AN = 2AK$, find the values of $\angle B$ and $\angle C$.

Proposed by Mahdi Etesami Fard

3. In the picture below, we know that $AB = CD$ and $BC = 2AD$. Prove that $\angle BAD = 30^\circ$.



Proposed by Morteza Saghafian

4. In rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD, AD respectively such that the area of triangles AQM, BMN, CNP, DPQ are equal. Prove that the quadrilateral $MNPQ$ is parallelogram.

Proposed by Mahdi Etesami Fard

5. Do there exist 6 circles in the plane such that every circle passes through exactly 3 centers of other circles?

Proposed by Morteza Saghafian

Solutions of 2nd Iranian Geometry Olympiad (Elementary)

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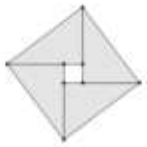

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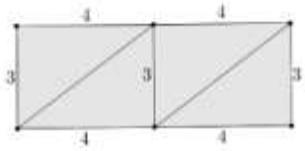
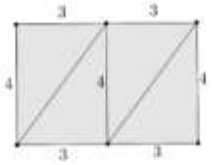
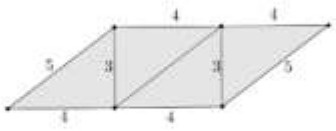
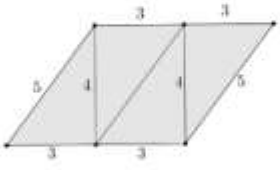
Proposed by Mahdi Etesami Fard

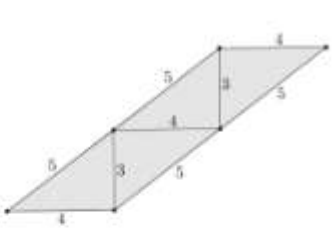
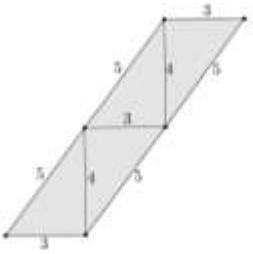
Solution.









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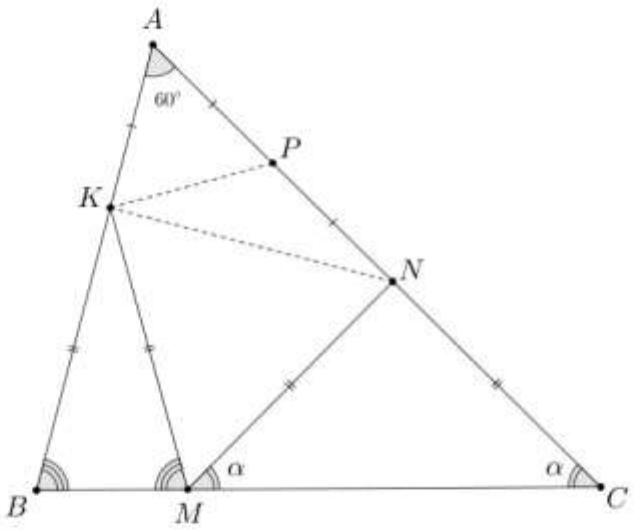
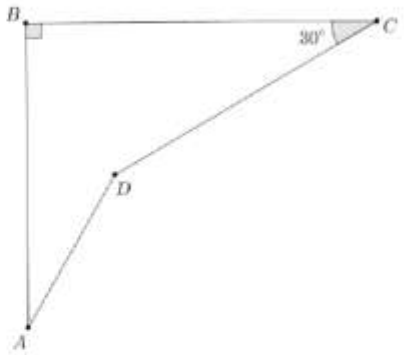
Proposed by Mahdi Etesami Fard

Solution.

Suppose the point P be the midpoint of AN. Therefore $AK = AP = AN$ and so we can say $\triangle APK$ is the equilateral triangle. So $\angle ANK = \frac{\angle KPA}{2} = 30^\circ$ Let $\angle ACB = \angle NMC = \alpha$. Therfore $\angle ABC = \angle KMB = 120^\circ - \alpha$. So $\angle KMN = 60^\circ$. Therefore $\triangle KMN$ is the equilateral triangle. Now we know that $\angle MNA = 90^\circ$. Therefore $\alpha = 45^\circ$. So we have $\angle C = 45^\circ$ and $\angle B = 75^\circ$.



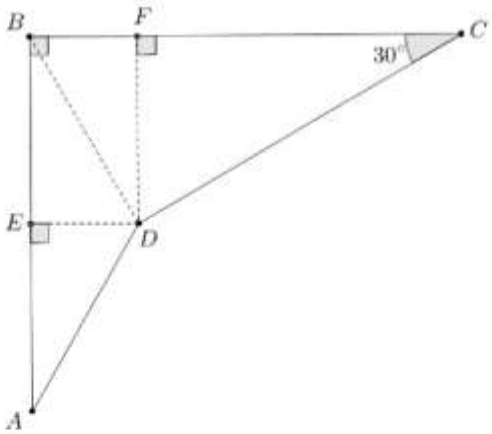
3. In the picture below, we know that $AB = CD$ and $BC = 2AD$. Prove that $\angle BAD = 30°$.



Proposed by Morteza Saghafian

Solution-1.

Let two points E and F on BC and AB respectively such that $DF \perp BC$ and $DE \perp AB$. We can say $DF = \frac{DC}{2} = \frac{AB}{2}$.(because of $\angle BCD = 30°$ and $\angle DFC = 90°$) Also we know that $DF = BE$, therfore DE is the perpendicular bisector of AB. So $BD = AD$.



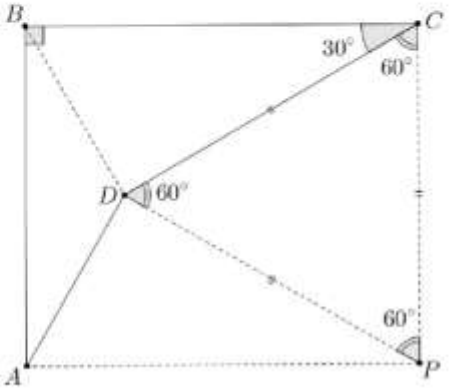
Let H be a point on CD such that $BH \perp CD$. therefore $BH = \frac{BC}{2} = BD$, so we can say $D \equiv H$ and $\angle BDC = 90°$. Therefore $\angle ABD = \angle BAD = 30°$.

Solution-2.

Suppose that P is the point such that triangle DCP is Equilateral. We know that $PC \perp BC$ and $PC = CD = AB$, therfore quadrilateral $ABCP$ is Rectangular.

$$\Rightarrow \quad \angle APD = \angle APC - \angle DPC = 90° - 60° = 30°$$

In other hand, $DP = DC$ and $AP = BC$. So $\triangle ADP$ and $\triangle BDC$ are congruent. Therfore $AD = BD$.



Let the point H on CD such that $BH \perp CD$. therefore $BH = \frac{BC}{2} = BD$, so we can say $D \equiv H$ and $\angle BDC = 90°$. Therefore $\angle ABD = \angle BAD = 30°$.

4. In rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD, AD respectively such that the area of triangles AQM, BMN, CNP, DPQ are equal. Prove that the quadrilateral $MNPQ$ is parallelogram.

Proposed by Mahdi Etesami Fard

Solution.

Let $AB = CD = a, AD = BC = b$ and $AM = x, AQ = z, PC = y, NC = t$. If $x \neq y$, we can assume that $x > y$. We know that:

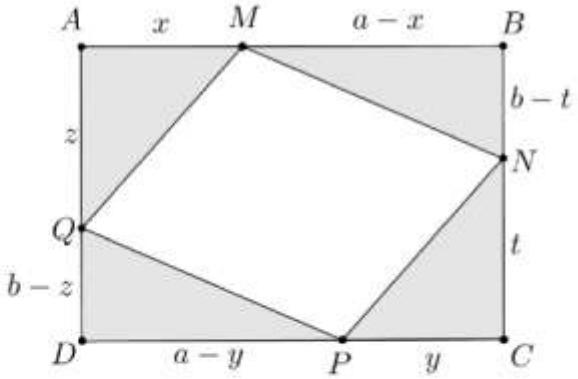
$$y < x \quad \Rightarrow \quad a - x < a - y \quad (1)$$

$$S_{AQM} = S_{CNP} \quad \Rightarrow \quad zx = yt \quad \Rightarrow \quad z < t \quad \Rightarrow \quad b - t < b - z \quad (2)$$

According to inequality 1, 2:

$$(a - x)(b - t) < (a - y)(b - z) \quad \Rightarrow \quad S_{BMN} < S_{DPQ}$$

it's a contradiction. Therfore $x = y$, so $z = t$ Now we can say two triangles AMQ and CPN are congruent. Therefore $MQ = NP$ and similarly $MN = PQ$. So the quadrilateral $MNPQ$ is parallelogram.



Comment.

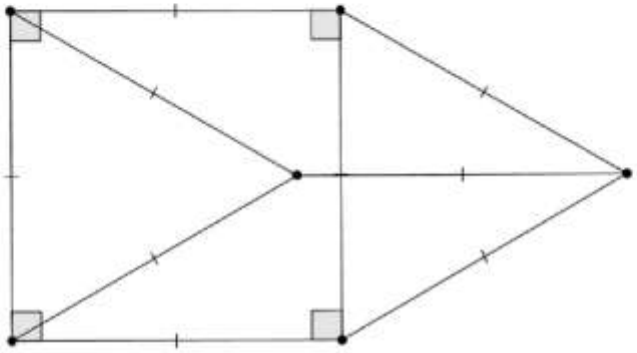
If quadrilateral $ABCD$ be the parallelogram, similarly we can show that quadrilateral $MNPQ$ is parallelogram.

5. Do there exist 6 circles in the plane such that every circle passes through exactly 3 centers of other circles?

Proposed by Morteza Saghafian

Solution.

in the picture below, we have 6 points in the plane such that for every point there exists exactly 3 other points on a circle with radius 1 centimeter.

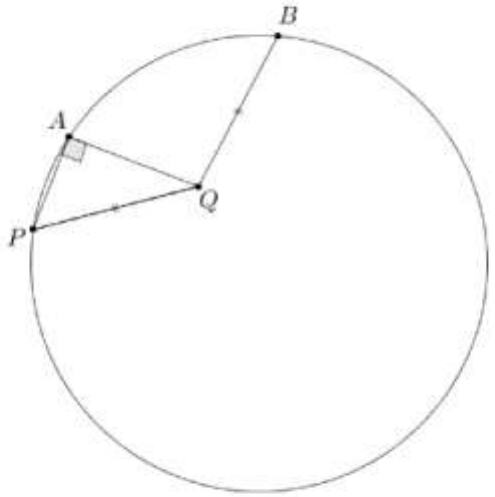


2nd Iranian Geometry Olympiad

September 2015

Problems of 2nd Iranian Geometry Olympiad (Medium)

1. In picture below, the points P, A, B lie on a circle. The point Q lie inside the circle such that $\angle PAQ = 90^\circ$ and $PQ = BQ$. Prove that the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB.



Proposed by Davood Vakili

2. In acute-angle triangle ABC, BH is the altitude of the vertex B. The points D and E are midpoints of AB and AC respectively. Suppose that F be the reflection of H to ED. Prove that the line BF passes through circumcenter of $\triangle ABC$.

Proposed by Davood Vakili

3. In triangle ABC, the points M, N, K are the midpoints of BC, AC, AB respectively. Let ω_B and ω_C be two semicirculars with diameter AC and AB outside the triangle respectively. Suppose that MK and MN intersect ω_C and ω_B at X and Y respectively. If the point Z be the intersection of the tangent to ω_C and ω_B in X and Y respectively, prove that: $AZ \perp BC$

Proposed by Mahdi Etesami Fard

4. Suppose that ABC be the equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC (the arc witch A doesn't lie). Tangent to ω in P intersects extension of AB and AC at K and L respectively. Show that: $\angle KOL > 90^\circ$

Proposed by Iman Maghsoudi

5. a) Do there exist 5 circles in the plane such that every circle passes through exactly 3 center of other circles?

b) Do there exist 6 circles in the plane such that every circle passes through exactly 3 center of other circles?

Proposed by Morteza Saghafian

Solutions of 2nd Iranian Geometry Olympiad (Medium)

1. In picture below, the points P, A, B lie on a circle. The point Q lie inside the circle such that $\angle PAQ = 90°$ and $PQ = BQ$. Prove that the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB. .

Proposed by Davood Vakili

Solution-1.

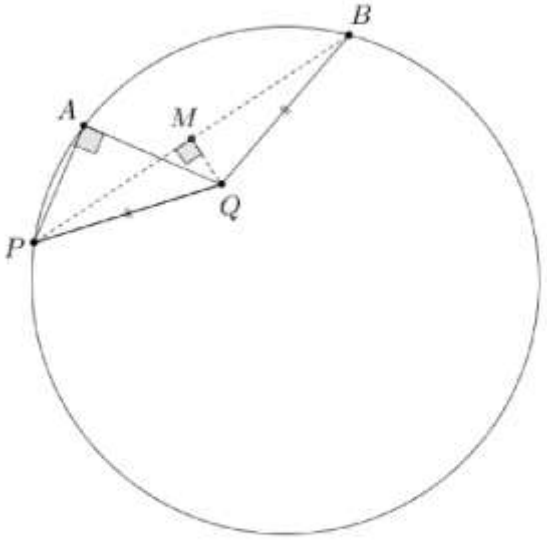
Let point M be the midpoint of PB. So we can say $\angle PMQ = 90°$ and we know that $\angle PAQ = 90°$, therefore quadrilateral $PAMQ$ is cyclic. Therefore:

$$\angle APM = \angle AQM$$

In the other hand:

$$\angle AQB - \angle AQP = \angle PQM + \angle AQM - \angle AQP = 2\angle AQM$$

So we can say that the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB.



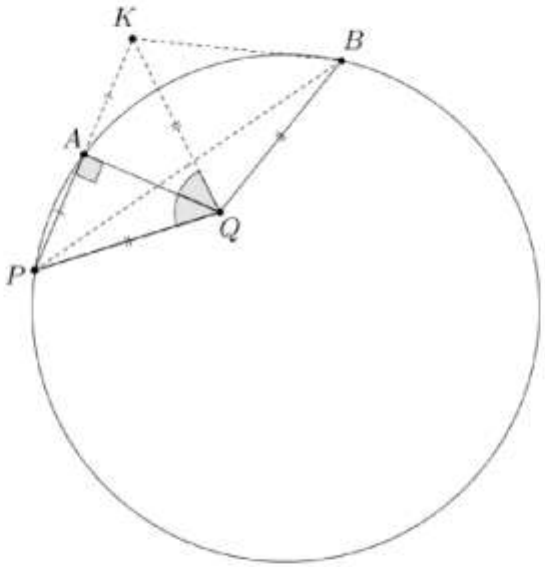
Solution-2.
Let the point K be the reflection of P to AQ. We have to show:

$$2\angle APB = \angle AQB - \angle AQP$$

Now we know that AQ is the perpendicular bisector of PK. So $\angle AQP = \angle AQK$ and $PQ = KQ = BQ$, therefore the point Q is the circumcenter of triangle PKB. We know that:

$$2\angle APB = \angle KQB = \angle AQB - \angle AQK = \angle AQB - \angle AQP$$

Therefore the subtract $\angle AQB$ from $\angle PQA$ is equal to arc AB.

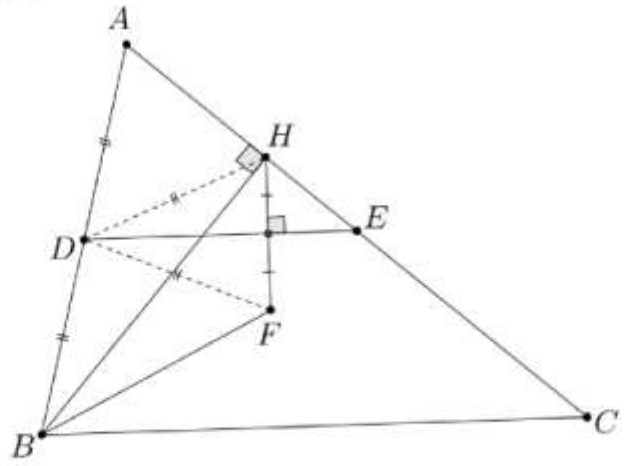


2. In acute-angle triangle ABC, BH is the altitude of the vertex B. The points D and E are midpoints of AB and AC respectively. Suppose that F be the reflection of H to ED. Prove that the line BF passes through circumcenter of $\triangle ABC$.

Proposed by Davood Vakili

Solution-1.

The circumcenter of $\triangle ABC$ denote by O. We know that $\angle OBA = 90^\circ - \angle C$, therfore we have to show that $\angle FBA = 90^\circ - \angle C$. We know that $AD = BD = DH$, also $DH = DF$.



Therfore quadrilateral $AHFB$ is cyclic (with circumcenter D)

$$\Rightarrow \quad \angle FBA = \angle FHE = 90^\circ - \angle DEH \quad , \quad DE \parallel BC \quad \Rightarrow \quad \angle DEH = \angle C$$

$$\Rightarrow \quad \angle FBA = 90^\circ - \angle C$$

Solution-2.

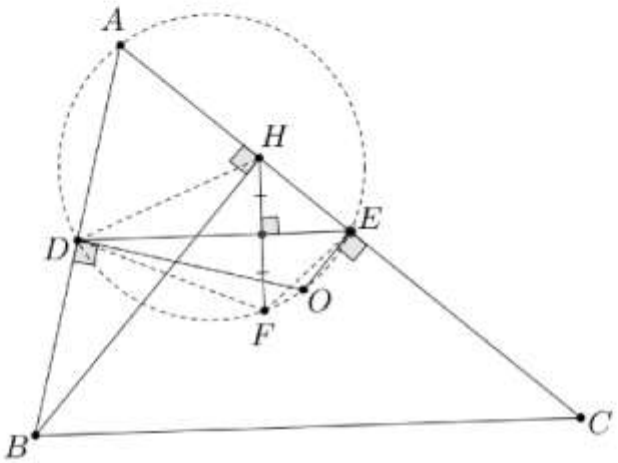
The circumcenter of $\triangle ABC$ denote by O. We know that quadrilateral $ADOE$ is cyclic. Also we know that $AD = HD = DB$, therefore:

$$\angle A = \angle DHA = 180^\circ - \angle DHE = 180^\circ - \angle DFE \quad \Rightarrow \quad ADFE : cyclic$$

So we can say $ADFOE$ is cyclic, therefore quadrilateral $DFOE$ is cyclic.

$$\angle C = \angle DEA = \angle DEF = \angle DOF$$

In the other hand: $\angle C = \angle DOB$ so $\angle DOF = \angle DOB$, therefore B, F, O are collinear.



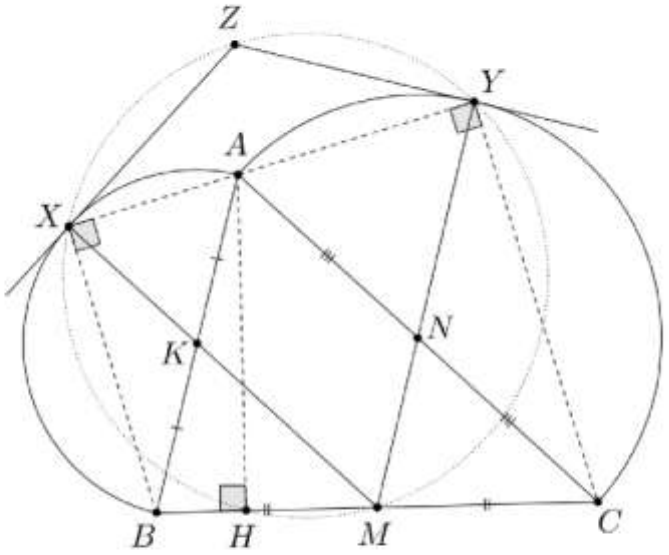
3. In triangle ABC, the points M, N, K are the midpoints of BC, AC, AB respectively. Let ω_B and ω_C be two semicirculars with diameter AC and AB outside the triangle respectively. Suppose that MK and MN intersect ω_C and ω_B at X and Y respectively. If the point Z be the intersection of the tangent to ω_C and ω_B in X and Y respectively, prove that: $AZ \perp BC$

Proposed by Mahdi Etesami Fard

Solution-1.

Let point H on BC such that $AH \perp BC$. Therefore quadrilaterals $AXBH$ and $AYCH$ are cyclic. We know that KM and MN are parallel to AC and AB respectively. So we can say $\angle AKX = \angle ANY = \angle A$, therefore $\angle ABX = \angle ACY = \frac{\angle A}{2}$ and $\angle XAB = \angle YAC = 90^\circ - \frac{\angle A}{2}$. So X, A, Y are collinear.

$$\angle AHX = \angle ABX = \frac{\angle A}{2} \, , \angle AHY = \angle ACY = \frac{\angle A}{2} \Rightarrow \angle XHY = \angle XMY = \angle A$$



Therefore quadrilateral $XHMY$ is cyclic. Also we know that $\angle MXZ = \angle MYZ = 90^\circ$, therefore quadrilateral $MXZY$ is cyclic. So we can say $ZXHMY$ is cyclic. therfore quadrilateral $HXZY$ is cyclic.

In the other hand: $\angle ZYX = \angle ACY = \frac{\angle A}{2}$

$$\angle ZHX = \angle ZYX = \frac{\angle A}{2}\ ,\ \angle AHX = \frac{\angle A}{2} \quad \Rightarrow \quad \angle ZHX = \angle AHX$$

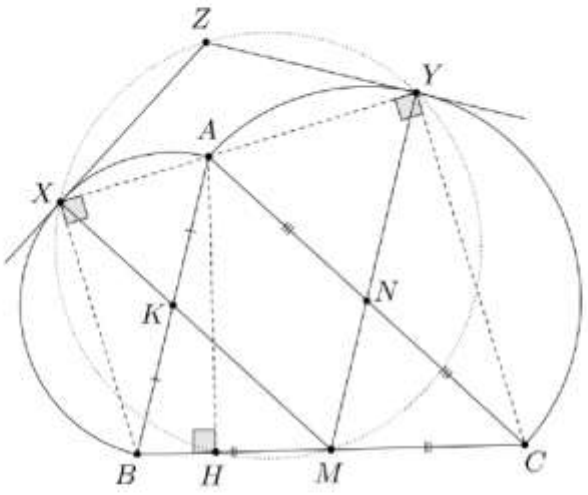
So the points Z, A, H are collinear, therefore $AZ \perp BC$.

Solution-2.

Let point H on BC such that $AH \perp BC$. We know that KM and MN are parallel to AC and AB respectively. So we can say $\angle AKX = \angle ANY = \angle A$, therefore $\angle ABX = \angle ACY = \frac{\angle A}{2}$ and $\angle XAB = \angle YAC = 90° - \frac{\angle A}{2}$. So X, A, Y are collinear.

$$\Rightarrow \quad \angle ZXY = \angle ZYX = \frac{\angle A}{2} \quad \Rightarrow \quad ZX = ZY$$

So the point Z lie on the radical axis of two these semicirculars. Also we know that the line AH is the radical axis of two these semicirculars. Therefore the points Z, A, H are collinear, therefore $AZ \perp BC$.



4. Suppose that ABC be the equilateral triangle with circumcircle ω and circumcenter O. Let P be the point on the arc BC (the arc witch A doesn't lie). Tangent to ω in P intersects extension of AB and AC at K and L respectively. Show that: $\angle KOL > 90^\circ$

Proposed by Iman Maghsoudi

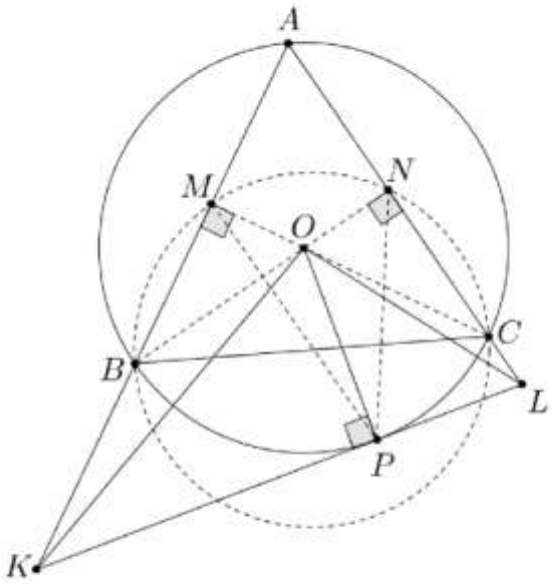
Solution-1.

Suppose that M and N be the midpoints of AB and AC respectively. We know that quadrilateral $BMNC$ is cyclic. Also $\angle BPC = 120^\circ > 90^\circ$, so we can say the point P is in the circumcircle of quadrilateral $BMNC$. Therefore: $\angle MPN > \angle MBN = 30^\circ$

In the other hand, quadrilaterals $TMOP$ and $NOPC$ are cyclic. Therefore:

$$\angle MKO = \angle MPO \ , \ \angle NLO = \angle NPO \ \Rightarrow \ \angle AKO + \angle ALO = \angle MPN > 30^\circ$$

$$\Rightarrow \angle KOL = \angle A + \angle AKO + \angle ALO > 90^\circ$$



Solution-2.

Suppose that $\angle KOL \leq 90°$, therfore $KL^2 \leq OK^2 + OL^2$. Assume that R is the radius of a circumcircle $\triangle ABC$. Let $BK = x$ and $LC = y$ and $AB = AC = BC = a$. According to law of cosines in triangle AKL, we have:

$$KL^2 = AK^2 + AL^2 - AK.AL.cos(\angle A) \quad \Rightarrow \quad KL^2 = (a+x)^2 + (a+y)^2 - (a+x)(a+y)$$

In the other hand:

$$KB.KA = OK^2 - R^2 \quad \Rightarrow \quad OK^2 = R^2 + x(a+x)$$

$$LC.LA = OL^2 - R^2 \quad \Rightarrow \quad OL^2 = R^2 + y(a+y)$$

We know that $KL^2 \leq OK^2 + OL^2$ and $a = R\sqrt{3}$, therfore:

$$(a+x)^2 + (a+y)^2 - (a+x)(a+y) \leq 2R^2 + x(a+x) + y(a+y)$$

$$\Rightarrow \quad R^2 \leq xy \quad (1)$$

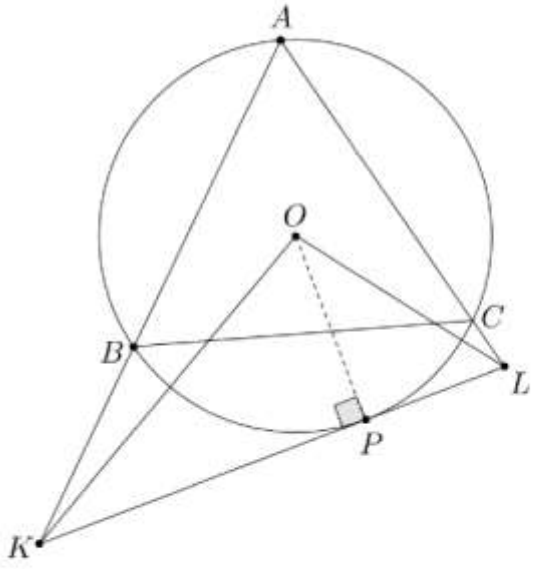
KL is tangent to circumcircle of $\triangle ABC$ at P. So we have:

$$KP^2 = KB.KA = x(a+x) > x^2 \quad \Rightarrow \quad KP > x \quad (2)$$

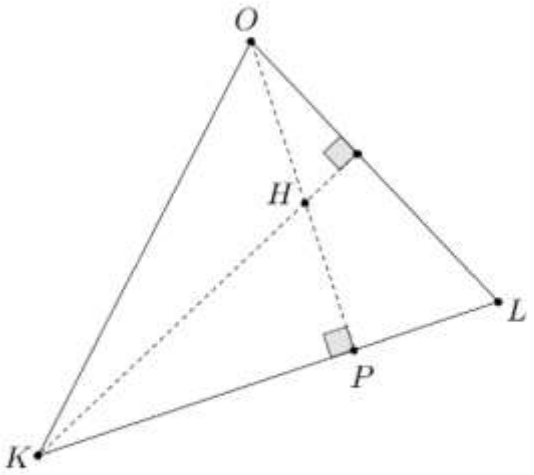
$$LP^2 = LC.LA = y(a+y) > y^2 \quad \Rightarrow \quad LP > y \quad (3)$$

According to inequality 2, 3 we can say: $xy < KP.LP$ (4)

Now According to inequality 1, 4 we have: $R^2 < KP.LP$ (5)



We know that $\angle KOL \leq 90°$, therefore KOL is acute-triangle. Suppose that H is orthocenter of $\triangle KOL$. So the point H lie on OP and we can say $HP \leq OP$.



In other hand, $\angle HKP = \angle POL$ and $\angle KHP = \angle OLP$, therefore two triangles THP and OPL are similar. So we have:

$$\frac{KP}{HP} = \frac{OP}{LP} \quad \Rightarrow \quad KP.LP = HP.OP \leq OP^2 = R^2$$

But according to inequality 5, we have $R^2 < KP.LP$ and it's a contradiction. Therfore $\angle KOL > 90°$.

5. a) Do there exist 5 circles in the plane such that every circle passes through exactly 3 center of other circles?

b) Do there exist 6 circles in the plane such that every circle passes through exactly 3 center of other circles?

Proposed by Morteza Saghafian

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a)Solution.

There aren't such 5 circles. Suppose that these circles exists, therefore their centers are 5 points that each point has same distance from 3 other points and has diffrent distance from the remaining point. We draw an arrow from each point to it's diffrent distance point.

- lemma 1. We don't have two points such O_i, O_j that each one is the diffrent distance point of the other one.

proof. If we have such thing then O_i and O_j both have same distance to the remaining points, therefore both of them are circumcenter of the remaining points, which is wrong.

- lemma 2. We don't have 4 points such O_i, O_j, O_k, O_l that O_i, O_j put their arrow in O_k and O_K puts it's arrow in O_l.

proof. If we name the remaining point O_m then the distances of O_i from O_j, O_l, O_m are equal and the distances of O_j from O_i, O_l, O_m are equal. Therefore each of O_l, O_m is the diffrent distance point of another which is wrong (according to lemma 1).

so each point sends an arrow and recives an arrow. Because of lemma 1 we don't have 3 or 4 points cycles. Therefore we only have one 5 points cycle. So each pair of these 5 points should have equal distance. which is impossible.

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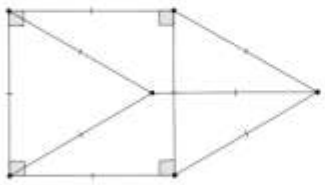
b)Solution.

in the picture below, we have 6 points in the plane such that for every point there exists exactly 3 other points on a circle with radius 1 centimeter.