SEEMOUS CYPRUS 2021
Mathematical Olympiad for University Students

Language: **English**

Wednesday, July 21, 2021

Problem 1. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous strictly increasing function such that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 1\,.$$

(a) Prove that the sequence $(x_n)_{n\geqslant 1}$ defined by

$$x_n = f\left(\frac{1}{1}\right) + f\left(\frac{1}{2}\right) + \cdots + f\left(\frac{1}{n}\right) - \int_1^n f\left(\frac{1}{x}\right)\mathrm{d}x$$

is convergent.

(b) Find the limit of the sequence $(y_n)_{n\geqslant 1}$ defined by

$$y_n = f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{2021n}\right).$$

Problem 2. Let $n \geqslant 2$ be a positive integer and let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix such that $A^2 = -I_n$. If $B \in \mathcal{M}_n(\mathbb{R})$ and $AB = BA$, prove that $\det B \geqslant 0$.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix such that $(AA^*)^2 = A^*A$, where $A^* = \left(\overline{A}\right)^t$ denotes the Hermitian transpose (i.e., the conjugate transpose) of A.

(a) Prove that $AA^* = A^*A$.

(b) Show that the non-zero eigenvalues of A have modulus one.

Problem 4. For $p \in \mathbb{R}$, let $(a_n)_{n\geqslant 1}$ be the sequence defined by

$$a_n = \frac{1}{n^p}\int_0^n \left|\sin(\pi x)\right|^x \mathrm{d}x\,.$$

Determine all possible values of p for which the series $\sum\limits_{n=1}^{\infty} a_n$ converges.

Language: English *Time: 5 hours*

Each problem is worth 10 points

Solution - Problem 1a:

We write

$$x_n = \sum_{k=1}^{n-1} \left(f\left(\frac{1}{k}\right) - \int_k^{k+1} f\left(\frac{1}{x}\right) \mathrm{d}x \right) + f\left(\frac{1}{n}\right).$$

Because f is increasing, for all $k \geqslant 1$ and $x \in [k, k+1]$ we have

$$f\left(\frac{1}{k+1}\right) \leqslant f\left(\frac{1}{x}\right) \leqslant f\left(\frac{1}{k}\right)$$

and therefore

$$f\left(\frac{1}{k+1}\right) \leqslant \int_k^{k+1} f\left(\frac{1}{x}\right) \mathrm{d}x \leqslant f\left(\frac{1}{k}\right) \tag{1}$$

Summing up for $k = 1$ up to $n-1$ we obtain

$$f\left(\frac{1}{n}\right) \leqslant x_n \leqslant f(1).$$

Since f is increasing then x_n is bounded below by $f(0)$.

It is easy to see that x_n is decreasing since using (1) we have:

$$x_{n+1} - x_n = f\left(\frac{1}{n+1}\right) - \int_n^{n+1} f\left(\frac{1}{x}\right) \mathrm{d}x \leqslant 0.$$

We conclude that (x_n) is convergent to some $\ell \in \mathbb{R}$.

Solution 1 - Problem 1b:

Since $\lim\limits_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $1-\varepsilon < \frac{f(x)}{x} < 1+\varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$ and every $k \geqslant 1$ we have $0 < \frac{1}{n+k} < \frac{1}{n} < \delta$ and therefore

$$(1-\varepsilon)\frac{1}{n+k} < f\left(\frac{1}{n+k}\right) < (1+\varepsilon)\frac{1}{n+k}.$$

Summing up the above inequalities from $k = 1$ to $2020n$ we get

$$(1-\varepsilon)S_n < f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{2021n}\right) < (1+\varepsilon)S_n,$$

where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2021n}.$$

It is well-known that $\lim\limits_{n\to\infty} S_n = \ln(2021)$ so since ε is arbitrary, we get that $\lim\limits_{n\to\infty} y_n = \ln 2021$.

Solution 2 - Problem 1b:

Since

$$y_n = x_{2021n} - x_n + \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x\,,$$

from part (a), it is enough to find

$$\lim_{n\to\infty} \int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x.$$

With the change of variable $x = \dfrac{1}{t}$ we obtain

$$\int_n^{2021n} f\left(\frac{1}{x}\right) \mathrm{d}x = \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2}\,\mathrm{d}t\,.$$

Since $\lim\limits_{x\to 0^+} \frac{f(x)}{x} = 1$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $1-\varepsilon < \frac{f(x)}{x} < 1+\varepsilon$ for every $0 < x < \delta$. In particular, for every $n > \frac{1}{\delta}$, we have $0 < \frac{1}{2021n} < \frac{1}{n} < \delta$ and therefore

$$(1-\varepsilon)\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\,\mathrm{d}t \leqslant \int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{f(t)}{t^2}\,\mathrm{d}t \leqslant (1+\varepsilon)\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\,\mathrm{d}t\,.$$

Since ε is arbitrary, and since

$$\int_{\frac{1}{2021n}}^{\frac{1}{n}} \frac{1}{t}\,\mathrm{d}t = \ln(2021n) - \ln n = \ln 2021\,,$$

we conclude that

$$\lim_{n\to\infty} y_n = \ln 2021\,.$$

Solution - Problem 2:

Since $A^2 = -I_n$, the only possible eigenvalues of A are $\pm i$. Since also $A \in \mathcal{M}_n(\mathbb{R})$ then $n = 2k$ and A has k eigenvalues equal to i and k eigenvalues equal to $-i$. Its minimal polynomial is $x^2 + 1$ which has distinct roots, therefore A is diagonalizable and is therefore similar to

$$X = \begin{bmatrix} iI_k & 0_k \\ 0_k & -iI_k \end{bmatrix}.$$

Similarly, if $P = \begin{bmatrix} 0_k & I_k \\ -I_k & 0_k \end{bmatrix}$, then P is also a real matrix with $P^2 = -I_n$ and so P is also similar to X. Therefore A and P are similar and so there is an invertible matrix $U \in \mathcal{M}_n(\mathbb{R})$ such that $P = U^{-1}AU$. For $C = U^{-1}BU \in \mathcal{M}_n(\mathbb{R})$ we get

$$CP = U^{-1}BAU \quad \text{and} \quad PC = U^{-1}ABU\,. \tag{1}$$

Since $AB = BA$, by (1) it follows that $CP = PC$.

Writing C into block form $C = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}$, where $X, Y, Z, T \in \mathcal{M}_k(\mathbb{R})$ and using $CP = PC$, it follows that $X = T$ and $Z = -Y$. Hence $C = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$. We now see that

$$\begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \begin{vmatrix} X + iY & Y - iX \\ -Y & X \end{vmatrix} = \begin{vmatrix} X + iY & (Y - iX) - i(X + iY) \\ -Y & X - iY \end{vmatrix} = \begin{vmatrix} X + iY & 0 \\ -Y & X - iY \end{vmatrix}.$$

Therefore

$$\det B = \det C = \begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = \det(X - iY)\det(X + iY) = |\det(X + iY)|^2 \geqslant 0.$$

Alternative Solution - Problem 2

Let λ be a real eigenvalue of B and let G_λ be its generalized eigenspace considered as a real vector space. I.e.

$$G_\lambda = \{\mathbf{v} \in \mathbb{R}^n : (B - \lambda I_n)^n \mathbf{v} = \mathbf{0}\}\,.$$

We have $AB^2 = (AB)B = (BA)B = B(AB) = B(BA) = B^2A$. Inductively we get $AB^k = B^kA$ for every natural number k and from this we deduce that $Ap(B) = p(B)A$ for every polynomial $p(x)$. In particular, $A(B - \lambda I_n)^n = (B - \lambda I_n)^n A$.

Now if $\mathbf{v} \in G_\lambda$, then $(B - \lambda I_n)^n(A\mathbf{v}) = A(B - \lambda I_n)^n \mathbf{v} = \mathbf{0}$, so $A\mathbf{v} \in G_\lambda$. Therefore we can define the linear map $\alpha : G_\lambda \to G_\lambda$ by $\alpha(\mathbf{v}) = A\mathbf{v}$.

Pick a basis of G_λ and let A' be the matrix of α with respect to this basis. Then $A' \in \mathcal{M}_n(\mathbb{R})$ and $(A')^2 = -I_{n'}$, where $n' = \dim(G_\lambda)$. As in the previous solution, we get that n' is even.

Since $\dim(G_\lambda)$ is even for every real eigenvalue of B and since its complex eigenvalues come in conjugate pairs, then $\det(B) \geqslant 0$.

Solution - Problem 3:

(a) The matrix AA^* is Hermitian and all its eigenvalues are non-negative real numbers.

If $\lambda \in \sigma(AA^*)$, then $\lambda^2 \in \sigma\left((AA^*)^2\right) = \sigma\left(A^*A\right) = \sigma(AA^*)$, hence $\lambda^2 \in \sigma(AA^*)$. It follows by induction that $\lambda^{2^k} \in \sigma(AA^*)$, for all $k \in \mathbb{N}$. Since $\lambda \geqslant 0$, the last relation assures us that $\lambda \in \{0,1\}$, so AA^* will have eigenvalues 0 or 1. On the other hand, since AA^* is Hermitian, it is also diagonalizable, thus

$$AA^* = U^{-1}\begin{bmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix} U\,.$$

Using the above statement, we conclude that

$$A^*A = (AA^*)^2 = AA^*\,.$$

(b) Using (a), the equality of our hypothesis can be transformed into $A^*A \cdot (AA^* - I_n) = O_n$. Letting $B = A \cdot (AA^* - I_n)$ we obtain

$$B^*B = (AA^* - I_n)A^*A(AA^* - I_n) = O_n$$

which gives $B = O_n$. Thus

$$A^2A^* = A\,. \tag{1}$$

Since $A^*A = AA^*$, it follows that the matrix A is normal, hence it is a unitary diagonalizable matrix. It follows that there is an unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that $A = U^*DU$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then $A^2A^* = U^*D^2UU^*\overline{D}U = U^*D^2\overline{D}U$ and using (1) we get

$$\begin{aligned} A^2A^* = A \iff D^2\overline{D} = D &\iff \lambda_i^2 \cdot \overline{\lambda_i} = \lambda_i \text{ for all } i \in \{1,2,\ldots,n\} \\ &\iff \lambda_i(|\lambda_i|^2 - 1) = 0 \text{ for all } i \in \{1,2,\ldots,n\}\,. \end{aligned}$$

Hence the conclusion.

Alternative Solution - Problem 3

(a) Let $X = AA^*$ and $Y = A^*A$. Since X is Hermitian, it is diagonalizable so $P^{-1}XP = D$ for some matrices P, D with D diagonal. Let $Z = P^{-1}YP$. The initial condition gives $Z = D^2$. Since X and Y have the same characteristic polynomial, so do $Z = D^2$ and D. As in the original proof we deduce that every entry of D must be 0 or 1. Then $Z = D$ and so $X = Y$ as required.

(b) Writing $A = U^*DU$ as in the original proof and using $(AA^*)^2 = A^*A$ (rather than $A^2A^* = A$) we get $(D\overline{D})^2 = \overline{D}D$. From this we get that $|\lambda|^4 = |\lambda|^2$ for each eigenvalue λ of A and the conclusion follows.

Solution - Problem 4:

For every positive integer n, let

$$I_n = \int_0^n \big|\sin(\pi x)\big|^x \,\mathrm{d}x = \sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^x \,\mathrm{d}x\,.$$

Then we have

$$\sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^{k+1} \,\mathrm{d}x < I_n < \sum_{k=0}^{n-1} \int_k^{k+1} \big|\sin(\pi x)\big|^{k} \,\mathrm{d}x\,.$$

Substituting $t = \pi x - k\pi$, we deduce that

$$\int_k^{k+1} \big|\sin(\pi x)\big|^m \,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin^m t\,\mathrm{d}t$$

for every nonnegative integer m. Therefore

$$\frac{1}{\pi}\sum_{k=1}^{n} J_k < I_n < \frac{1}{\pi}\sum_{k=0}^{n-1} J_k\,, \tag{1}$$

where $J_k = \displaystyle\int_0^\pi \sin^k t\,\mathrm{d}t$. For $k \geqslant 2$, integration by parts yields

$$\begin{aligned}
J_k &= \int_0^\pi (-\cos t)' \sin^{k-1} t\,\mathrm{d}t \\
&= \Big[-\cos t \sin^{k-1} t\Big]_0^\pi + (k-1)\int_0^\pi \sin^{k-2} t \cos^2 t\,\mathrm{d}t \\
&= 0 + (k-1)\int_0^\pi \sin^{k-2} t(1-\sin^2 t)\,\mathrm{d}t \\
&= (k-1)J_{k-2} - (k-1)J_k\,,
\end{aligned}$$

whence

$$J_k = \frac{k-1}{k}\,J_{k-2}\,.$$

Since $J_0 = \pi$ and $J_1 = 2$, we obtain

$$J_{2k} = \pi\,\frac{(2k-1)!!}{(2k)!!} \quad \text{and} \quad J_{2k+1} = 2\,\frac{(2k)!!}{(2k+1)!!}\,.$$

We observe that

$$J_{2k-1}J_{2k} = \frac{2\pi}{2k} \quad \text{and} \quad J_{2k}J_{2k+1} = \frac{2\pi}{2k+1}\,.$$

Since (J_n) is a decreasing sequence, we deduce that

$$\frac{2\pi}{2k+1} = J_{2k}J_{2k+1} \leqslant J_{2k}^2 \leqslant J_{2k-1}J_{2k} = \frac{2\pi}{2k}$$

It follows that $\sqrt{2\pi}\sqrt{\frac{2k}{2k+1}} = \sqrt{2k}J_{2k} \leqslant \sqrt{2\pi}$ and therefore

$$\lim_{k\to\infty} \sqrt{2k}J_{2k} = \sqrt{2\pi}\,. \tag{2}$$

Similarly $\sqrt{2\pi}\sqrt{\frac{2k+1}{2k+2}} \leqslant \sqrt{2k+1}J_{2k+1} \leqslant \sqrt{2\pi}$ and therefore

$$\lim_{k\to\infty} \sqrt{2k+1}J_{2k+1} = \sqrt{2\pi}\,. \tag{3}$$

By (2) and (3) it follows that

$$\lim_{n\to\infty} \sqrt{n}\,J_n = \sqrt{2\pi}\,. \tag{4}$$

By virtue of (4) and the Cesàro-Stolz theorem we have

$$\begin{aligned}\lim_{n\to\infty} \frac{J_1 + \cdots + J_n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{J_{n+1}}{\sqrt{n+1} - \sqrt{n}} \\ &= \lim_{n\to\infty} \left(\sqrt{n+1} + \sqrt{n}\right) J_{n+1} \\ &= 2\sqrt{2\pi}\,.\end{aligned} \tag{5}$$

Now relations (1) and (5) ensure that

$$\lim_{n\to\infty} \frac{I_n}{\sqrt{n}} = \frac{1}{\pi} \cdot 2\sqrt{2\pi} = 2\sqrt{\frac{2}{\pi}}\,.$$

Taking into consideration that

$$a_n = \frac{I_n}{n^p} = \frac{I_n}{\sqrt{n}} \cdot \frac{1}{n^{p-\frac{1}{2}}}\,,$$

we deduce that the series $\sum_{n=1}^{\infty} a_n$ has the same nature as $\sum_{n=1}^{\infty} \frac{1}{n^{p-\frac{1}{2}}}$. In conclusion, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $p > \frac{3}{2}$.

Comments.

(1) One could use Wallis' formula or Stirling's Approximation in order to deduce (4).

(2) One could avoid the use of Cesàro-Stolz as follows: By (4) we have $J_n = \Theta(\frac{1}{\sqrt{n}})$. Since also (e.g. by considering Riemann sums) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} = \Theta(\sqrt{n})$ then $a_n = \Theta\left(\frac{1}{n^{p-1/2}}\right)$ and the conclusion follows as before.