JBMO ShortLists 2012

– Algebra

1 Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{b}+\frac{a}{c}+\frac{c}{b}+\frac{c}{a}+\frac{b}{c}+\frac{b}{a}+6 \geq 2\sqrt{2}\left(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\right).$$

When does equality hold?

2 Let a , b , c be positive real numbers such that $abc=1$. Show that :

$$\frac{1}{a^3+bc}+\frac{1}{b^3+ca}+\frac{1}{c^3+ab} \leq \frac{(ab+bc+ca)^2}{6}$$

3 Let a , b , c be positive real numbers such that $a+b+c=a^2+b^2+c^2$. Prove that :

$$\frac{a^2}{a^2+ab}+\frac{b^2}{b^2+bc}+\frac{c^2}{c^2+ca} \geq \frac{a+b+c}{2}$$

4 Solve the following equation for $x, y, z \in \mathbb{N}$:

$$\left(1+\frac{x}{y+z}\right)^2+\left(1+\frac{y}{z+x}\right)^2+\left(1+\frac{z}{x+y}\right)^2=\frac{27}{4}$$

5 Find the largest positive integer n for which the inequality

$$\frac{a+b+c}{abc+1}+\sqrt[n]{abc} \leq \frac{5}{2}$$

holds true for all $a, b, c \in [0,1]$. Here we make the convention $\sqrt[1]{abc}=abc.$

– Geometry

1 Let ABC be an equilateral triangle , and P be a point on the circumcircle of the triangle but distinct from A ,B and C. The lines through P and parallel to BC , CA , AB intersect the lines CA , AB , BC at M , N and Q respectively .Prove that M , N and Q are collinear .

2 Let ABC be an isosceles triangle with $AB = AC$. Let also ω be a circle of center K tangent to the line AC at C which intersects the segment BC again at H . Prove that $HK \perp AB$.

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3 Let AB and CD be chords in a circle of center O with A, B, C, D distinct , and with the lines AB and CD meeting at a right angle at point E. Let also M and N be the midpoints of AC and BD respectively . If $MN \perp OE$, prove that $AD \parallel BC$.

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4 Let ABC be an acute-angled triangle with circumcircle ω , and let O , H be the triangle's circumcenter and orthocenter respectively . Let also A' be the point where the angle bisector of the angle BAC meets ω . If $A'H = AH$, then find the measure of the angle BAC.

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5 Let the circles k_1 and k_2 intersect at two points A and B, and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate the angle NMB.

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6 Let O_1 be a point in the exterior of the circle ω of center O and radius R , and let O_1N , O_1D be the tangent segments from O_1 to the circle. On the segment O_1N consider the point B such that $BN = R$ .Let the line from B parallel to ON intersect the segment O_1D at C . If A is a point on the segment O_1D other than C so that $BC = BA = a$, and if the incircle of the triangle ABC has radius r , then find the area of $\triangle ABC$ in terms of a, R, r.

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7 Let $MNPQ$ be a square of side length 1 , and A, B, C, D points on the sides MN, NP, PQ and QM respectively such that $AC \cdot BD = \frac{5}{4}$. Can the set $\{AB, BC, CD, DA\}$ be partitioned into two subsets S_1 and S_2 of two elements each , so that each one has the sum of his elements a positive integer?

– Combinatorics

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1 Along a round table are arranged 11 cards with the names (all distinct) of the 11 members of the 16^{th} JBMO Problem Selection Committee . The cards are arranged in a regular polygon manner . Assume that in the first meeting of the Committee none of its 11 members sits in front of the card with his name . Is it possible to rotate the table by some angle so that at the end at least two members sit in front of the card with their names ?

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2 On a board there are n nails, each two connected by a rope. Each rope is colored in one of n given distinct colors. For each three distinct colors, there exist three nails connected with ropes of these three colors.
a) Can n be 6 ?
b) Can n be 7 ?

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3 In a circle of diameter 1 consider 65 points, no three of them collinear. Prove that there exist three among these points which are the vertices of a triangle with area less than or equal to $\frac{1}{72}$.

–	Number Theory
1	If a , b are integers and $s = a^3 + b^3 - 60ab(a+b) \geq 2012$, find the least possible value of s.
2	Do there exist prime numbers p and q such that $p^2(p^3-1) = q(q+1)$?
3	Decipher the equality : $$(\overline{VER} - \overline{IA}) = G^{R^E}(\overline{GRE} + \overline{ECE})$$ assuming that the number $\overline{GREECE}$ has a maximum value .Each letter corresponds to a unique digit from 0 to 9 and different letters correspond to different digits . It's also supposed that all the letters G ,E ,V and I are different from 0.
4	Determine all triples (m,n,p) satisfying : $$n^{2p} = m^2 + n^2 + p + 1$$ where m and n are integers and p is a prime number.
5	Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.
6	If a , b , c , d are integers and $A = 2(a-2b+c)^4 + 2(b-2c+a)^4 + 2(c-2a+b)^4$, $B = d(d+1)(d+2)(d+3)+1$, then prove that $\left(\sqrt{A}+1\right)^2 + B$ cannot be a perfect square.
7	Find all $a, b, c \in \mathbb{N}$ for which $$1997^a + 15^b = 2012^c$$