2009 Balkan MO Shortlist

– Algebra

A1 Let $N \in \mathbb{N}$ and $x_k \in [-1,1]$, $1 \leq k \leq N$ such that $\sum_{k=1}^{N} x_k = s$. Find all possible values of $\sum_{k=1}^{N} |x_k|$

A2 Let $ABCD$ be a square and points $M \in BC$, $N \in CD$, $P \in DA$, such that $\angle BAM = x$, $\angle CMN = 2x$, $\angle DNP = 3x$

- Show that, for any $x \in (0, \frac{\pi}{8})$, such a configuration exists
- Determine the number of angles $x \in (0, \frac{\pi}{8})$ for which $\angle APB = 4x$

A3 Denote by $S(x)$ the sum of digits of positive integer x written in decimal notation. For k a fixed positive integer, define a sequence $(x_n)_{n\geq 1}$ by $x_1 = 1$ and $x_{n+1} = S(kx_n)$ for all positive integers n. Prove that $x_n < 27\sqrt{k}$ for all positive integer n.

A4 Denote by S the set of all positive integers. Find all functions $f : S \to S$ such that

$$f(f^2(m) + 2f^2(n)) = m^2 + 2n^2$$

for all $m, n \in S$.

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A5 Given the monic polynomial

$$P(x) = x^N + a_{N-1}x^{N-1} + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$$

of even degree $N = 2n$ and having all real positive roots x_i, for $1 \leq i \leq N$. Prove, for any $c \in [0, \min_{1\leq i\leq N}\{x_i\})$, the following inequality

$$c + \sqrt[N]{P(c)} \leq \sqrt[N]{a_0}$$

A6 We denote the set of nonzero integers and the set of non-negative integers with $\mathbb{Z}^*$ and $\mathbb{N}_0$, respectively. Find all functions $f : \mathbb{Z}^* \to \mathbb{N}_0$ such that: $a)$ $f(a+b) \geq min(f(a), f(b))$ for all a, b in $\mathbb{Z}^*$ for which $a+b$ is in $\mathbb{Z}^*$. $b)$ $f(ab) = f(a) + f(b)$ for all a, b in $\mathbb{Z}^*$.

A7 Let $n \geq 2$ be a positive integer and

$$P(x) = c_0X^n + c_1X^{n-1} + \ldots + c_{n-1}X + c_n$$

be a polynomial with integer coefficients, such that $\mid c_n \mid$ is a prime number and

$$|c_0| + |c_1| + \ldots + |c_{n-1}| < |c_n|$$

Prove that the polynomial $P(X)$ is irreducible in the $\mathbb{Z}[x]$

A8 For every positive integer m and for all non-negative real numbers x, y, z denote

$$K_m = x(x-y)^m(x-z)^m + y(y-x)^m(y-z)^m + z(z-x)^m(z-y)^m$$

- Prove that $K_m \geq 0$ for every odd positive integer m
- Let $M = \prod_{cyc}(x-y)^2$. Prove, $K_7 + M^2K_1 \geq MK_4$

– Geometry

G1 In the triangle $ABC, \angle BAC$ is acute, the angle bisector of $\angle BAC$ meets BC at D, K is the foot of the perpendicular from B to AC, and $\angle ADB = 45^o$. Point P lies between K and C such that $\angle KDP = 30^o$. Point Q lies on the ray DP such that $DQ = DK$. The perpendicular at P to AC meets KD at L. Prove that $PL^2 = DQ \cdot PQ$.

G2 If $ABCDEF$ is a convex cyclic hexagon, then its diagonals AD, BE, CF are concurrent if and only if $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$.

Alternative version. Let $ABCDEF$ be a hexagon inscribed in a circle. Then, the lines AD, BE, CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

G3 Let $ABCD$ be a convex quadrilateral, and P be a point in its interior. The projections of P on the sides of the quadrilateral lie on a circle with center O. Show that O lies on the line through the midpoints of AC and BD.

G4 Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

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G5 Let $ABCD$ be a convex quadrilateral and S an arbitrary point in its interior. Let also E be the symmetric point of S with respect to the midpoint K of the side AB and let Z be the symmetric point of S with respect to the midpoint L of the side CD. Prove that $(AECZ) = (EBZD) = (ABCD)$.

G6 Two circles O_1 and O_2 intersect each other at M and N. The common tangent to two circles nearer to M touch O_1 and O_2 at A and B respectively. Let C and D be the reflection of A and B respectively with respect to M. The circumcircle of the triangle DCM intersect circles O_1 and O_2 respectively at points E and F (both distinct from M). Show that the circumcircles of triangles MEF and NEF have same radius length.

– Combinatorics

C1 A 9×12 rectangle is partitioned into unit squares. The centers of all the unit squares, except for the four corner squares and eight squares sharing a common side with one of them, are coloured red. Is it possible to label these red centres $C_1, C_2, \ldots, C_{96}$ in such way that the following to conditions are both fulfilled
i) the distances $C_1C_2, \ldots, C_{95}C_{96}, C_{96}C_1$ are all equal to $\sqrt{13}$,
ii) the closed broken line $C_1C_2 \ldots C_{96}C_1$ has a centre of symmetry?

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C2 Let $A_1, A_2, \ldots, A_m$ be subsets of the set $\{1, 2, \ldots, n\}$, such that the cardinal of each subset A_i, such $1 \leq i \leq m$ is not divisible by 30, while the cardinal of each of the subsets $A_i \cap A_j$ for $1 \leq i, j \leq m$, $i \neq j$ is divisible by 30. Prove

$$2m - \left\lfloor \frac{m}{30} \right\rfloor \leq 3n$$

– Number Theory

N1 Solve the given equation in integers

$$y^3 = 8x^6 + 2x^3y - y^2$$

N2 Solve the equation

$$3^x - 5^y = z^2.$$

in positive integers.

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N3 Determine all integers $1 \leq m, 1 \leq n \leq 2009$, for which

$$\prod_{i=1}^{n} \left(i^3 + 1\right) = m^2$$