XV International Zhautykov Olympiad in Mathematics
Almaty, 2019
January 11, 9.00-13.30

№1. Prove that there are at least 100! ways to partition the number 100! into summands from the set $\{1!, 2!, 3!, \ldots, 99!\}$. (Partitions differing in the order of summands are considered the same; any summand can be taken multiple times. We remind that $n! = 1 \cdot 2 \cdot \ldots \cdot n$.)

Solution. Let us prove by induction on $n \geqslant 4$ that there are at least $n!$ ways to partition the number $n!$ into summands from $\{1!, 2!, \ldots, (n-1)!\}$.

For $n = 4$, if we use only the summands $1!, 2!$ there are 13 ways to partition $4!$ as $2!$ can be used from 0 to 12 times. If $3!$ is used 1 time, then $4! - 3! = 18$ can be partitioned using $1!, 2!$ in 10 ways. We get at least one more partition if we use $3!$ two times. So, there are at least 24 such partitions as needed.

Suppose now the statement holds for n and let us prove it for $n+1$. To partition $(n+1)!$, the summand $n!$ can be used i times for $0 \leqslant i \leqslant n$. By the hypothesis, for every such i, the remaining number $(n+1)! - i \cdot n! = (n+1-i) \cdot n!$ can be partitioned into the summands $\{1!, \ldots, (n-1)!\}$ in at least $n!$ ways as follows. For any partition of $n!$ take each summand appearing say k times and write it $(n+1-i)k$ times. Hence we obtain at least $(n+1) \cdot n! = (n+1)!$ ways to partition the number $(n+1)!$ as desired. The original problem follows for $n = 100$ then.

№2. Find the largest real C such that for all pairwise distinct positive real $a_1, a_2, \ldots, a_{2019}$ the following inequality holds

$$\frac{a_1}{|a_2 - a_3|} + \frac{a_2}{|a_3 - a_4|} + \ldots + \frac{a_{2018}}{|a_{2019} - a_1|} + \frac{a_{2019}}{|a_1 - a_2|} > C.$$

2. **The answer** is 1010.

Solution. Without loss of generality we assume that $\min(a_1, a_2, \ldots, a_{2019}) = a_1$. Note that if a, b, c ($b \neq c$) are positive, then $\frac{a}{|b-c|} > \min(\frac{a}{b}, \frac{a}{c})$. Hence

$$S = \frac{a_1}{|a_2 - a_3|} + \cdots + \frac{a_{2019}}{|a_1 - a_2|} > 0 + \min\left(\frac{a_2}{a_3}, \frac{a_2}{a_4}\right) + \cdots + \min\left(\frac{a_{2017}}{a_{2018}}, \frac{a_{2017}}{a_{2019}}\right) + \frac{a_{2018}}{a_{2019}} + \frac{a_{2019}}{a_2} = T.$$

Take $i_0 = 2$ and for each $\ell \geqslant 0$ let $i_{\ell+1} = i_\ell + 1$ if $a_{i_\ell+1} > a_{i_\ell+2}$ and $i_{\ell+1} = i_\ell + 2$ otherwise. There is an integral k such that $i_k < 2018$ and $i_{k+1} \geqslant 2018$. Then

$$T \geqslant \frac{a_2}{a_{i_1}} + \frac{a_{i_1}}{a_{i_2}} + \cdots + \frac{a_{i_k}}{a_{i_{k+1}}} + \frac{a_{2018}}{a_{2019}} + \frac{a_{2019}}{a_2} = A. \tag{1}$$

We have $1 \leqslant i_{\ell+1} - i_\ell \leqslant 2$, therefore $i_{k+1} \in \{2018, 2019\}$.

Since

$$2018 \leqslant i_{k+1} = i_0 + (i_1 - i_0) + \cdots + (i_{k+1} - i_k) \leqslant 2(k+2), \tag{2}$$

it follows that $k \geqslant 1007$. Consider two cases.

(i) $k = 1007$. Then in the inequality (2) we have equalities everywhere, in particular $i_{k+1} = 2018$. Applying AM–GM inequality for $k+3$ numbers to (1) we obtain $A \geqslant k+3 \geqslant 1010$.

(ii) $k \geqslant 1008$. If $i_{k+1} = 2018$ then we get $A \geqslant k+3 \geqslant 1011$ by the same argument as in the case (i). If $i_{k+1} = 2019$ then applying AM–GM inequality to $k+2$ summands in (1) (that is, to all the summands except $\frac{a_{2018}}{a_{2019}}$) we get $A \geqslant k+2 \geqslant 1010$.

So we have $S > T \geqslant A \geqslant 1010$. For $a_1 = 1+\varepsilon$, $a_2 = \varepsilon$, $a_3 = 1+2\varepsilon$, $a_4 = 2\varepsilon$, $\ldots$, $a_{2016} = 1008\varepsilon$, $a_{2017} = 1+1009\varepsilon$, $a_{2018} = \varepsilon^2$, $a_{2019} = 1$ we obtain $S = 1009 + 1008\varepsilon + \frac{1008\varepsilon}{1+1009\varepsilon-\varepsilon^2} + \frac{1+1009\varepsilon}{1-\varepsilon^2}$. Then $\lim\limits_{\varepsilon \to 0} S = 1010$, which means that the constant 1010 cannot be increased.

№3. The extension of median CM of the triangle ABC intersects its circumcircle ω at N. Let P and Q be the points on the rays CA and CB respectively such that $PM \parallel BN$ and $QM \parallel AN$. Let X and Y be the points on the segments PM and QM respectively such that PY and QX are tangent to ω. The segments PY and QX intersect at Z. Prove that the quadrilateral $MXZY$ is circumscribed.

Solution.

Lemma. The points K and L lie on the sides BC and AC of a triangle ABC. The segments AK and BL intersect at D. Then the quadrilateral $CKDL$ is circumscribed if and only if $AC - BC = AD - BD$.

Proof. Let $CKDL$ be circumscribed and its incircle touches LC, CK, KD, DL at X, Y, Z, T respectively (see Fig. 1). Then

$$AC - BC = AX - BY = AZ - BT = AD - BD.$$

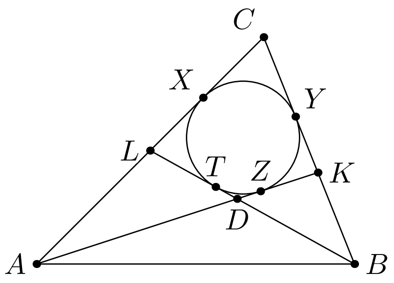

Рис. 1

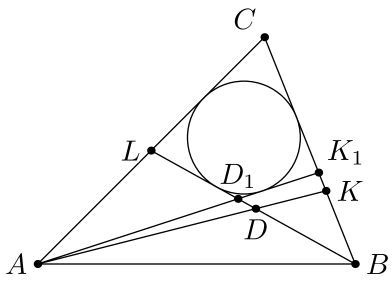

Рис. 2

Now suppose that $AC - BC = AD - BD$. Let the tangent to the incircle of BLC different from AC meets the segments BL and BC at D_1 and K_1 respectively. If $K = K_1$ then the lemma is proved. Otherwise $AD_1 - BD_1 = AC - BC = AD - BD$ or $AD_1 - BD_1 = AD - BD$. In the case when D lies on the segment BD_1 (see Fig. 2) we have

$$AD_1 - BD_1 = AD - BD \Rightarrow AD_1 - AD = BD_1 - BD \Rightarrow AD_1 - AD = DD_1.$$

But the last equation contradicts the triangle inequality, since $AD_1 - AD < DD_1$. The case when D is outside the segment BD_1 is similar.

Back to the solution of the problem, let PY and QX touch ω at Y_1 and X_1 respectively. Since $ACBN$ is cyclic and $PM \parallel BN$ we have $\angle ACN = \angle ABN = \angle AMP$, i. e. the circumcircle of $\triangle AMC$ is tangent to the line PM. Thus $PM^2 = PA \cdot PC$. But $PA \cdot PC = PY_1^2$, and therefore $PM = PY_1$. In the same way we have $QM = QX_1$. Obviously $ZX_1 = ZY_1$. It remains to note that the desired result follows from the Lemma because

$$PM - QM = PY_1 - QX_1 = (PZ + ZY_1) - (QZ + ZX_1) = PZ - QZ \quad \Rightarrow \quad PM - QM = PZ - QZ.$$

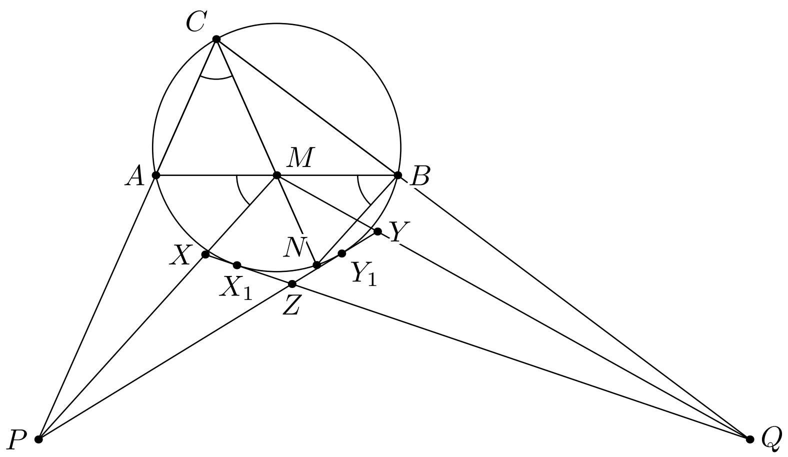

Рис. 3

Note. This solution does not use the comdition that M is the midpoint of AB.

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Second day. Solutions

№4. An isosceles triangle ABC with $AC = BC$ is given. Point D is chosen on the side AC. The circle S_1 of radius R with the center O_1 touches the segment AD and the extensions of BA and BD over the points A and D, respectively. The circle S_2 of radius $2R$ with the center O_2 touches the segment DC and the extensions of BD and BC over the points D and C, respectively. Let the tangent to the circumcircle of the triangle BO_1O_2 at the point O_2 intersect the line BA at point F. Prove that $O_1F = O_1O_2$.

Solution By condition, in the triangle ABC we have $\angle A = \angle B$. It is evident that $\angle O_1BO_2 = \angle B/2$. Let ℓ be the straight line passing through O_2 parallel to AC. By the problem condition ℓ touches S_1 (say, at a point N). Let also K be the tangency point of S_1 and BA. Then the clockwise rotation about the point O_1 through the angle NO_1K transposes ℓ to BA and thus transposes the point O_2 to some point $O \in BA$. Hence $O_1O = O_1O_2$ and $\angle OO_1O_2 = \angle NO_1K = 180° - \angle A = 180° - \angle B$, so $\angle O_1O_2O = \angle B/2 = \angle O_1BO_2$. The latter does mean that the line O_2O is the tangent to the circumcircle of $\triangle BO_1O_2$. Hence $F = O$, and $O_1F = O_1O_2$, as was to be proved.

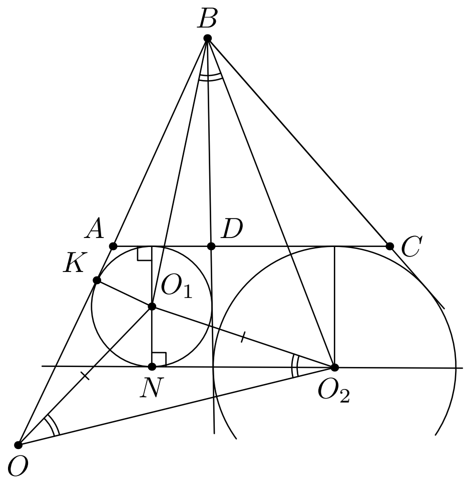

Рис. 1

№5. Let $n > 1$ be a positive integer. A function $f : I \to \mathbb{Z}$ is given, where I is the set of all integers coprime with n. ($\mathbb{Z}$ is the set of integers). A positive integer k is called a *period* of the function f if $f(a) = f(b)$ for all $a, b \in I$ such that $a \equiv b \pmod{k}$. It is known that n is a period of f. Prove that the minimal period of the function f divides all its periods.

Example. For $n = 6$, the function f with period 6 is defined entirely by its values $f(1)$ and $f(5)$. If $f(1) = f(5)$, then the function has minimal period $P_{\min} = 1$, and if $f(1) \neq f(5)$, then $P_{\min} = 3$.

№6. On a polynomial of degree three it is allowed to perform the following two operations arbitrarily many times:

(i) reverse the order of its coefficients including zeroes (for instance, from the polynomial $x^3 - 2x^2 - 3$ we can obtain $-3x^3 - 2x + 1$);

(ii) change polynomial $P(x)$ to the polynomial $P(x+1)$.

Is it possible to obtain the polynomial $x^3 - 3x^2 + 3x - 3$ from the polynomial $x^3 - 2$?

The **answer** is no.

Solution I. The original polynomial $x^3 - 2$ has a unique real root. The two transformations clearly preserve this property. If α is the only real root of $P(x)$, then the first operation produces a polynomial with root $\frac{1}{\alpha}$, and the second operation gives a polynomial with root $\alpha - 1$. Since the root of the original polynomial is $\sqrt[3]{2}$, and thar of the resulting polynomial is $1 + \sqrt[3]{2}$, the problem is reduced to the question whether it is possible to obtain the latter number from the former by operations $x \mapsto \frac{1}{x}$ and $x \mapsto x - 1$. Let us apply one more operation $x \mapsto x - 1$ (so as to transform $\sqrt[3]{2}$ to itself) and reverse all the operations. It appears then that the number $\sqrt[3]{2}$ is transformed to itself by several operations of the form $x \mapsto \frac{1}{x}$ and $x \mapsto x + 1$. It is easy to see that the composition of any number of such operations is a fractional-linear function $x \mapsto \frac{ax+b}{cx+d}$, where a, b, c, d are non-negative integers and $ad - bc = 1$. Each operation $x \mapsto x + 1$ increases $a + b + c + d$, and, since we started with this operation, the resulting function is not identical. Thus $\sqrt[3]{2}$ is transformed to itself by some such composition. This means however that $\sqrt[3]{2}$ is a root of non-zero polynomial $x(cx+d) - ax - b$ with integral coefficients and degree at most 2, which is impossible.

Solution II. The original polynomial has one real and two conjugate complex roots. We have seen above that under the two operations these roots are subject to transforms $x \mapsto \frac{1}{x}$ and $x \mapsto x - 1$. Note that both imaginary roots of the original polynomial have negative real part. It is easy to check that this property is preserved under the two operations. However the real parts of all the roots of the desired polynomial are positive, a contradiction.

Solution III. For a polynomial $P(x) = ax^3 + bx^2 + cx + d$ we define $\Delta(P) = 3ad - bc$. The first operation transforms $P(x)$ to $dx^3 + cx^2 + bx + a$ and does not change Δ. The second operation transforms $P(x)$ to $Q(x) = ax^3 + (b+3a)x^2 + (c+3a+2b)x + (d+a+b+c)$, for which $\Delta(Q) = 3(d+a+b+c)a - - (b+3a)(c+3a+2b) = \Delta(P) - (2b^2 + 6ab + 6a^2) < \Delta(P)$. Thus the permitted operation can not increase Δ. On the other hand, for the original polynomial $\Delta(P) = -6$, and for the resulting polynomial it must be 0.