Mediterranean Mathematics Olympiad 2021

1 Determine the smallest positive integer M with the following property:
For every choice of integers a, b, c, there exists a polynomial $P(x)$ with integer coefficients so that $P(1) = aM$ and $P(2) = bM$ and $P(4) = cM$.

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2 For every sequence $p_1 < p_2 < \cdots < p_8$ of eight prime numbers, determine the largest integer N for which the following equation has no solution in positive integers $x_1, \ldots, x_8$:

$$p_1\, p_2\, \cdots\, p_8 \left(\frac{x_1}{p_1} + \frac{x_2}{p_2} + \cdots + \frac{x_8}{p_8} \right) = N$$

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3 Let ABC be an equiangular triangle with circumcircle ω. Let point $F \in AB$ and point $E \in AC$ so that $\angle ABE + \angle ACF = 60°$. The circumcircle of triangle AFE intersects the circle ω in the point D. The halflines DE and DF intersect the line through B and C in the points X and Y. Prove that the incenter of the triangle DXY is independent of the choice of E and F.

(The angles in the problem statement are not directed. It is assumed that E and F are chosen in such a way that the halflines DE and DF indeed intersect the line through B and C.)

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4 Let x_1, x_2, x_3, x_4, x_5 ve non-negative real numbers, so that $x_1 \le 4$ and $x_1 + x_2 \le 13$ and $x_1 + x_2 + x_3 \le 29$ and $x_1 + x_2 + x_3 + x_4 \le 54$ and $x_1 + x_2 + x_3 + x_4 + x_5 \le 90$.
Prove that $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5} \le 20$.
